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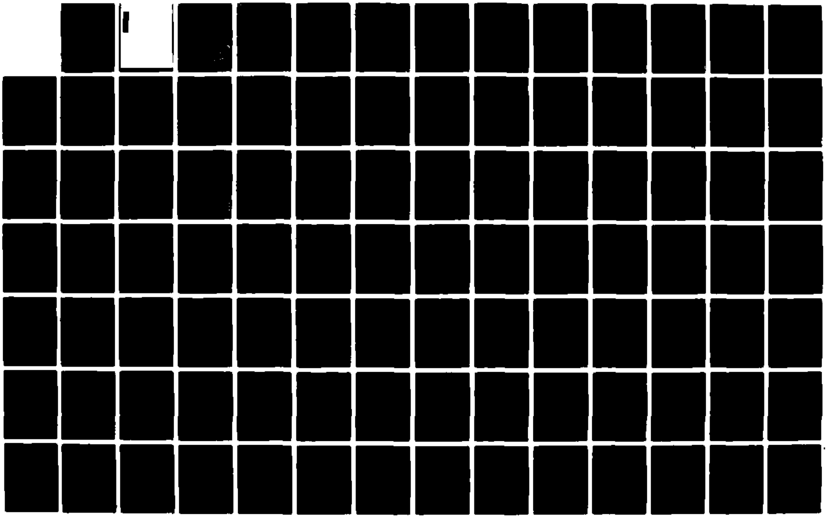

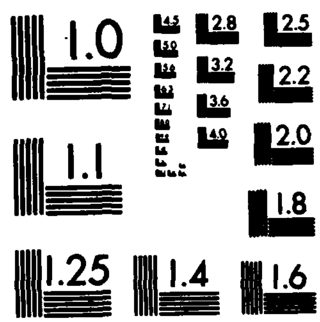

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Singular perturbations and order reduction in control theory

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SINGULAR PERTURBATIONS AND TIME SCALES IN MODELING AND CONTROL OF DYNAMIC SYSTEMS

Petar V. Kokotovic, Editor

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November 1980

PREFACE

The purpose of this volume is twofold. First, it reports on recent developments of singular perturbation and two-time-scale methods for modeling, analysis and design of control systems. The results obtained in the last five years are summarized in thirty papers which appeared in the period of 1976-1980. Second, it responds to the need for a comprehensive and systematic treatment of this rapidly developing field of research. For this reason, seven earlier papers are included and the whole collection is organized in a logical rather than chronological order. The First three sections deal with modeling and analysis, while the subsequent four sections are devoted to the design and optimization of linear, nonlinear and stochastic control systems. The last two sections treat large scale system problems with multiple controllers and incomplete models.

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SECTION 1

SURVEY AND INTRODUCTION

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Singular Perturbations and Order Reduction in Control Theory — An Overview* †

P. V. KOKOTOVIC,‡ R. E. O'MALLEY, JR.§ and P. SANNUTI||

Singular perturbation methods are physically motivated tools for order reduction, separation of time scales and other simplifications in control system analysis and design.

Summary—Recent results on singular perturbations are surveyed as a tool for model order reduction and separation of time scales in control system design. Conceptual and computational simplifications of design procedures are examined by a discussion of their basic assumptions. Over 100 references are organized into several problem areas. The content of main theorems is presented in a tutorial form aimed at a broad audience of engineers and applied mathematicians interested in control, estimation and optimization of dynamic systems.

INTRODUCTION

ALTHOUGH many control theory concepts are valid for any system order, their actual use is limited to low order models. In optimization of dynamic systems the 'curse of dimensionality' is not only in a formidable amount of computation, but also in the ill-conditioned initial and two point boundary value problems. The interaction of fast and slow phenomena in high-order systems results in 'stiff' numerical problems which require expensive integration routines.

The singular perturbation approach outlined in this survey alleviates both dimensionality and stiffness difficulties. It lowers the model order by first neglecting the fast phenomena. It then improves the approximation by reintroducing their effect as 'boundary layer' corrections calculated in separate time scales. Further improvements are possible by asymptotic expansion methods. In addition to being helpful in design procedures, the singular perturbation approach is an indispensable tool for analytical investigations of robustness of system properties, behavior of optimal controls near singular arcs, and other effects of intentional or unintentional changes of system order.

This paper is a tutorial survey of recent works on singular perturbations in control and estimation theory. Only several other references are mentioned to establish mathematical background and illustrate related approaches. Among surveys and monographs providing a broader view of the field are [A1–10].

ORDER REDUCTION

Suppose that a dynamic system is modeled by

$$\dot{x} = f(x, z, u, t, \mu), \quad (1)$$

$$\mu\dot{z} = g(x, z, u, t, \mu), \quad (2)$$

where $\mu > 0$ is a scalar and x, z and u are n-, m-, and r-dimensional vectors, respectively. For $\mu = 0$, the order $n+m$ of (1, 2) reduces to n, that is (2) becomes

$$0 = g(\bar{x}, \bar{z}, \bar{u}, t, 0) \quad (3)$$

and the substitution of a root of (3),

$$\bar{z} = \phi(\bar{x}, \bar{u}, t), \quad (4)$$

into (1) yields a 'reduced' model

$$\dot{\bar{x}} = f[\bar{x}, \phi(\bar{x}, \bar{u}, t), \bar{u}, t, 0] \equiv \bar{f}(\bar{x}, \bar{u}, t). \quad (5)$$

The use of $\mu = 0$ is formal since then $\dot{z} = g/\mu$ in (2) may be unbounded for $g \neq 0$. An analysis validating this order reduction procedure is outlined in the next section where it also becomes apparent that a reduced model (4) represents slow and neglects fast phenomena in (1, 2). In this respect the singular perturbation approach is related to familiar "dominant mode" techniques [B2, E4] which neglect "high-frequency" parts and retain "low-frequency" parts of models.

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We note that (3) may have several roots each resulting in a different reduced model (4). Most of the available theory is restricted to models (4) corresponding to real and distinct roots of (3), along which $\partial g/\partial z$ is nonsingular. At points where $\partial g/\partial z$ is singular, z may jump from one root to another [C6]. In the special case when g is linear in z the reduced model (4) is unique. For a linear system

$$\dot{x} = A_{11}x + A_{12}z + B_1 u, \tag{6}$$

$$\mu\dot{z} = A_{21}x + A_{22}z + B_2 u \tag{7}$$

the root (4) is

$$\bar{z} = -A_{22}^{-1}A_{21}\bar{x} - A_{22}^{-1}B_2 u, \tag{8}$$

yielding the reduced model

$$\dot{\bar{x}} = (A_{11} - A_{12}A_{22}^{-1}A_{21})\bar{x} + (B_1 - A_{12}A_{22}^{-1}B_2)\bar{u}. \tag{9}$$

In applications, models of various physical systems are put in form (1), (2) by expressing small time constants T_i, small masses m_j, large gains K_l, etc., as $T_i = c_i\mu$, $m_j = c_j\mu$, $K_l = c_l/\mu$, etc., where c_i, c_j, c_l are known coefficients [A5, B5]. In power system models μ can represent machine reactances or transients in voltage regulators [B8], in industrial control systems it may represent time constants of drives and actuators [B11], in biochemical models μ can indicate a small quantity of an enzyme [B4], in a flexible booster model μ is due to bending modes [B3] and in nuclear reactor models it is due to fast neutrons [B7, 9, 12]. Singular perturbations are extensively used in aircraft and rocket flight models [B6, 10, 13, 16], and in chemical reaction diffusion theory [B14, 15]. Other order reduction techniques [B17] can be interpreted as singular perturbations [B18].

INITIAL VALUE PROBLEMS

When does a reduced solution $\bar{x}, \bar{z}$ approximate the original solution x, z and in what sense? For clarity we begin with the linear system (6, 7), assuming that it is time invariant and that $u = 0$. To exhibit the error $z - \bar{z} = z + A_{22}^{-1}A_{21}\bar{x}$ let

$$\eta = z + A_{22}^{-1}A_{21}x + \mu M_1 x \tag{10}$$

and choose M_1 such that the substitution of (10) into (6), (7) separates the η-subsystem as

$$\dot{x} = (A_{11} - A_{12}A_{22}^{-1}A_{21} + \mu M_2)x + A_{12}\eta, \tag{11}$$

$$\mu\eta = (A_{22} + \mu M_3)\eta. \tag{12}$$

It is easily shown that there exists $\mu^* > 0$ such that $M_i = M_i(\mu)$, $i = 1, 2, 3$, are bounded for all $\mu \in [0, \mu^*]$. For $\mu \to 0$ the eigenvalues of the independent η-subsystem (12) tend to infinity like the eigenvalues of $(1/\mu)A_{22}$. Thus (12) is the 'fast' part of (6, 7). It can be written as

$$\frac{d\eta(\tau)}{d\tau} = (A_{22} + \mu M_3)\eta(\tau), \tag{13}$$

where τ is the 'stretched time scale' defined for all $\mu \geq 0$,

$$\tau = \frac{t - t_0}{\mu}, \quad \tau = 0 \text{ at } t = t_0. \tag{14}$$

The system (13) depends continuously on μ and at $\mu = 0$ it becomes

$$\frac{d\eta(\tau)}{d\tau} = A_{22}\eta(\tau). \tag{15}$$

From (8) and (10) at $\mu = 0$ the initial condition for (15) is

$$\eta(0) = z(t_0) - \bar{z}(t_0). \tag{16}$$

The solution $\eta(\tau)$ of the 'fast' subsystem (13) is the input to the 'slow' subsystem (11). The homogeneous part of (11) is an $O(\mu)$ perturbation† of the reduced model (9) with $u = 0$. If the eigenvalues of A_{22} all have negative real parts, then $\eta(\tau) \to 0$ as $\tau \to \infty$, that is for μ small η as a function of t rapidly decays away from t_0. Under this condition, integration by parts in the variation of parameters formula for the solution of (11) yields

$$x(t) = \bar{x}(t) + O(\mu) \tag{17}$$

and, on substitution into (10),

$$z(t) = \bar{z}(t) + \eta(\tau) + O(\mu). \tag{18}$$

Thus the reduced model state $\bar{x}(t)$ approximates the x-part of the actual state, while to approximate its z-part we need both $\bar{z}(t)$ from (8) and $\eta(\tau)$ from (14). The 'boundary layer' correction $\eta(\tau)$ is significant only during a short interval $[t_0, t_1]$ after which

$$z(t) = \bar{z}(t) + O(\mu). \tag{19}$$

A remarkable property of the singularly perturbed model (1, 2) is that the structure of the approximation (17, 18) remains the same for time-varying and nonlinear systems. This is established by a fundamental theorem due to Tihonov [C1], whose essential conditions are imposed on a 'boundary layer' system for $\eta = z - \bar{z}$

$$\frac{d\eta}{d\tau} = g[\bar{x}, \bar{z} + \eta(\tau), \bar{u}, t, 0], \tag{20}$$

a nonlinear analog of (15). By virtue of (3) an equilibrium of (20) is $\eta = 0$. Assuming the existence and smoothness of $\bar{x}(t)$, $\bar{z}(t)$ for $t \in [t_0, T]$, the conditions imposed on (20) are, first, that $\eta = 0$ be an asymptotically stable equilibrium of (20) at $\bar{x}(t_0)$, $\bar{z}(t_0)$, $\bar{u}(t_0)$, t_0 with $\eta(0) = z(t_0) - \bar{z}(t_0)$ belonging to its domain of attraction; second, that for all $t \in [t_0, T]$ the eigenvalues of $\partial g/\partial z$ along $\bar{x}(t)$, $\bar{z}(t)$, $\bar{u}(t)$ all have real parts less than a fixed negative number. Then (17, 18) hold for all $t \in [t_0, T]$ and (19) holds for all $t \in [t_1, T]$.

† A function of μ is denoted by $O(\mu^k)$ when for all $\mu \in [0, \mu^*]$ its norm is less than $c\mu^k$, where $c > 0$, $\mu^* > 0$ and k are some constants.

The proof of this theorem is found in [A1, 8; C1–3] and, under slightly weaker conditions, in [C4]. The separation of time scales is exemplified by the fact that in the boundary layer system the variables $\bar{x}, \bar{z}, \bar{u}$ and t are fixed parameters. The boundary layer correction $\eta(\tau)$ used in (18) is the solution of (20) with (16), where $\bar{x}, \bar{z}, \bar{u}$ and t are fixed at their values for $t = t_0$.

Expressions (17) and (18) represent $O(\mu)$ approximations of $x(t)$, $z(t)$. If f and g in (1), (2) possess $k+2$ derivatives in their arguments, then $x(t)$, $z(t)$ can be approximated up to $O(\mu^k)$ using series with terms depending on t and terms depending on τ. These terms can be generated by methods in [A4, 8, 10; C4, 5].

BOUNDARY VALUE PROBLEMS

In boundary value problems when $z(t)$ is specified at both $t = t_0$ and $t = T$, two boundary layer correction terms η_L and η_R are needed to compensate for $z(t_0) - \bar{z}(t_0)$ and $z(T) - \bar{z}(T)$, respectively. The correction η_L is the same as η in the initial value problems. To define η_R an additional stretched variable is introduced for all $\mu \geqslant 0$,

$$\sigma = (t-T)/\mu, \quad \sigma = 0 \text{ at } t = T, \tag{21}$$

and (20) is rewritten in σ-scale with $\bar{x}$, $\bar{z}$, $\bar{u}$ and t fixed at their values for $t = T$. Then $\eta_R = \eta_R(\sigma)$ is its solution for $\eta_R(0) = z(T) - \bar{z}(T)$. The approximation of $z(t)$ is sought in the form

$$z(t) = \bar{z}(t) + \eta_L(\tau) + \eta_R(\sigma) + O(\mu) \tag{22}$$

such that η_L and η_R decay exponentially as $\tau \to \infty$ and $\sigma \to -\infty$, that is their norms satisfy the 'dichotomy condition'

$$\left.\begin{aligned} \|\eta_L\| &\leqslant c_1 \exp(-c_2\tau) \quad \text{for } 0 \leqslant \tau < \infty, \\ \|\eta_R\| &\leqslant c_3 \exp(c_4\sigma) \quad \text{for } -\infty < \sigma \leqslant 0, \end{aligned}\right\} \tag{23}$$

where $c_1, \ldots, c_4$ are positive constants. A simple illustration is again the linear system (12). Its solutions in τ and σ scales at $\mu = 0$ are

$$\left.\begin{aligned} \eta_L(\tau) &= \exp(A_{22}\tau)\,\eta_L(0), \\ \eta_R(\sigma) &= \exp(A_{22}\sigma)\,\eta_R(0). \end{aligned}\right\} \tag{24}$$

Let the first k eigenvalues of A_{22} have negative real parts and the remaining $m-k$ eigenvalues positive real parts. Then (23) will result if $\eta_L(0) = z(t_0) - \bar{z}(t_0)$ belongs to the eigenspace corresponding to the first k-eigenvalues of A_{22}, and if $\eta_R(0) = z(T) - \bar{z}(T)$ belongs to the eigenspace corresponding to the remaining $m-k$ eigenvalues of A_{22}. Under this condition (17) and (22) hold for all $t \in [t_0, T]$, while (19) holds for $t_0 < t_1 < t < t_2 > T$. In some problems the initial conditions are always in the required subspaces due to the physical nature of the variables η_L and η_R. In others, they have to be set there by the user. Since it cannot be done exactly, such problems may appear ill-posed. Fortunately, it follows from [G6, 13] that control problems allowing combined open loop–feedback realizations are well posed in this sense.

In nonlinear problems $\partial g/\partial z$ along $\bar{x}(t)$, $\bar{z}(t)$, $\bar{u}(t)$ is assumed to possess the above eigenvalue distribution throughout the interval $[t_0, T]$. Also $z(t_0) - \bar{z}(t_0)$ and $z(T) - \bar{z}(T)$ are restricted to be on manifolds for which the equilibrium $\eta = 0$ of (20) is attractive in forward and reverse directions of t, respectively. Then (17) and (22) hold for all $t \in [t_0, T]$. Higher order approximations are possible by asymptotic expansions [A4, 8, 10; C4].

In a wider class of 'matched' expansion methods [A3, 9] other conditions for 'matching' of 'outer' (slow) and 'inner' (fast) terms are used. They are often motivated by specific applications, such as in inter-planetary guidance problems [D6]. The conditions outlined here originate from [A1; D1–5] and can be found in more recent works [A8, 10; D7, 8] and, in a compact form, in [D9]. These conditions are particularly sui[illegible] for optimal control problems whose Hamilt[illegible]n symmetry is related to the dichotomy (23). [illegible]tical implications of this relationship are [illegible]ussed in the section on 'Trajectory Optimiza[illegible]'.

STABILITY AND STABIL[illegible]LITY

In approximations discussed so far stability requirements were imposed only on (20), and the reduced solution $\bar{x}(t)$ was permitted to be unstable. In infinite time-interval problems it is of interest to establish stability properties of $x(t)$, $z(t)$ from stability properties of $\bar{x}(t)$ and $\eta(\tau)$. Several such results are available.

For linear time-invariant systems a stability result immediately follows from the upper triangular form of the system (11, 12). Its $m+n$ eigenvalues are perturbations of the n eigenvalues of $A_{11} - A_{12}A_{22}^{-1}A_{21}$ and of the m eigenvalues of $(1/\mu)A_{22}$. If the real parts of these eigenvalues are negative,

$$\operatorname{Re}\lambda\{A_{22}\} < 0, \quad \operatorname{Re}\lambda\{A_{11} - A_{12}A_{22}^{-1}A_{21}\} < 0, \tag{25}$$

that is, if the reduced solution $\bar{x}(t)$ and the boundary layer correction $\eta(\tau)$ are asymptotically stable, then there exists $\mu^* > 0$ such that the original solution $x(t)$, $z(t)$ is asymptotically stable for all $\mu \in [0, \mu^*]$. For linear time-varying systems a similar condition is derived in [E1, 7], assuming that the reduced model be uniformly asymptotically stable and that for $t \geqslant t_0$ the eigenvalues of $A_{22}(t)$ have real parts less than a fixed negative number $-\delta$. This contrasts with the general case in which $\operatorname{Re}\lambda\{F(t)\} < -\delta$ for all t does not imply stability of $\dot{v} = F(t)v$.

In nonlinear systems the first requirement of (25) is imposed on the eigenvalues of $\partial g/\partial z$ evaluated along $\bar{x}(t)$, $\bar{z}(t)$, $\bar{u}(t)$ for all $t \geqslant t_0$. In addition,

$\bar{x}(t)$, $\bar{z}(t)$ and

$$F(t) = \frac{\partial f}{\partial x} - \frac{\partial f}{\partial z}\left(\frac{\partial g}{\partial z}\right)^{-1}\frac{\partial g}{\partial x} \tag{26}$$

evaluated along $\bar{x}(t)$, $\bar{z}(t)$, $\bar{u}(t)$ are assumed to have finite limits $\bar{x}(\infty)$, $\bar{z}(\infty)$ and $F(\infty)$ as $t\to\infty$, where $\mathrm{Re}\,\lambda\{F(\infty)\}<0$. Then, if $x(t_0)$ and $z(t_0)$ are in the appropriate domain of attraction, the limits $x(t)$ and $z(t)$ as $t\to\infty$ are

$$x(t)\to\bar{x}(\infty)+O(\mu), \quad z(t)\to\bar{z}(\infty)+O(\mu). \tag{27}$$

This is the content of the stability theorem in [E10], whose proof, along with an estimate of the domain of attraction, is given in [E12]. Alternative sets of conditions are given in [E1]. In [E5, 6] similar conditions are employed to analyze stability of networks with parasitics, while a problem of absolute stability is discussed in [E8] and stability bounds for μ are estimated in [E9]. Some early results on stability of control systems with infinite gain coefficients are found in [B1]. Related theorems on linear systems with slowly varying coefficients are found in [E2, 3, 11], [E4, Section 32] and [E16, pp. 125–128].

A general stabilizability condition for linear time-varying systems is formulated in [G6]. Special cases for linear time-invariant systems, are discussed in [E13–15].

REGULATORS AND RICCATI EQUATIONS

Among the most actively investigated singularly perturbed optimal control problems is the general linear–quadratic regulator problem. For brevity we consider only the time-invariant case. When the system (6), (7) is optimized with respect to

$$J = \frac{1}{2}\int_0^\infty (y'y+u'\mathrm{R}u)\,\mathrm{d}t, \tag{28}$$

where $y = C_1x+C_2z$ and $R>0$, then to implement the optimal control

$$u = -R^{-1}[B_1' \quad B_2'/\mu]K\begin{bmatrix} x \\ z \end{bmatrix} \tag{29}$$

we have to solve

$$K\begin{bmatrix} A_{11} & A_{12} \\ A_{21}/\mu & A_{22}/\mu \end{bmatrix} + \begin{bmatrix} A_{11}' & A_{21}'/\mu \\ A_{12}' & A_{22}'/\mu \end{bmatrix}K - K\begin{bmatrix} B_1 \\ B_2/\mu \end{bmatrix}R^{-1}[B_1' \quad B_2'/\mu]K + C'C = 0, \tag{30}$$

where $C = [C_1 \quad C_2]$. To avoid unboundedness as $\mu\to0$ the solution is sought in the form

$$K = K(\mu) = \begin{bmatrix} K_{11}(\mu) & \mu K_{12}(\mu) \\ \mu K_{12}'(\mu) & \mu K_{22}(\mu) \end{bmatrix} \tag{31}$$

which permits us to set $\mu = 0$ in (30). At $\mu = 0$ an $m\times m$ equation for $\bar{K}_{22}$,

$$\bar{K}_{22}A_{22}+A_{22}'\bar{K}_{22}-\bar{K}_{22}S_2\bar{K}_{22}+C_2'C_2 = 0, \tag{32}$$

where $S_2 = B_2R^{-1}B_2'$ separates from the $(n+m)\times(n+m)$ equation (30). If A_{22}, B_2 is a stabilizable pair, and if A_{22}, C_2 is a detectable pair, then a unique positive semidefinite solution $\bar{K}_{22}$ exists and the eigenvalues of $A_{22}-S_2\bar{K}_{22}$ have negative real parts. Another result of the substitution of (31) into (30) is that at $\mu = 0$ it is possible to express $\bar{K}_{12}$ in terms of $\bar{K}_{11}$ and $\bar{K}_{22}$, and to obtain an $n\times n$ equation for K_{11},

$$\bar{K}_{11}\hat{A}+\hat{A}'\bar{K}_{11}-\bar{K}_{11}\hat{B}R^{-1}\hat{B}'\bar{K}_{11}+\hat{C}'\hat{C} = 0. \tag{33}$$

The expressions for $\hat{A}$, $\hat{B}$ and $\hat{C}$ are given in [F5]. An interpretation of (32) and (33) is that (32) yields a 'boundary layer regulator' for the fast variable $\eta(\tau)$, and (33) yields the regulator for the reduced state variable $\bar{x}(t)$. For $\hat{A}$, $\hat{B}$ stabilizable and $\hat{A}$, $\hat{C}$ detectable, the implicit function theorem applied to (30) with (31) shows that

$$K_{ij} = \bar{K}_{ij}+O(\mu), \quad i,j = 1,2. \tag{34}$$

Not only are the approximations $\bar{K}_{ij}$ calculated from lower order equations, but in addition the ill-conditioning of (30) has been removed.

If $\bar{K}_{ij}$ are used instead of K_{ij} the system (6), (7) with feedback control (29) becomes

$$\dot{x} = (A_{11}-S_1\bar{K}_{11}-S\bar{K}_{12}')x+(A_{12}-S\bar{K}_{22})z, \tag{35}$$

$$\mu\dot{z} = (A_{21}-S'\bar{K}_{11}-S_2\bar{K}_{12}')x+(A_{22}-S_2\bar{K}_{22})z, \tag{36}$$

where $S_1 = B_1R^{-1}B_1'$ and $S = B_1R^{-1}B_2'$. If this system is asymptotically stable, then because of (34), its solution $x(t)$, $z(t)$ is within $O(\mu)$ of the optimal solution. The stability condition (25) can now be applied to the feedback system (35), (36). The boundary layer stability condition is satisfied by $A_{22}-S_2\bar{K}_{22}$. The condition for the reduced system is satisfied by the solution of (33). Thus (35), (36) is a near-optimal system.

The singularly perturbed regulator problem was posed in [F1] with $C_2 = 0$ and A_{22} stable, which gave $\bar{K}_{22} = 0$. The general time-varying problem was treated in [F3, 5] using the notion of boundary layer controllability–observability. These results and extensions [F6, 7, 9, 10, 13] are based on the singularly perturbed differential Riccati equation. An alternative approach via boundary value problems is presented in [G8, 19], its relationship with the Riccati approach is analyzed in [F12]. In [F2] it was shown that the reduced Riccati equation (33) can also be obtained from the reduced model (9). Asymptotic expansions are constructed in [F6, 7] and applied to a 17th order power station model in [F8]. Two other order reduction techniques [F4, 11] lead to equations similar to (33)

and it would be of interest to investigate their relationship with the singular perturbation approach.

TRAJECTORY OPTIMIZATION

In trajectory optimization problems for the system (1), (2) some conditions are imposed on x, z at both $t = t_0$ and $t = T$, and a control $u(t)$ is sought to minimize the performance index

$$J = \int_{t_0}^{T} V(x, z, u, t)\, dt. \tag{37}$$

An optimal solution must satisfy $H_u = 0$ and

$$\dot{x} = H_p, \quad \dot{p} = -H_x, \tag{38}$$

$$\mu\dot{z} = H_q, \quad \mu\dot{q} = -H_z, \tag{39}$$

with $2n+2m$ boundary conditions. Here $H_u, H_x, H_z, H_p = f$, $H_q = g$, denote the partial derivatives of the Hamiltonian $H = V + p'f + q'g$, and the adjoint variables for (1) and (2) are p and μq, respectively. At $\mu = 0$ we use $H_q = 0$ and $H_z = 0$ to eliminate z and q from (38) and to get the reduced system

$$\dot{\bar{x}} = \bar{H}_p, \quad \dot{\bar{p}} = -\bar{H}_x \tag{40}$$

for which only $2n$ conditions can be imposed. Suppose that they are uniquely satisfied by a continuously differentiable reduced solution $\bar{x}(t)$, $\bar{p}(t)$. Since the reduced variables $\bar{z}(t), \bar{q}(t)$ obtained from $H_q = 0$, $H_z = 0$ may not satisfy the remaining $2m$ conditions, corrections $\eta_L(\tau), \eta_R(\sigma)$ for z, and $\rho_L(\tau), \rho_R(\sigma)$ for q, are to be determined from appropriately defined layer systems

$$\frac{d\eta_L}{d\tau} = \bar{H}_q(\eta_L, \rho_L), \quad \frac{d\rho_L}{d\tau} = -\bar{H}_z(\eta_L, \rho_L), \tag{41}$$

$$\frac{d\eta_R}{d\sigma} = \bar{H}_q(\eta_R, \rho_R), \quad \frac{d\rho_R}{d\sigma} = -\bar{H}_z(\eta_R, \rho_R), \tag{42}$$

where (41) is used at $t = t_0$ and (42) at $t = T$. To be specific consider the problem with fixed end points.

$$z(t_0) = z^0, \quad z(T) = z^T. \tag{43}$$

Then the initial values for η_L and η_R are

$$\eta_L(0) = z^0 - \bar{z}(t_0), \quad \eta_R(0) = z^T - \bar{z}(T) \tag{44}$$

and the additional boundary conditions are

$$\eta_L, \rho_L \to 0, \quad \tau \to \infty; \quad \eta_R, \rho_R \to 0, \quad \sigma \to -\infty. \tag{45}$$

Existence of optimal solutions and their approximation by reduced solutions have been investigated in [G1, 3, 9] and extended in [G16, 17] by a construction of asymptotic expansions. Unfortunately, the applicability of these results is restricted by the requirement that $\eta_L(0)$ and $\eta_R(0)$ be sufficiently small. To what extent such restrictions can be avoided in a general nonlinear problem (1), (2) and (37) is still an open question. Results without restrictions on z^0, z^T are available for linear time-varying systems [G6, 8, 13, 19] and for a special class of nonlinear systems [G14, 15, 20]. They are briefly outlined here.

Let the performance index be (28), but on the interval $[t_0, T]$, and consider the trajectory optimization problem for (6), (7) allowing that the matrices in (6), (7) and (28) be time varying. Using a 'dichotomy transformation' proposed in [G6]

$$x = l_1 + r_1, \quad z = l_2 + r_2, \tag{46}$$

$$\begin{bmatrix} p \\ q \end{bmatrix} = P(t)\begin{bmatrix} l_1 \\ l_2 \end{bmatrix} + N(t)\begin{bmatrix} r_1 \\ r_2 \end{bmatrix}, \tag{47}$$

where $P(t)$ is a positive definite and $N(t)$ is a negative definite solution of a differential equation analogous to (30), we transform (41), (42) into two separate 'layer regulator systems'

$$\frac{d\eta_L}{d\tau} = [A_{22}(t_0) - S_{22}(t_0)P_{22}(t_0)]\,\eta_L, \tag{48}$$

$$\frac{d\eta_R}{d\sigma} = [A_{22}(T) - S_{22}(T)N_{22}(T)]\,\eta_R, \tag{49}$$

where $\eta_L = l_2 - \bar{l}_2$, $\eta_R = r_2 - \bar{r}_2$ and $P_{22}(t_0)$, $N_{22}(T)$ are the positive and the negative definite roots of (32) at t_0 and T. If for all $t \in [t_0, T]$

$$\operatorname{rank}[B_2, A_{22}B_2, \ldots, A_{22}^{m-1}B_2] = m, \tag{50}$$

$$\operatorname{rank}[C_2', A_{22}'C_2' \ldots, A_{22}'^{m-1}C_2'] = m, \tag{51}$$

then the approximations (17), (22) and

$$p(t) = \bar{p}(t) + O(\mu), \tag{52}$$

$$q(t) = \bar{q}(t) + P_{22}(t_0)\eta_L + N_{22}(T)\eta_R + O(\mu) \tag{53}$$

hold for arbitrary boundary values z^0, z^T since (48), (49) satisfy the dichotomy condition (23). A less restrictive stabilizability–detectability condition can be used instead of (50), (51). This result of [G13] delineates a class of well-posed singularly perturbed trajectory optimization problems. The use of $-R^{-1}B_2'P_{22}z + u^0$ results in a stable feedback realization of the initial layer and $u^0 = R^{-1}B_2^1(P_{22} - N_{22})r_2$ is the open-loop control of the end-layer. An 'inverse' Riccati approach to the linear fixed end-point problem is developed in [F9]. In [G8] a different set of conditions is derived and asymptotic expansions are constructed for the linear boundary value problem.

In [G14, 15] the above results have been extended to the nonlinear problem

$$\dot{x} = f_1(x,t) + A_{12}(x,t)z + B_1(x,t)u, \tag{54}$$

$$\mu\dot{z} = g_1(x,t) + A_{22}(x,t)z + B_2(x,t)u, \tag{55}$$

$$J = \frac{1}{2}\int_{t_0}^{T} [v_1(x,t) + z'C_2'(x,t)C_2(x,t)z + u'R(x,t)u]\,dt. \tag{56}$$

It is shown in [G15] that, if the matrices in (32) and (48)–(51) are interpreted as the matrices of (54)–(56) evaluated along $\bar{x}(t)$, then (50), (51) are sufficient for the approximation (17), (22), (52), (53) to hold for (54)–(56) with arbitrary z^0, z^T. The conditions derived in [G14] extend the results of [G8] to (54)–(56). Among other works on trajectory optimization, [G18] shows that (40) can also be obtained from the reduced system, [G1] analyzes the scalar problem, [G2, B5] give approximations without layer corrections and [G10] makes an attempt to include control inequality constraints. Applications to aircraft control problems are discussed in [G4, 5, 11, 12] and in [B6, 10, 13, 16]. A class of singular problems is analyzed in [G22]. A result on periodic controls appears in [G21]. An application to a pursuit–evasion problem is discussed in [G2, 3].

CONTROLLABILITY AND TIME OPTIMAL CONTROL

In the design of time-optimal controls difficulties with high-order systems are considerable even in the linear time-invariant problems. A simplified design procedure has been developed in [H1, 2, 3]. The discussion here is based on [H2], where also the following controllability result is obtained. The use of (10) and

$$\xi = x - \mu A_{12} A_{22}^{-1} \eta + O(\mu^2) \qquad (57)$$

transforms (6), (7) into

$$\dot{\xi} = [\bar{A} + O(\mu)]\,\xi + [\bar{B} + O(\mu)]\,u, \qquad (58)$$

$$\mu\dot{\eta} = [A_{22} + O(\mu)]\,\eta + [B_2 + O(\mu)]\,u, \qquad (59)$$

where $\bar{A} = A_{11} - A_{12} A_{22}^{-1} A_{21}$, $\bar{B} = B_1 - A_{12} A_{22}^{-1} B_2$, see (9).

It follows from (58), (59) that for μ small the controllability of the reduced and the boundary layer systems, that is of the pairs $\bar{A}$, $\bar{B}$ and A_{22}, B_2, implies the controllability of the original system (6), (7).

In the time-optimal control problem a control u, subject to constraint $|u_i| \le 1$, $i = 1, \ldots, r$, is to transfer the state of (6), (7) from $x(0) = x^0$, $z(0) = z^0$ to $x(T) = 0$, $z(T) = 0$ in minimum time T. Equivalently the problem can be solved in terms of ξ and η. A control steering ξ, η to zero in minimum time is of the form

$$u = -\operatorname{sgn}\{\bar{B}' \exp[\bar{A}'(T-t)]p + B_2' \exp(-A_{22}\sigma) q\}, \qquad (60)$$

where σ is as in (21), p and q are constant vectors and $O(\mu)$ terms have been neglected. When the eigenvalues of A_{22} all have negative real parts, the term depending on σ is significant only near T. For some $\sigma^* < 0$ and $0 \le t \le T + \mu\sigma^*$ the control (60) can be approximated by

$$\bar{u} = -\operatorname{sgn}\{\bar{B}' \exp[\bar{A}'(T-t)]\bar{p}\}, \qquad (61)$$

which is interpreted as a time-optimal control for the reduced system (9), steering $\bar{x}$ to zero. For $T + \mu\sigma^* < t \le T$ the control (60) is approximated by

$$u_\sigma = -\operatorname{sgn}\{\bar{B}'\bar{p} + B_2' \exp(-A_{22}\sigma) q\}. \qquad (62)$$

We note from (8) that, after the last switching of $\bar{u}$, z may be far from the origin and the boundary layer control u_σ is needed to correct this error.

This separation of slow and fast switchings was first analyzed for single-input systems in [H1], and then generalized in [H2]. A special case when (7) is due to actuator dynamics is discussed in [H3]. An iterative method based on these results is developed in [H4].

FILTERING AND SMOOTHING

Results on singular perturbation of linear–quadratic regulator problems should have their counterparts in the linear–quadratic–Gaussian filtering and smoothing problems. Preliminary investigations along this line have been reported in [I1, 3–6]. The analysis in [I6] shows that the duality is not complete and the singularly perturbed filtering and smoothing problems require separate treatment and cautious interpretation. The analysis is more complicated since the white noise input process u in (58), (59) 'fluctuates' faster than the fast part η of the state no matter how small $\mu > 0$ is. In the limit, η becomes a white noise process whose covariance is the same as the covariance of the reduced solution η, and the integral error covariance of $\eta(t) - \bar{\eta}(t)$ tends to zero. Thus, as an input to a slow system, $\bar{\eta}(t)$ can replace $\eta(t)$, but not as an approximation for each t. Pursuing such considerations it is shown in [I6] that a filtering (or smoothing) problem for the system (6), (7) can be obtained by solving two lower order problems in separate time scales.

An example given in [I2] indicates that deterministic observers also can be approached as singular perturbations. Control problems with small noise are treated in [I7, 8].

CHEAP CONTROL AND SINGULAR ARCS

In singular perturbation problems considered so far a small parameter μ multiplies derivatives and the differential order is reduced when $\mu = 0$. Another sign of singular perturbation phenomena is a characteristic lowering of dimensionality for the limiting problem, such as in limit approaches to singular optimal controls [J1]. An example of these problems is

$$\dot{x} = Ax + Bu, \quad x(0) = x^0, \qquad (63)$$

$$J = \frac{1}{2}\int_0^T (x'Qx + \mu^2 u'Ru)\,dt, \qquad (64)$$

where J is to be minimized for μ small. In [L2] analogous problems for systems governed by partial differential equations are called 'cheap control' problems since the cost of the control u is cheap relative to that of the state x (for $Q>0$). Other applications include study of limiting possibilities for regulators and filters [J2, 5; I8].

When $\mu=0$, the resulting problem is a well-known singular problem [J3] whose solution satisfies the singular arc condition

$$B'K_0 = 0 \tag{65}$$

for $t>0$ and the appropriate Riccati gain K_0. Motion is thereby restricted to a manifold of dimension at most $n-r$. By obtaining the asymptotic solution of (63), (64) as $\mu \to 0$, we show how this reduction in order comes about and, simultaneously, discover the nature of the initial control impulse. For $\mu>0$, the feedback control is

$$u = -\frac{1}{\mu^2} R^{-1} B' Kx, \tag{66}$$

where $K \geqslant 0$ satisfies the singularly perturbed problem

$$\mu^2 \frac{dK}{dt} + \mu^2 (KA + A'K + Q) = KBR^{-1}B'K,$$

$$K(T) = 0. \tag{67}$$

The limiting solution K_0 of (67) within $(0, T)$ satisfies the singular arc condition (65). An asymptotic solution of (67) is complicated and considerably different, however, in a hierarchy of cases: Case 1 where $B'QB>0$, Case 2 where $B'QB=0$ and $B_1'QB_1>0$ for $B_1 = AB - \dot{B}$. This reflects the situation for the singular arc problem [J3, 4] where the initial optimal control successively becomes increasingly impulsive and the singular arc increasingly restrictive. A singular perturbation analysis in [J6–10] reveals the detailed structure of these phenomena. Its use for numerical solution of ill-conditioned Riccati equation is discussed in [J11].

TIME-DELAY SYSTEMS

The difficulties incumbent with control systems having time delays have motivated various approximations. When the delay is small, it is often neglected and a tractable 'nominal' problem is solved. Such design procedures can be justified in terms of singular perturbation methods. Boundary layer phenomena do occur, although they are not of lowest order importance. Interesting and significant extensions are to problems with both small parameters multiplying derivatives and small delays. Discussions with applications to nuclear reactor models occur in [K1–4]. In [K5] a method is proposed replacing several small time constants by a single time delay.

DISTRIBUTED PARAMETER SYSTEMS

From the results of [L1, 3] it can be expected that the singular perturbation techniques will be among the main tools for asymptotic analysis and design of optimal control of distributed parameter systems. Several generalizations of the finite dimensional linear-quadratic problems are available. In particular, a distributed parameter analog of the method [F5, 7] is developed in [L3] for systems described by singularly perturbed parabolic differential equations.

CONCLUSION

It seems that, instead of giving a short summary of solved problems, the conclusion of a survey of a new direction of research should concentrate on missing links, restrictive assumptions and hints of new problems. Starting with order reduction the need for a systematic modeling procedure to formulate the model (1), (2) is apparent. Conversely, this model is expected to interpret other order reduction procedures as limit processes. In initial and boundary value problems, controllability and stabilizability studies may relax the restrictions of stable initial and final manifolds. Although optimal regulators seem a solved problem, there remains a desire to reduce the dimensionality of the feedback matrix. In trajectory optimization, restrictions on norms of boundary layer jumps should be, and very likely can be, removed for a wider class of Hamiltonian systems. The only result with constrained control is the linear time-optimal control. Various generalizations to other bang–bang controls are visible.

In addition to linear regulators, other optimum feedback design problems need to be solved. Order reduction in dynamic programming and Hamilton–Jacobi optimization approaches would result in even bigger conceptual and computational simplifications. Singularly perturbed filtering, smoothing, singular arc, distributed systems and time-delay problems require further exploration. More work on numerical aspects of these problems is also needed. What has been surveyed here is only a first step.

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Singular Perturbation and Iterative Separation of Time Scales*

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Based on singular perturbations concepts, an iterative method for separation of time scales removes inconsistencies of the classical quasi-steady-state approach, and it systematically improves the accuracy of lower order models.

Key Words—Computational methods; time scale modeling; system order reduction; iterative methods; power system simulation; nonlinear systems.

Abstract—This tutorial paper presents an iterative method for the separation of slow and fast modes, which removes the inconsistencies of the classical quasi-steady-state approach and systematically improves the accuracy of the lower order models. It also serves as a self-contained introduction to singular perturbations. State variable reformulation and time scale identification are discussed and illustrated with power system examples. A correction procedure for nonlinear systems is also presented.

NOTATION

D	machine damping
δ	machine angle
e'_d	component of voltage behind transient reactance due to quadrature axis flux linkages (not with the field windings)
e'_q	component of voltage behind transient reactance due to direct axis flux linkages (with the field windings)
E_{fd}	exciter output voltage
H	machine inertia constant
K_A	voltage regulator gain
K_E	exciter gain
K_F	feedback compensator gain
l_1, l_2	leakage inductances
L_1, L_2	self inductances
λ_1, λ_2	flux linkages
N_1, N_2	turns ratios
R_f	feedback compensator state
R_1, R_2	transformer resistances
$S_e(E_{fd})$	exciter saturation
T_A	voltage regulator time constant
T'_{do}	open circuit direct axis time constant
T_E	exciter time constant
T_F	feedback compensator time constant
T'_{qo}	open circuit quadrature axis time constant
V_R	voltage regulator output
ω	machine speed
X_d	direct axis synchronous reactance
X_q	quadrature axis synchronous reactance.

1. INTRODUCTION

REALISTIC models of large scale systems involve interacting dynamic phenomena of widely different speeds. In a power system model, for example, voltage and frequency transients range from intervals of seconds, corresponding to generator voltage regulator, speed governor action and shaft energy storage, to several minutes, corresponding to load voltage regulator action, prime mover fuel transfer times and thermal energy storage (Luini, Schulz and Turner, 1975). Since such models are of high order and numerically stiff, order reduction and separation of time scales are often made using aggregation, modal analysis and similar techniques (Sandell and coworkers, 1978; Undrill and Turner, 1971). The underlying assumption is that during the fast transients the slow variables remain constant and that by the time their changes become noticeable, the fast transients have already reached their quasi-steady-states ('qss'). Based on this qss assumption and experience, the state variables are divided into n 'slow' states x and m 'fast' states z, that is the full scale model is

$$\frac{dx}{dt} = f(x, z, t), \qquad x(t_0) = x^0 \tag{1}$$

$$\frac{dz}{dt} = G(x, z, t), \qquad z(t_0) = z^0. \tag{2}$$

Then the only states used for short term studies are z, disregarding (1) and considering the states x as constant parameters. In long term studies the only states used are x and the differential equations for z are reduced to algebraic or

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transcendental equations by formally setting $\dot{z}=0$. The qss model is thus

$$\frac{dx_s}{dt}=f(x_s, z_s, t) \qquad x(t_0)=x^0 \tag{3}$$

$$0=G(x_s, z_s, t). \tag{4}$$

Examples of power system models derived in such a fashion are many, as illustrated in Alder and Nolan (1976). An inconsistency of this classical qss approach is the requirement that z_s equal a constant, as implied by $dz_s/dt=0$, is violated by (4) which defines z_s as a time varying quantity. Furthermore, the initial condition for z had to be dropped in (4), since there is no freedom to satisfy it. If a qss model fails to provide a good approximation of the actual solution $x(t)$ and $z(t)$, there is no provision for improving the approximation.

This tutorial paper presents an iterative scheme for the separation of slow and fast modes which removes the inconsistencies of the classical qss approach and systematically improves the accuracy of the lower order models. It modifies the qss assumption into the multi-time scale property of singularly perturbed systems (Kokotovic, O'Malley and Sannuti, 1976; Chow, Allemong and Kokotovic, 1978) and applies the modified qss assumption at each iteration step to the model obtained from the previous step. The accuracy of the models for the slow and the fast modes is improved at each step and they are further separated from each other. The iterations are related to, but simpler to interpret than standard asymptotic expansion methods (Hoppensteadt, 1974; O'Malley, 1974). The successive use of the modified qss assumption can be followed without any background in singular perturbation theory. The iterations offer more freedom to select various, possibly nonuniform, combinations of correction terms. Finally, the tutorial use of the classical qss assumption as an introduction to singular perturbations clarifies the relationship between the classical model reduction and the singular perturbations method. The classical approach can now be justified and improved to any degree of accuracy.

2. SINGULAR PERTURBATIONS AND TIME SCALES

Assuming that t is properly scaled for the slow phenomena, let us introduce a new time variable τ and scale it for the fast phenomena. For example, if t is in minutes, τ can be in seconds. The ratio of the time scales, in this case 1/60, is in general a small positive parameter ε. This parameter will be the main tool for our asymptotic analysis. Using ε the new time variable τ is defined by

$$\tau=(t-t')/\varepsilon \tag{5}$$

and its initial instant $\tau=0$ is chosen to correspond to a particular instant t' in t time scale.

The wider the separation of the time scales, such as seconds and hours, the smaller ε will be. On the other hand, the smaller ε is, the larger τ will be for a given $(t-t')$ interval. In the limit as $\varepsilon\to 0$ even a short interval in t is 'stretched' to an infinite interval in τ. When τ is sufficiently large, the fast phenomena have adequate time to reach their steady-states. This, however, does not contradict the assumption that $(t-t')$ is sufficiently short to consider the slow variables as constants. Thus, the limit of $\varepsilon\to 0$ is equivalent to the qss assumption, but without its inconsistencies.

A more difficult task is to reformulate the model (1), (2) to incorporate the scaling (5). If it is known that the dynamics of the states z are $1/\varepsilon$ times faster than x, then $\dot{z}$ is about $1/\varepsilon$ times larger than $\dot{x}$ and G can be rescaled as

$$g=\varepsilon G \tag{6}$$

such that f and g are of the same order of magnitude. The model (1), (2) then becomes

$$\frac{dx}{dt}=f(x, z, t) \qquad x(t_0)=x^0 \tag{7}$$

$$\varepsilon\frac{dz}{dt}=g(x, z, t) \qquad z(t_0)=z^0. \tag{8}$$

The above qualitative reasoning is based on some empirical estimates of dx/dt and dz/dt. When this information is not available, then physical parameters such as time constants, loop gains and energy storage constants (masses, inductances, etc.) are examined to determine which states are slow and which are fast. Not every choice of state variables will be separable in this sense. Where separable, a model (7), (8) will be obtained by expressing the small time constants and the inverses of the high gain coefficients as multiples of a single small parameter ε [see Section 3 and the companion paper (Winkelman and co-workers, 1980)].

In the limit $\varepsilon\to 0$, the model (7), (8), being in the t time scale, defines the quasi-steady-states $x_s(t)$, $z_s(t)$ as

$$\frac{dx_s}{dt}=f(x_s, z_s, t) \qquad x_s(t_0)=x^0 \tag{9}$$

$$0=g(x_s, z_s, t). \tag{10}$$

Although this is the same qss model (3), (4), its origin and meaning are different. The crucial difference is that $dz_s/dt \neq 0$ as required by (10) is not contradicted by $\varepsilon(dz_s/dt) = 0$ which is now due to $\varepsilon = 0$, and not $dz_s/dt = 0$. To obtain the fast parts of x and z we rewrite (7), (8) in the fast time scale τ

$$\frac{dx}{d\tau} = \varepsilon f(x, z, t' + \varepsilon\tau) \tag{11}$$

$$\frac{dz}{d\tau} = g(x, z, t' + \varepsilon\tau) \tag{12}$$

and again examine the limit as $\varepsilon \to 0$. Then $dx/d\tau = 0$, that is x is constant in the fast time scale. This implies that as $\varepsilon \to 0$ the only fast variations are the deviations of z from its quasi-steady-state z_s. Denoting them by $z_f = z - z_s$ and letting $\varepsilon = 0$ in (11), (12), we obtain

$$\frac{dz_f}{d\tau} = g(x^0, z_s^0 + z_f(\tau), t_0), \; z_f(0) = z^0 - z_s^0. \tag{13}$$

The fixed instant t' has been chosen to be t_0 and hence the model constants are t_0, x^0, $z_s^0 = z_s(t_0)$, which is suitable for the fast phenomena occurring near t_0.

Using (9), (10) as the slow model and (13) as the fast model one expects to approximate x and z by

$$x(t) \cong x_s(t) \tag{14}$$

$$z(t) \cong z_s(t) + z_f\left(\frac{t - t_0}{\varepsilon}\right) \tag{15}$$

where $z_f(\tau)$ is expressed in the t time scale. When is such an approximation valid? How can it be further improved? Singular perturbations addresses these issues much better than other model simplification methods. The tool at hand, not present in other methods, is the scaling parameter ε.

Recent results by Chow, Allemong and Kokotovic (1978) show that systems with lightly damped high frequency oscillatory modes can also be expressed in the form of (7), (8). The reduction procedure (9)–(15) and the iterative separation method discussed in Section 4 also hold for these systems. However, the interpretations of the reduction process for these two types of systems are quite different. With well damped fast modes, the state z rapidly reaches its quasi-steady-state z_s. When the state z exhibits high frequency oscillations, the state x is approximated by the slow subsystem (9), (10) due to the 'averaging' or filtering effect of the slow subsystem. Furthermore, for lightly damped oscillatory systems, the validity of the approximation (14), (15) is only up to a finite time t which depends on the accuracy of the high frequency being approximated.

While the full order models (7), (8) and (11), (12) are exact, the separated lower order models (9), (10) and (13) are in error because they assume $\varepsilon = 0$, instead of the actual $\varepsilon > 0$. This parameter change is called 'singular' and it results in an inherent perturbation in model order. The approximation (14), (15) can now be improved by constructing asymptotic expansions in ε. It is crucial that each expansion term is calculated at $\varepsilon = 0$, retaining the advantage of having separate lower order models. The standard expansion techniques are described in Hoppensteadt (1974) and O'Malley (1974). Our iterative technique is presented in Section 4, illustrated with a power system example in Section 5 and extended to a class of nonlinear systems in Section 6.

3. SEPARABLE AND MIXED STATES

Before proceeding to the iterative separation of time scales, we illustrate the state separation problem by two elementary examples. In the IEEE type 1 voltage regulator (IEEE Committee Report, 1968) commonly used state variables are separable, that is the fast parts of some states are small compared with their slow parts. The model (7), (8) can be obtained without redefinition of the state variables. On the other hand the commonly used state variables in a transformer model are not separable. In this 'mixed' case another choice of state variables exists for which the model is in the form (7), (8).

Voltage regulator. We use the standard model in Fig. 1 with the exciter saturation $S_E(E_{fd}) = A_{sat}\exp[B_{sat}E_{fd}]$ retained but limit type nonlinearities neglected. The numerical values are given in Table 1. The feedback compensator is

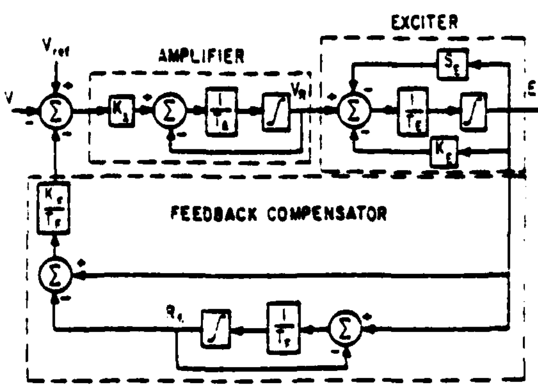

FIG. 1. IEEE type 1 voltage regulator.

TABLE 1. VOLTAGE REGULATOR CONSTANTS

T_A = 0.06 s	K_E = −0.0445
T_E = 0.5 s	K_F = 0.16
T_F = 1.0 s	A_{SAT} = 0.001123
K_A = 25.0	B_{SAT} = 0.3043

represented by two parallel paths in order to exhibit R_f. The state equations are

$$\frac{dV_R}{dt}=\frac{K_A}{T_A}\left[\frac{K_F}{T_F}(R_f-E_{fd})-\frac{V_R}{K_A}+\Delta V\right] \quad (16)$$

$$\frac{dE_{fd}}{dt}=\frac{1}{T_E}[V_R-K_E E_{fd}-S_E(E_{fd})] \quad (17)$$

$$\frac{dR_f}{dt}=\frac{1}{T_F}(E_{fd}-R_f) \quad (18)$$

where $\Delta V=V_{ref}-V$. To separate the states let us identify the fast and slow loops in Fig. 1. Since T_A is much smaller than T_E and T_F, the amplifier loop is fast and its state V_R will have a non-negligible fast part. The nature of E_{fd} is less obvious. If it has a fast part, it would not pass through the low pass R_f-path (T_F is large), but it would pass through the parallel path. We therefore disregard the R_f-path and examine the remaining system. Using K_E' as some linearized equivalent of the exciter gain, we see that the loop gain $K_A T_F/K_E' K_F$ is high because K_A is large and $K_E' K_F$ is small. The conclusion is that E_{fd} as a signal in a high gain loop will have a fast part. The remaining state R_f is a candidate for a slow state.

Our next step is to examine whether this choice of state variables can be scaled for (7), (8). Since the fast phenomena are caused by the smallness of $1/K_A=0.04$ and $T_A=0.06$, we take $\varepsilon=0.04$, that is

$$\frac{1}{K_A}=\varepsilon \quad \text{and} \quad T_A=1.5\varepsilon. \quad (19)$$

In addition to the time scaling, the states must be scaled to allow a meaningful limit as $\varepsilon\to 0$. It is apparent from Fig. 1 that $V_R\to\infty$ if $K_A\to\infty$, that is if $\varepsilon\to 0$. Hence we will use εV_R as a fast state variable. With

$$R_f=x,\ E_{fd}=z_1,\ V_R=z_2 \quad (20)$$

and with the given numerical values, the voltage regulator model in the form (7), (8) is

$$\frac{dx}{dt}=z_1-x \quad (21)$$

$$\varepsilon\frac{dz_1}{dt}=2[z_2-0.0445\varepsilon z_1-\varepsilon S_E(z_1)] \quad (22)$$

$$\varepsilon\frac{dz_2}{dt}=\frac{1}{1.5}[0.16(x-z_1)-z_2+\Delta V]. \quad (23)$$

Applying the reduction procedure (9)–(15), the slow model (9), (10) is

$$\frac{dx_s}{dt}=\frac{1}{0.16}\Delta V \quad (24)$$

$$z_{1s}=x_s+\frac{\Delta V}{0.16}$$

$$z_{2s}=0$$

and the fast model is

$$\frac{dz_{1f}}{d\tau}=2z_{2f} \quad \frac{dz_{2f}}{d\tau}=\frac{-1}{1.5}(0.16z_{1f}+z_{2f}). \quad (25)$$

To get an idea how this approximates the full scale model, we linearize (21), (22), (23) and compare its eigenvalues with those of (24) and (25) multiplied by $1/\varepsilon=25$. They compare very closely as 0.00916 to 0, and $-8.80\pm j8.45$ to $-8.33\pm j7.99$.

Transformer. In the coupled circuit in Fig. 2

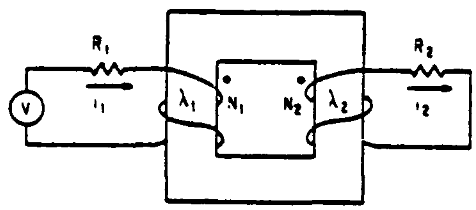

FIG. 2. Transformer model.

the ratio of leakage inductances ℓ_1, ℓ_2 and the self inductances L_1, L_2 is assumed to be the same small parameter

$$\frac{\ell_1}{L_1}=\frac{\ell_2}{L_2}=\mu. \quad (26)$$

Using the flux linkages

$$\lambda_1=(1+\mu)L_1 i_1-\frac{N_1}{N_2}L_2 i_2$$
$$\lambda_2=-\frac{N_2}{N_1}L_1 i_1+(1+\mu)L_2 i_2 \quad (27)$$

as the state variables and eliminating i_1, i_2 from

(27) using equations

$$\frac{d\lambda_1}{dt} = -R_1 i_1 + v \qquad \frac{d\lambda_2}{dt} = -R_2 i_2 \tag{28}$$

we obtain the model

$$\varepsilon\frac{d\lambda_1}{dt} = -\frac{\sqrt{1+\varepsilon}}{T_1}\lambda_1 - \frac{h}{T_2}\lambda_2 + \varepsilon v \tag{29}$$

$$\varepsilon\frac{d\lambda_2}{dt} = -\frac{1}{hT_1}\lambda_1 - \frac{\sqrt{1+\varepsilon}}{T_2}\lambda_2 \tag{30}$$

where $h = T_2 N_1/T_1 N_2$, $T_1 = L_1/R_1$, $T_2 = L_2/R_2$. The small parameter $\varepsilon = 2\mu + \mu^2$ multiplies both derivatives, and hence both λ_1 and λ_2 are fast, that is when we, for example, take $\lambda_1 = -1$, $\lambda_2 = 0$, $v = 0$, both derivatives will tend to infinity if $\varepsilon \to 0$. However the system matrix becomes singular indicating the possibility of a hidden slow phenomenon. Since both λ_1 and λ_2 are fast, we form the slow state by subtracting out the fast phenomena in λ_1 and λ_2. This is equivalent to defining a state x as a linear combination of λ_1 and λ_2 such that the derivative of x will not be multiplied by ε. In this case, an appropriate transformation is $x = \lambda_1 - h\lambda_2$, and the variable $z = \lambda_2$ is kept as the fast variable. Then (29), (30) becomes

$$\frac{dx}{dt} = -\frac{1}{2T_1}x - \frac{h}{2}\left(\frac{1}{T_1} + \frac{1}{T_2}\right)z + v \tag{31}$$

$$\varepsilon\frac{dz}{dt} = -\frac{1}{hT_1}x - \left(\frac{1}{T_1} + \frac{1}{T_2} + \frac{\varepsilon}{2T_2}\right)z \tag{32}$$

where $\sqrt{(1+\varepsilon)}$ is approximated by $1 + \varepsilon/2$. This is now a model of the type (7), (8) having the physically meaningful slow model $dx_s/dt = -x_s/(T_1 + T_2) + v$ for the flux linkage of an ideal transformer and the fast model $dz_f/dt = -(T_1^{-1} + T_2^{-1})z_f$, representing the flux leakage.

4. ITERATIVE SEPARATION OF TIME SCALES

As a special case of (7), (8) we consider a linear system

$$\dot{x} = Ax + Bz \qquad x(t_0) = x^0 \tag{33}$$

$$\varepsilon\dot{z} = Cx + Dz \qquad z(t_0) = z^0 \tag{34}$$

where d/dt is denoted by a dot and D^{-1} is assumed to exist. The qss assumption $\varepsilon\dot{z}_s = 0$, that is $0 = z_s + D^{-1}Cx_s$, yields $z_s = -D^{-1}Cx_s$. The true x, z will differ from x_s, z_s mainly by their fast parts. To find the fast part of z we introduce η_1 as the difference between z and z_s

$$\eta_1 = z + D^{-1}Cx \tag{35}$$

which transforms (33), (34) into

$$\dot{x} = (A - BD^{-1}C)x + B\eta_1 \triangleq A_1 x + B\eta_1 \tag{36}$$

$$\varepsilon\dot{\eta}_1 = \varepsilon D^{-1}CA_1 x + (D + \varepsilon D^{-1}CB)\eta_1 \triangleq C_1 x + D_1\eta_1. \tag{37}$$

This is a model of the type (7), (8) with η_1 playing the role of z. The crucial difference is however in the weaker presence of x in the η_1 equations where C_1 is $O(\varepsilon)$.* The qss of η_1 obtained from $0 = \eta_{1s} + D_1^{-1}C_1 x_s$ is only $O(\varepsilon)$, that is, η_1 is predominantly fast. Continuing this process we introduce

$$\eta_2 = \eta_1 + D_1^{-1}C_1 x \tag{38}$$

as the error due to the qss assumption $\varepsilon\dot{\eta}_{1s} = 0$ and substitute (38) into (36), (37). Repeating this step k times with

$$\eta_k = \eta_{k-1} + D_{k-1}^{-1}C_{k-1}x \qquad \eta_0 = z \tag{39}$$

we end up with the system

$$\dot{x} = A_k x + B\eta_k \tag{40}$$

$$\varepsilon\dot{\eta}_k = C_k x + D_k\eta_k \tag{41}$$

whose matrices are defined by

$$A_k = A_{k-1} - BD_{k-1}^{-1}C_{k-1} \qquad A_0 = A \tag{42}$$

$$C_k = \varepsilon D_{k-1}^{-1}C_{k-1}A_k \qquad C_0 = C \tag{43}$$

$$D_k = D_{k-1} + \varepsilon D_{k-1}^{-1}C_{k-1}B \qquad D_0 = D. \tag{44}$$

Again C_k has been reduced, this time to $O(\varepsilon^k)$. To recover z from η_k and x we observe from (39) that

$$\sum_{i=1}^{k}(\eta_i - \eta_{i-1}) = \eta_k - z = \left(\sum_{i=1}^{k} D_{i-1}^{-1}C_{i-1}\right)x. \tag{45}$$

Block diagram representations of the iterations are given in Fig. 3. The qss model is indicated in thick lines. The speed of integration in the fast loop is large due to its high gain $1/\varepsilon$. The input from x into each successive fast model is weaker. In the limit ($k = \infty$) the model becomes the fast-slow cascade in which A_∞ contains all the slow

*A vector or matrix function $\psi(\varepsilon)$ of a positive scalar ε is said to be $O(\varepsilon^k)$ if there exist positive constants c and ε^* such that $|\psi(\varepsilon)| \leq c\varepsilon^k$ for all $\varepsilon \leq \varepsilon^*$.

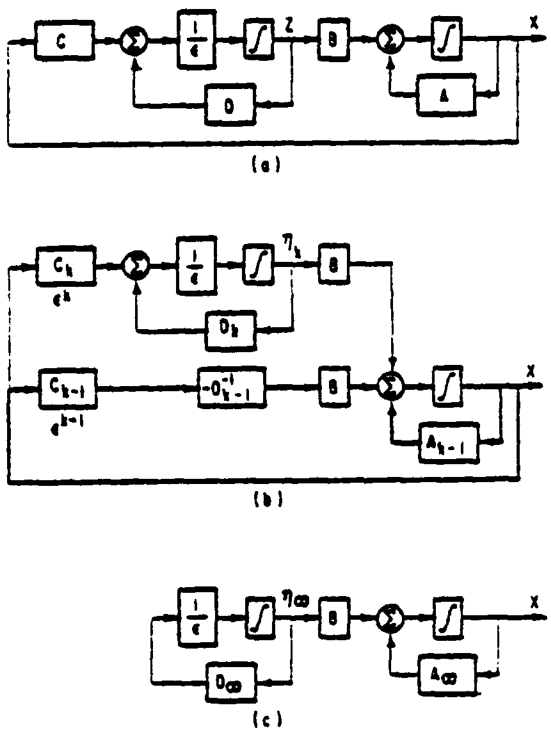

FIG. 3. Block diagram models.

modes and D_∞/ε all the fast modes. Using (42), (43) and (44) we can determine $A_\infty = A - BD^{-1}C + O(\varepsilon)$ and $D_\infty = D + O(\varepsilon)$, that is, the slow and the fast subsystem matrices which would have been obtained using (9), (10) and (13) as $O(\varepsilon)$ approximations of A_∞ and D_∞. Note that A_∞ and D_∞ are obtained in terms of the original subsystem matrices without ill-conditioned modal transformations. Another practical advantage over the modal method is that the physical meaning of the original state variables is preserved. From (45), the definition of x remains the same, while the new state variable η_k has the same meaning as z.

After k iterations the model (40), (41) still has the full fast input $B\eta_k$ into the slow subsystem which we now want to reduce. Expressing η_k from (41) in terms of $\dot\eta_k$ and x and substituting into (40)

$$\dot x - \varepsilon BD_k^{-1}\dot\eta_k = (A_k - BD_k^{-1}C_k)x \triangleq A_{k1}x \quad (46)$$

suggests

$$\xi_1 = x - \varepsilon BD_k^{-1}\eta_k \quad (47)$$

as the slow part of x. The slow subsystem then becomes

$$\dot\xi_1 = A_{k1}\xi_1 + \varepsilon A_{k1}BD_k^{-1}\eta_k \triangleq A_{k1}\xi_1 + B_{k1}\eta_k. \quad (48)$$

Since B_{k1} is $O(\varepsilon)$ the fast input has been reduced. Next we define ξ_2 as the slow part of ξ_1 etc., that is we construct the iterations

$$\xi_{j+1} = \xi_j - \varepsilon B_{kj}D_{kj}^{-1}\eta_k \qquad \xi_0 = x, \quad (49)$$

where

$$A_{kj+1} = A_{kj} - B_{kj}D_{kj}^{-1}C_k \qquad A_{k0} = A_k \quad (50)$$

$$B_{kj+1} = \varepsilon A_{kj+1}B_{kj}D_{kj}^{-1} \qquad B_{k0} = B \quad (51)$$

$$D_{kj+1} = D_{kj} + \varepsilon C_k B_{kj} D_{kj}^{-1} \qquad D_{k0} = D_k. \quad (52)$$

The weakening of the fast input has been accomplished since after each iteration B_{kj} is reduced by an order of ε and tends to zero as $j \to \infty$. In other words the slow and the fast subsystems of the resulting system

$$\dot\xi_j = A_{kj}\xi_j + B_{kj}\eta_k \qquad \xi_j(t_0) = \xi_j^0 \quad (53)$$

$$\varepsilon\dot\eta_k = C_k\xi_j + D_{kj}\eta_k \qquad \eta_k(t_0) = \eta_k^0 \quad (54)$$

are only weakly coupled because B_{kj} is $O(\varepsilon^j)$ and C_k is $O(\varepsilon^k)$. It is also easily seen that A_{kj}, D_{kj} are $O(\varepsilon^{k+j})$ approximations of A_∞, D_∞.

To recover x from ξ_j and η_k we observe from (47) that

$$\sum_{i=1}^{j}(\xi_i - \xi_{i-1}) = \xi_j - x = -\varepsilon\left(\sum_{i=1}^{j} B_{ki-1}D_{ki-1}\right)\eta_k. \quad (55)$$

Thus the slow variable ξ_j is the dominant part of x, whose fast part is $O(\varepsilon)$. It is of practical importance that ξ_j has the same physical meaning as x.

Remark. Observe that the recursions remain the same if we use C/ε and D/ε instead of C and D. This means that ε, which is crucial in the asymptotic analysis of validity and convergence, does not have to be explicitly identified in the iterations.

In conclusion our objective to reduce a system with coupled slow and fast parts has been met. In the transformed system (53), (54) the coupling terms B_{kj} and C_k are weak and can be neglected. Instead of the original full order system (53), (54) we will use the separate lower order subsystems

$$\dot{\bar\xi}_j = A_{kj}\bar\xi_j, \quad \varepsilon\dot{\bar\eta}_k = D_{kj}\bar\eta_k \quad (56)$$

with the initial value η_k^0 obtained from x^0, z^0 via (45) and ξ_j^0 obtained from x^0 and η_k^0 via (55). The simulation of $\bar\eta_k$ can be performed in the fast time scale τ.

The error $\xi_j(t) - \bar\xi_j(t)$ is $O(\varepsilon^j)$ while the error $\eta_k(t) - \bar\eta_k(t)$ is $O(\varepsilon^k)$. Using $\bar\xi_j(t)$, $\bar\eta_k(t)$ we obtain

the corresponding approximation of $x(t)$, $z(t)$ by evaluating $\hat{x}(t)$ from (55) and $\hat{z}(t)$ from (45). The error $x(t)-\hat{x}(t)$ is $O(\varepsilon^i)$ where $i=\min(j,k+1)$ while the error $z(t)-\hat{z}(t)$ is $O(\varepsilon^i)$ where $i=\min(j,k)$. In long term or short term studies a further simplification would be to keep only one of the two models (56). In general we need to compute four matrices A_{kj}, D_{kj} and the sums in (45) and (55). They can be generated by (42)–(44) and (50)–(52).

An alternative algorithm is presented in Kokotovic (1975) and is motivated by (45) and (55). Substitution of $\eta=z+Lx$ into (34) yields

$$\varepsilon\dot{\eta}=Mx+(D+\varepsilon LB)\eta \tag{57}$$

where $M=C-DL+\varepsilon L(A-BL)$. To completely decouple x from η in (57), we choose L such that $M=0$. The expression $M=0$ rewritten as $L=D^{-1}C+\varepsilon D^{-1}L(A-BL)$ suggests that L can be solved for iteratively as

$$L_{k+1}=D^{-1}C+\varepsilon D^{-1}L_k(A-BL_k)$$

where $L_1=D^{-1}C$. The system (33), (34) after k L-iterations has the form (40), (41), where the matrices are now defined as $A_k=A-BL_k$, $C_k=C-DL_k+L_kA_k$, and $D_k=D+L_kB$. Note that as $k\to\infty$, L_{k+1} converges to L when ε is sufficiently small.

Similarly, substitution of $\xi=x-\varepsilon H\eta_k$ into (41) yields

$$\dot{\xi}=(A_k-HC_k)\xi+N\eta_k \tag{58}$$

and we set $N=B-HD+\varepsilon(A_k-HC_k)H=0$ to decouple η_k from ξ. Rearranging the expression $N=0$ we obtain

$$H=BD_k^{-1}+\varepsilon(A_k-HC_k)HD_k^{-1}$$

and solve for H iteratively as

$$H_{kj+1}=BD_k^{-1}+\varepsilon(A_k-H_{kj}C_k)H_{kj}D_k^{-1}$$

where $H_{k1}=BD_k^{-1}$. The system (40), (41) after j H-iterations has the form (53), (54) where the new expressions for the matrices are $A_{kj}=A_k-H_{kj}C_k$, $B_{kj}=B-H_{kj}D_k+A_{kj}H_{kj}$ and $D_{kj}=D_k+C_kH_{kj}$.

The L- and H-iterations avoid repeated matrix inversions of D_k and D_{kj} which are required by the iterations (39) and (49). The development of (39), (40) complements the L-, H-iterations by showing that L and H are obtained from a succession of qss assumptions. In the computation of A_{kj}, D_{kj}, we will use the L-, H-iterations. Some other aspects of computing A_{kj}, D_{kj} are given in Anderson (1978).

5. AN ILLUSTRATIVE APPLICATION

In the companion paper (Winkelman and coworkers, 1980) a systematic separation procedure using the iterative scheme is proposed and applied to the two-time-scale and four-time-scale investigations of a 20th order model of a three-machine power system. Here we analyze a seventh order model of the single machine-infinite bus system in Fig. 4. A five cycle 3 phase fault is

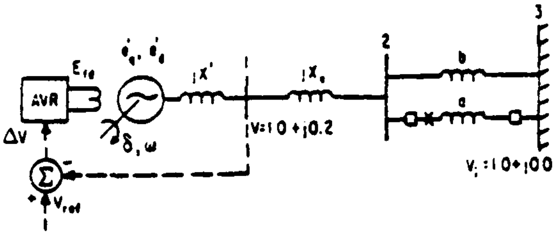

FIG. 4. Single machine—infinite bus system (AVR is the voltage regulator from Fig. 1).

applied on circuit 'a' close to bus 2, and is cleared by opening circuit 'a'. This example will illustrate some features of the separation procedure and introduce the time scales to form a basis for the analysis of the three machine system studied in the companion paper. The block AVR in Fig. 4 is the voltage regulator described in Section 3 and its model (16), (17), (18) will be retained with the generator terminal voltage V defined by

$$a^2V^2=(a-1)^2(e_q'^2+e_d'^2)+2(a-1)\times(e_d'\cos\delta-e_q'\sin\delta)V+V^2 \tag{59}$$

where $a=(X'Y)^{-1}$ and Y is the admittance of the transmission line. The four additional state equations

$$\dot{e}_q'=\frac{1}{T_{d0}'}\{-[1+(X_d-X')Y]e_q'-(X_d-X')YV\sin\delta+E_{fd}\} \tag{60}$$

$$\dot{e}_d'=\frac{1}{T_{q0}'}\{(X_q-X')YV\cos\delta-[1+(X_q-X')Y]e_d'\} \tag{61}$$

$$\dot{\delta}=377(\omega-1) \tag{62}$$

$$\dot{\omega}=\frac{1}{2H}\left[\frac{P_{in}}{\omega}-D(\omega-1)-YV(e_q'\cos\delta+e_d'\sin\delta)\right] \tag{63}$$

describe the flux linkage decay transients in the direct (d) and the quadrature (q) axes (60), (61) and the mechanical transient by the swing equations (62), (63). The numerical values in Table 2 are typical. It should be noted that in this problem formulation the quadrature axis leads the direct axis.

TABLE 2. SYNCHRONOUS MACHINE DATA

H	$= 5.0$ s	X_e	$= 0.25$ pu
D	$= 2.0$ pu	T'_{d0}	$= 5.0$ s
X_d	$= 1.2$ pu	T'_{q0}	$= 0.5$ s
X_q	$= 1.0$ pu	X_e	$= 0.01$ pu
X'	$= 0.25$ pu	X_b	$= 0.01$ pu

To determine the fast and slow states we first note that the earlier reason for V_R and E_{fd} to be fast remains valid in this enlarged system. The linearized swing equations (62), (63), with all the variables constant except for δ and ω, show a typical swing frequency of about 1.4 Hz. Hence both δ and ω will be fast. Finally for the flux linkage equations we note that the quadrature axis has a much smaller time constant (0.5 s) than the direct axis (5 s). Therefore we assume that e'_d is fast and e'_q is slow, and order the states as follows:

$$e'_q,\ R_f,\ e'_d,\ \delta,\ \omega,\ E_{fd},\ V_R \tag{64}$$

considering e'_q, R_f as slow and the remaining five variables as fast. Upon linearization of the nonlinear model at the nominal values given in Fig. 4, the system matrix is as follows:

$$\begin{bmatrix} -0.58 & 0 & 0 & -0.269 & 0 & 0.2 & 0 \\ 0 & -1.0 & 0 & 0 & 0 & 1.0 & 0 \\ 0 & 0 & -5.0 & 2.12 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 377 & 0 & 0 \\ -0.141 & 0 & 0.141 & -0.2 & -0.28 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.0838 & 2.0 \\ -173 & 66.7 & -116 & 40.9 & 0 & -66.7 & -16.7 \end{bmatrix} \tag{65}$$

The system eigenvalues $-0.36 \pm j0.56$, $-0.86 \pm j8.4$, -3.93, $-8.53 \pm j8.22$ also indicate that there should be two slow and five fast states. In view of the remark in Section 4 we proceed with iterations without an explicit value for ε.

The post-fault simulation results using the subsystems are quite revealing. The simple qss slow model causes large errors in the linearized states $\Delta e'_q$ and ΔR_f (Figs 5 and 6). After only one iteration of the slow and fast subsystems ($j = k = 1$) the error is practically unnoticeable. The response of fast state $\Delta\delta$ for both the uncorrected fast model and after one iteration is shown in Fig. 7. That is typical of all five fast states.

6. A CORRECTION METHOD FOR NONLINEAR SYSTEMS

With minor modifications the iterative procedure of Section 4 is applicable to linear time varying systems. It can also be extended to nonlinear systems of the type (7), (8) where f and g are continuous and differentiable in all its arguments by first linearizing (7), (8) along the trajectory (14), (15) and then applying the method for linear time varying systems. As in standard asymptotic expansion methods this requires

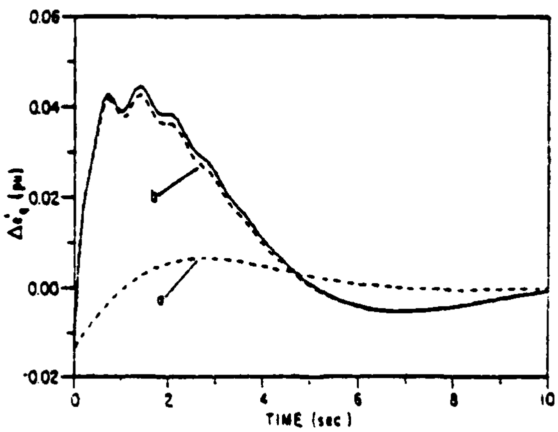

FIG. 5. Slow variable $\Delta e'_q$: exact (solid), qss approximation [dotted (a)] and after one iteration [dotted (b)].

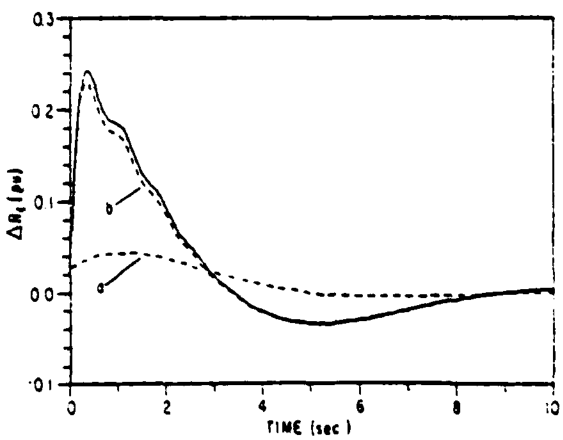

FIG. 6. Slow variable ΔR_f: exact (solid), qss approximation [dotted (a)] and after one iteration [dotted (b)].

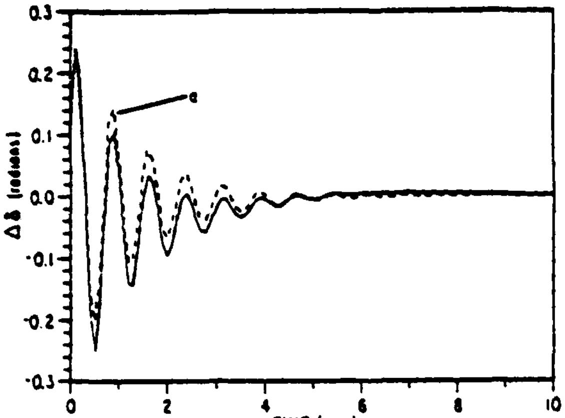

FIG. 7. Fast variable $\Delta\delta$: exact (solid), qss approximation [dotted (a)] and after one iteration (same as exact).

the time varying Jacobian matrices f_x, f_z, g_x, g_z and the inverse of g_z. From a computational point of view it is more desirable to deal with f and g directly. Allemong (1978) has proposed such a method for a class of nonlinear systems including power systems considered here and in the companion paper. In the following outline of the method we drop t from f and g and let $t_0 = 0$.

Section 2 discussed the problem of obtaining reduced order models for nonlinear systems without corrections. Equations (9), (10) and (13) yield these approximations. Using these solutions for x_s, z_s and z_f we proceed as follows.

Let $z_s = \psi(x_s)$ be a root of $g = 0$ in (10). Substitute

$$z = \psi(x) + \eta \tag{66}$$

into (7) and assuming that the nonlinearities in x and z are separable, (7) may be written as

$$\begin{aligned}\dot{x} &= f(x, \psi(x)) + f_2(\psi(x), \eta) \\ &\triangleq f_1(x) + f_2(\psi(x), \eta),\ x(0) = x^0.\end{aligned} \tag{67}$$

Note that neglecting f_2, which contains the dominant fast part, yields the slow subsystem (9), (10).

The integral form of (67) is

$$x = x^0 + k + \int_0^t f_1(x)\,dt + \int_0^t f_2(\psi(x), \eta)\,dt - k \tag{68}$$

where we have added and subtracted

$$k = \int_0^{\infty} f_2(z_s, z_f)\,dt \tag{69}$$

where z_s, z_f are obtained from (9), (10) and (13). To better approximate the state x we introduce the expression $x = \xi_1 + x_f$ and solve the slow part as

$$\dot{\xi}_1 = f_1(\xi_1),\ \xi_1(0) = x^0 + k \tag{70}$$

that is, we include the influence of the fast part as a shift k in the slow initial condition. The remaining terms in (68) are fast and are solved by approximating $\psi(x)$ with z_s and η_f by z_f, that is

$$\dot{x}_f = f_2(z_s, z_f),\ x_f(0) = -k \tag{71}$$

where z_s is known from (9), (10) and z_f from (13). If desired, the next step can be a further improvement of the fast subsystem

$$\dot{\eta}_1 = g(\xi_1 + x_f, z_s + \eta_1),\ \eta_1(0) = z^0 - z_s(0) \tag{72}$$

where ξ_1, x_f and z_s are now known from (10), (70) and (71) (Allemong, 1978).

This method has been tested on the single machine model (16)–(18), (60)–(63). The only types of nonlinearities in these equations are sine and cosine functions and the saturation $S_E(E_{fd})$. Limit type nonlinearities on V_R are not considered here. The uncorrected slow variables e'_{qs} and R_{fs} obtained from (9), (10) are shown in Figs 8 and 9. Then the corrected e'_{qsc} and R_{fsc} are solved from

$$\dot{e}'_{qsc} = -\frac{1}{T'_{d0}}\{[1 + (X_d - X')Y]e'_{qsc} + (X_d - X')YV\sin\delta_s - E_{fds}\} \tag{73}$$

$$\dot{R}_{fsc} = \frac{1}{T_F}(E_{fds} - R_{fsc}) \tag{74}$$

$$e'_{qsc}(0) = e'_q(0) + k_1,\ R_{fsc} = R_f(0) + k_2 \tag{75}$$

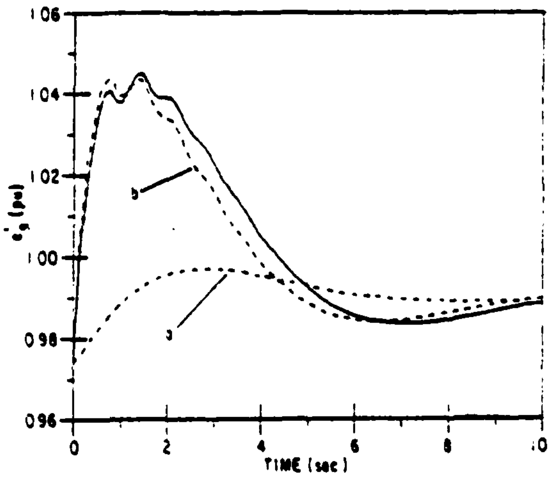

FIG. 8. Slow variable e'_q: exact (solid), qss approximation [dotted (a)] and after one iteration [dotted (b)].

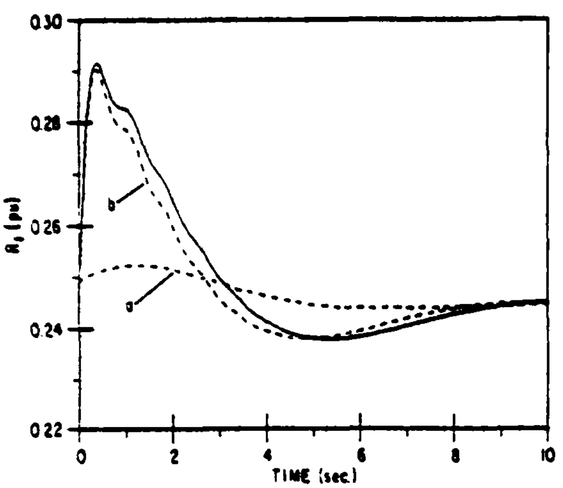

FIG. 9. Slow variable R_f: exact (solid), qss approximation [dotted (a)] and after one iteration [dotted (b)].

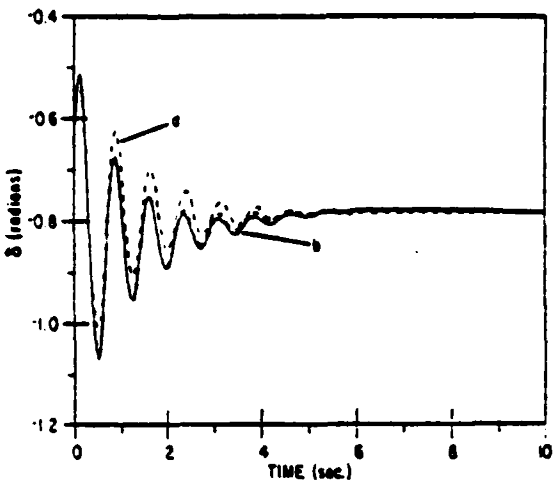

FIG. 10. Fast variable δ: exact (solid), qss approximation [dotted (a)] and after one iteration [dotted (b)].

$$k_1 = -\frac{(X_d - X')YV}{T'_{d0}} \int_0^{\infty} F(\delta_s, \delta_f)\,dt + \frac{1}{T'_{d0}} \int_0^{\infty} E_{fdf}\,dt \qquad (76)$$

$$k_2 = \frac{1}{T} \int_0^{\infty} E_{fdf}\,dt \qquad (77)$$

where

$$F(\delta_s, \delta_f) = \sin\delta_s(\cos\delta_f - 1) + \cos\delta_s \sin\delta_f. \qquad (78)$$

Their fast parts are solved from

$$\dot{e}'_{qf} = -\frac{1}{T'_{d0}}((X_d - X')YVF(\delta_s, \delta_f) - E_{fdf}),\ e'_{qf}(0) = -k_1 \qquad (79)$$

$$\dot{R}_{ff} = \frac{1}{T_F} E_{fdf}, \qquad R_{ff}(0) = -k_2 \qquad (80)$$

where δ_s, E_{fd_s} are known from (10) and δ_f, E_{fdf} are obtained from the uncorrected fast model (13). The corrected slow variables are in Figs 8 and 9 and the quality of the approximations is almost as good as in the linear case. The uncorrected and corrected fast variable δ is shown in Fig. 10. The behavior of other fast variables is similar. Although this method requires further testing, these first experiments are encouraging.

7. CONCLUSIONS

This paper discusses the application of singular perturbation methods to eliminate the inconsistencies of the classical quasi-steady-state approach to model reduction. The iterative method presented is a means for improving the accuracies of the reduced models. A correction method for nonlinear systems with sine, cosine and exponential nonlinearities is also presented. The modeling and separation of time scales are illustrated by several systems common in electrical engineering. Further application of the method is developed in the companion paper (Winkelman and co-workers, 1980).

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SUBSYSTEMS, TIME SCALES, AND MULTIMODELING

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ABSTRACT

Through a couple of naive examples the control theorists are invited to reexamine the role of modeling in the study of large scale dynamic systems. Instead of assuming the existence of "N diagonally dominant blocks," they should identify one strongly coupled slow core and N weakly coupled fast subsystems. This structure is exhibited with a physically meaningful choice of state variables. The controls are introduced following the recent concept of multimodeling.

INTRODUCTION

Most control studies of large scale systems start with a model possessing some known hierarchical or diagonal dominance properties. This assumption expresses our desire to escape the task of modeling. At the extreme are the researchers for whom it is more rational to design strategies for controlling an unknown plant, than to first spend some time developing a model for it. In simpler and smaller size systems a certain disregard of modeling is tolerable. The situation is different in large scale systems where the phenomena occurring are too rich to be handled by all-purpose control strategies. Consider, for instance, the stabilization strategies based on the assumption of diagonal dominance, and designed by vector Lyapunov function methods (Siljak, 1978). As the two power system examples in Jocic et al. (1978) and Grujic et al. (1979) show, the success of these strategies critically depends on what is modeled as a subsystem. If the subsystems are simply taken to be the individual generating units, the results are extremely conservative. With a careful choice of "coherent areas" as subsystems, the results become more meaningful.

Instead of assuming that an "off-the-shelf" model is already available in a neat weakly coupled form, a deeper understanding of the causes for weak coupling must be gained and used in modeling of subsystems. In this paper we make an attempt in this direction. We first examine the relationship of diagonal dominance and time scales in the decomposition of Markov chains and show that similar decompositions apply to electrical, mechanical, and electromechanical networks. We then outline a grouping procedure for determination of subsystems and separation of time scales. A general property of the considered systems is that they are strongly coupled in the slow time scale, and weakly coupled in the fast time scale. Due to this property every subsystem controller can neglect all other fast subsystems except for his own. This multimodeling situation is discussed in the last section of the paper. To highlight the ideas and avoid technicalities, the paper is written as an informal discussion of representative examples. More general and rigorous treatment can be found in quoted references.

SUBSYSTEMS AND TIME SCALES IN MARKOV CHAINS

In attempting to decompose a system into subsystems, the first step is to identify the units of the system and quantify their interactions. This is a nontrivial task and its outcome may have to be revised after subsequent steps. With units and interactions defined, the next step is to form the subsystems as groups of units. A criterion for this grouping may be to require that the "inner" interactions be stronger than the "outer" interactions, that is, a unit should be coupled more strongly with the units in its own subsystem, than with the other units. Representing the interactions of n units as the entries of an n×n matrix, the decomposition into N subsystems is considered to be possible if there is an ordering of the units for which the interaction matrix possesses N dominant diagonal blocks.

For dynamic systems we broaden this reasoning to include a separation of time scales. Since the aggregation of Markov chains is a particularly clear illustration of this, we begin with a singular perturbation interpretation of the results of Pervozvanski and Smirnov (1974), Gaitsgori and Pervozvanski (1975), and Delebecque and Quadrat (1978, 1980).

In a system described by a finite state Markov chain, the states are the units of the system and their interactions are the transition probabilities. If some of these interactions are weak, they can be neglected over shorter periods of time. For example, in the four-state chain in Fig. 1a, we may choose to neglect all the interactions smaller than or equal to .2. Then the states are grouped into two classes: 1,2 and 3,4. By increasing the self-interactions to compensate for the neglected weak interactions the two "fast" chains can be formed as in Fig. 1b.

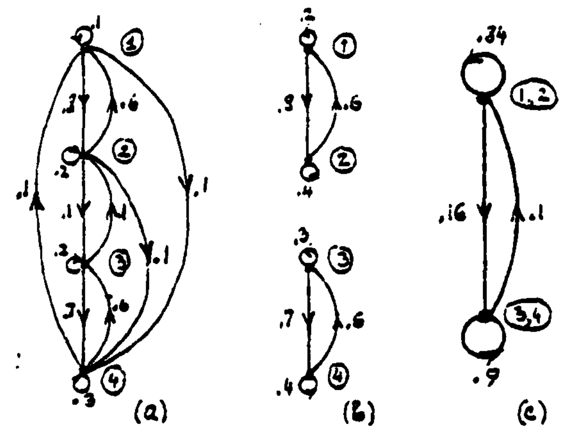

Fig. 1

This means that the last matrix in

$$\begin{bmatrix} .1 & .8 & 0 & .1 \\ .6 & .2 & .1 & .1 \\ 0 & .1 & .2 & .7 \\ .1 & 0 & .6 & .3 \end{bmatrix} = \begin{bmatrix} .2 & .8 & 0 & 0 \\ .6 & .4 & 0 & 0 \\ 0 & 0 & .3 & .7 \\ 0 & 0 & .6 & .4 \end{bmatrix} + \begin{bmatrix} -.1 & 0 & 0 & .1 \\ 0 & -.2 & .1 & .1 \\ 0 & .1 & -.1 & 0 \\ .1 & 0 & 0 & -.1 \end{bmatrix} \tag{1}$$

has been neglected.

Preparing for an asymptotic analysis, we represent all the weak interactions as multiples of a small positive scalar ε, that is we represent the last matrix in (1) by εA_1. Furthermore, we denote by $A_\varepsilon+I$ and A_0+I the transition matrices of the chains in Fig. 1a and b, respectively, observing that the row sums of A_ε, A_0, and εA, are all zero. Thus a general expression of the type (1) is

$$(A_\varepsilon+I) = (A_0+I) + \varepsilon A. \tag{2}$$

By this construction A_0 is made of diagonal blocks. Each block contributes one right and one left eigenvector of A_0 for $\lambda=0$. The right eigenvectors are of the form [00111000]', where ' denotes a transpose, and the number and the position of the ones are determined by the dimension and the position of the diagonal block in A_0. Similarly, the left eigenvectors are of the form $[0\ 0\ q_i q_{i+1}\ q_{i+2}\ 0\ 0\ 0]$ where q_i, q_{i+1}, q_{i+2} are the stationary probabilities for $\varepsilon=0$ of the states i, i+1, i+2 in the same class and, hence, their sum is one. For an n-state chain with N blocks the right eigenvectors of A_0 for $\lambda=0$ form an n×N matrix R and the left eigenvectors form an N×n matrix Q. In our example (1) we have

$$R = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}, \quad Q = \begin{bmatrix} \frac{3}{7} & \frac{4}{7} & 0 & 0 \\ 0 & 0 & \frac{7}{13} & \frac{6}{13} \end{bmatrix}. \tag{3}$$

To summarize, the right and left null spaces of A_0 are characterized by

$$A_0 R = 0, \quad QA_0 = 0, \quad QR = I_N \tag{4}$$

where I_N is the N×N identity.

Let us now form a continuous time model of the chain. Assuming that $(A_\varepsilon+I)$ is the transition matrix in "fast" time t_f, the transition matrix in slow time $t=\varepsilon t_f$ is $\frac{1}{\varepsilon}(A_\varepsilon+I)$. For example, if $\varepsilon=\frac{1}{7}$ and if t_f is in days, then t will be in weeks. Since we are interested in the longer term behavior of the chain, we will use the model in slow time

$$\dot{p} = \frac{1}{\varepsilon} pA_\varepsilon = p(\frac{1}{\varepsilon} A_0 + A_1) \tag{5}$$

where the n-row p(t) is the probability distribution at time t. If the initial distribution p(0) is far from $p(0)A_0=0$, then $\frac{1}{\varepsilon} pA_0 \gg pA_1$ and the initial fast transient can be approximately determined from $\dot{p} \approx \frac{1}{\varepsilon} pA_0$. Since A_0 is block-diagonal, the fast transient is formed of separate transients within the classes. After some t the probability p(t) will be ε close to the composite of the stationary distributions within the N classes. From then on pA_1 is no longer negligible with respect to $\frac{1}{\varepsilon} pA_0$ and the transitions between the classes must be taken into account. To do this let us introduce y_i - the aggregate probability for the state to be in the ith class. The N-row of the aggregate probabilities is

$$y = pR. \tag{6}$$

In our example (1) this simply means $y_1 = p_1+p_2$, $y_2 = p_3+p_4$. The probability p_j to be in state j can be expressed as the probability y_k to be in its class times the probability to be in that state when in the class. If the latter is approximated by the stationary probability then $p_j \approx y_k q_j$. This motivates the representation of p as

$$p = yQ + zW, \quad WR = 0 \tag{7}$$

where z is an (n-N)-row and the choice of the constant (n-N)×n matrix W is in agreement with (4) and (6). Intuitively the term represents *fast fluctuations* around yQ. If we multiply (7) by an n×(n-N) matrix S such that

$$QS = 0, \quad WS = I_{n-N}, \tag{8}$$

then the result is

$$z = pS. \tag{9}$$

We need to choose S for a meaningful definition of z, satisfying (8). In our example such a choice is

$$S = \begin{bmatrix} \frac{1}{q_1} & 0 \\ \frac{1}{q_2} & 0 \\ 0 & -\frac{1}{q_3} \\ 0 & -\frac{1}{q_4} \end{bmatrix}, \quad \begin{aligned} z_1 &= \frac{p_1}{q_1} - \frac{p_2}{q_2} \\ z_2 &= \frac{p_3}{q_3} - \frac{p_4}{q_4} \end{aligned} \tag{10}$$

The corresponding W is then

$$W = \begin{bmatrix} q_1q_2 & -q_1q_2 & 0 & 0 \\ 0 & 0 & q_3q_4 & -q_3q_4 \end{bmatrix}. \tag{11}$$

In general, fast variables z_i should be defined as weighted differences of probabilities within a class.

We now use (7) to express (5) in terms of y and z. After simple manipulations we obtain

$$\dot{y} = yQA_1R + zWA_1R \tag{12}$$

$$\varepsilon\dot{z} = \varepsilon yQA_1S + zF \tag{13}$$

where the (n-N)×(n-N) matrix

$$F = W(A_0+\varepsilon A_1)S \tag{14}$$

is stable. By having transformed (5) into the standard singular perturbation form (12), (13), we have accomplished one of the goals of this section. It is straightforward now to analyze the time scale properties of (12), (13) using asymptotic or iterative techniques, such as in Kokotovic, O'Malley, and Sannuti (1976) and Kokotovic et al. (1980). The slow subsystem of (12), (13) is

$$\dot{\bar{y}} = \bar{y}(QA_1R - \varepsilon WA_1RQA_1SF^{-1}) = \bar{y}\,\bar{\Lambda}. \tag{15}$$

For $\varepsilon=0$ it reduces to the aggregate proposed by Smirnov and Parvozvanski (1974). The transition matrix $(\bar{\Lambda}+I_N)$ represents an aggregate chain whose states are the classes of the original chain. For our example(1) this matrix is

$$(\bar{\Lambda}+I_N) = \begin{bmatrix} .84 & .16 \\ .1 & .9 \end{bmatrix} \tag{16}$$

and the aggregate chain is shown in Fig. 1c. The fast fluctuations are approximately governed by

$$\frac{dz_f}{d\tau_f} = z_f F = -z_f \begin{bmatrix} .136 & .014 \\ .023 & .1 \end{bmatrix} \tag{17}$$

where the numerical values are for our example (1). It is crucial to point out that due to the form (14) of F, its diagonal blocks are indeed dominant. The eigenvalues of F in the example -.14 and -0.09 are close to its diagonal elements. Hence, (17) describes N separate fluctuations within each of the N classes. Even though transition probabilities as large as 0.2 have been neglected, the approximation is excellent. The stationary probability distribution $p = [.18\ \ .21\ \ .28\ \ .33]$ is approximated by $\bar{y}Q = [.17\ \ .22\ \ .28\ \ .33]$.

Our final conclusion is that the original chain (5) should be decomposed not into N, but into N+1 subsystems. One of them is the slow subsystem which defines the Markov chain of N strongly coupled aggregate states. The remaining N fast subsystems are not Markov chains, but represent internal fluctuations within the N classes.

NETWORKS, SUBNETWORKS, AND TIME SCALES

A similar reasoning can be used to determine time scales and subsystems of electrical, mechanical, and electromechanical networks.

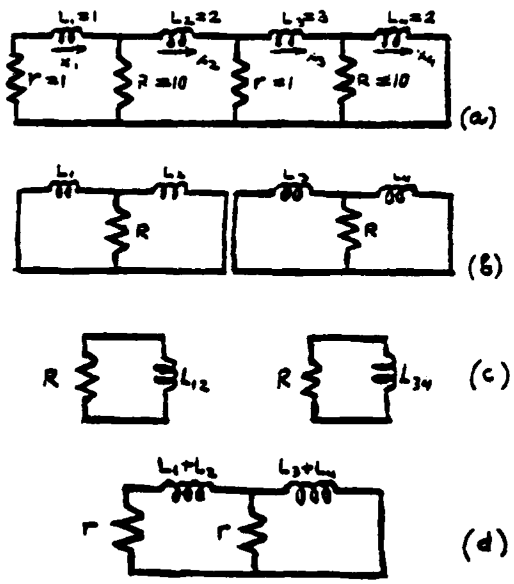

Fig. 2

As an illustration we consider a simple RL-network in Fig. 2a, where the inductors are of the same order of magnitude, and the nonuniformity of interactions is due to the fact that the resistors R are much larger than the resistors r. This nonuniformity suggests that there may exist a way to approximate the network by some simpler subnetworks. The system equation with the inductor currents as the states is

$$\dot{x} = A_\varepsilon x = \begin{bmatrix} -\frac{r+R}{L_1} & \frac{R}{L_1} & 0 & 0 \\ \frac{R}{L_2} & -\frac{r+R}{L_2} & \frac{r}{L_2} & 0 \\ 0 & \frac{r}{L_3} & -\frac{r+R}{L_3} & \frac{R}{L_3} \\ 0 & 0 & \frac{R}{L_4} & -\frac{R}{L_4} \end{bmatrix} x \tag{18}$$

where denoting $r = \varepsilon R$ the matrix A_ε is expressed as $A_\varepsilon = A_0 + \varepsilon A_1$, that is

$$A_\varepsilon = \begin{bmatrix} -\frac{R}{L_1} & \frac{R}{L_1} & 0 & 0 \\ \frac{R}{L_2} & -\frac{R}{L_2} & 0 & 0 \\ 0 & 0 & -\frac{R}{L_3} & \frac{R}{L_3} \\ 0 & 0 & \frac{R}{L_4} & -\frac{R}{L_4} \end{bmatrix} + \varepsilon \begin{bmatrix} -\frac{R}{L_1} & 0 & 0 & 0 \\ 0 & -\frac{R}{L_2} & \frac{R}{L_2} & 0 \\ 0 & \frac{R}{L_3} & -\frac{R}{L_3} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{19}$$

We note that A_0 represents the two subnetworks in Fig. 2b, which, due to the all-inductor loops, possess equilibrium subspaces. If the initial currents are far from these subspaces, $x_1(0) \neq x_2(0)$ and $x_3(0) \neq x_4(0)$, the currents $(x_1 - x_2)_o$ and $(x_3 - x_4)_o$ will flow through the resistors R, governed by

$$(\dot{x}_1 - \dot{x}_2)_o = -\frac{R}{L_{12}}(x_1 - x_2)_o \qquad (\dot{x}_3 - \dot{x}_4)_o = -\frac{R}{L_{34}}(x_3 - x_4)_o, \tag{20}$$

where $L_{jk} = \frac{L_j L_k}{L_j + L_k}$ and $(\ )_o$ indicates that only A_o, that is $\varepsilon = 0$, is considered. The subsystems (20) are "fast" because R is large, and the corresponding "fast" subnetworks are shown in Fig. 2c. When the subnetwork equilibria are approximately reached, the state x is close to the null-space of A_0 and, hence, $\varepsilon A_1 x$ is no longer negligible with respect to $A_0 x$. In the long term, the state of (18) will continue to be in the neighborhood of the null-sapce of A_0, whose basis we denote by V, an n×N matrix which in this example has the columns [1100]' and [0011]'. Representing x as

$$x = Vy + Hz, \quad A_0 V = 0 \tag{21}$$

and introducing matrices M and P satisfying

$$MV = I_N, \quad MH = 0, \quad PH = I_{n-N}, \quad PV = 0 \tag{22}$$

we obtain

$$y = Mx, \qquad z = Px \tag{23}$$

where y will be the slow and z the fast variables. Since we have already observed that $x_1 - x_2$ and $x_3 - x_4$ are fast, let us use this observation and (22) to determine P. For our example this will be

$$P = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}; \quad \begin{matrix} z_1 = x_1 - x_2 \\ z_2 = x_3 - x_4 \end{matrix} \tag{24}$$

which satisfies $PV = 0$. Then using H and M in the form

$$H = \begin{bmatrix} h_1 & 0 \\ h_2 & 0 \\ 0 & h_3 \\ 0 & h_4 \end{bmatrix}, \quad M = \begin{bmatrix} m_1 & m_2 & 0 & 0 \\ 0 & 0 & m_1 & m_2 \end{bmatrix} \tag{25}$$

we see from the first three conditions of (22) that

$$\begin{aligned} h_1 - h_2 = 1, \quad h_1 = m_2, \quad m_1 + m_2 = 1 \\ h_3 - h_4 = 1, \quad h_3 = m_4, \quad m_3 + m_4 = 1. \end{aligned} \tag{26}$$

Physically meaningful quantities which satisfy these relations are

$$m_1 = \frac{L_1}{L_1+L_2}, \quad m_2 = \frac{L_2}{L_1+L_2}, \quad m_3 = \frac{L_3}{L_3+L_4}, \quad m_4 = \frac{L_4}{L_3+L_4}. \tag{27}$$

Thus we see that our "slow" currents are the weighted sums of the currents within the subnetworks

$$\begin{aligned} y_1 &= \frac{L_1}{L_1+L_2} x_1 + \frac{L_2}{L_1+L_2} x_2 \\ y_2 &= \frac{L_3}{L_3+L_4} x_3 + \frac{L_4}{L_3+L_4} x_4. \end{aligned} \tag{28}$$

It is worth noting that y_1 and y_2 are not exactly analogous to the Markov chain example, where $y_1 = p_1 + p_2$ and $y_2 = p_3 + p_4$, nor is (24) analogous to (10). However, had we taken the fluxes as the state variables instead of the currents, the analogy would have been complete. Applying the transformation (24), (28) to the original system (18) and expressing $R = \frac{r}{\varepsilon}$ we obtain the singularly perturbed system analogous to (12), (13)

$$\begin{aligned} \dot{y}_1 &= -\frac{2r}{L_1+L_2} y_1 + \frac{r}{L_1+L_2} y_2 + a_1 z_1 + a_2 z_2 \\ \dot{y}_2 &= \frac{r}{L_3+L_4} y_1 - \frac{r}{L_3+L_4} y_2 + a_3 z_1 + a_4 z_2 \\ \varepsilon \dot{z}_1 &= \varepsilon a_5 y_1 + \varepsilon a_6 y_2 - r\left(\frac{1}{L_1} + \frac{1}{L_2}\right) z_1 + \varepsilon a_7 z_2 \\ \varepsilon \dot{z}_2 &= \varepsilon a_8 y_1 + \varepsilon a_9 y_2 + \varepsilon a_{10} z_1 - r\left(\frac{1}{L_3} + \frac{1}{L_4}\right) z_2 \end{aligned} \tag{29}$$

where the coefficients $a_1, \ldots, a_{10}$ are of the form r divided by appropriate inductances. The first point we want to make is that the fast subsystem matrix of (29) is diagonally dominant. Its diagonal entries $-R\left(\frac{1}{L_1} + \frac{1}{L_2}\right)$, $-R\left(\frac{1}{L_3} + \frac{1}{L_4}\right)$ represent the subnetworks in Fig. 2c, as expected. The second important point is that, up to an ε-error, the slow subsystem represents the subnetwork in Fig. 2d. Thus, the subnetworks of the original network in Fig. 2a are the two fast subnetworks in Fig. 2c and one slow subnetwork in Fig. 2d. A simple rule is apparent: the fast subnetworks are obtained by considering the small resistors r as short circuits, while the slow subnetwork is obtained by considering the large resistors R as open circuits. It may come as a surprise that this simple rule yields an excellent approximation, as it can be seen from the comparison of the actual network eigenvalues with the subnetwork eigenvalues

$-\lambda_i$(network)	.08	.77	8.47	15.8
$-\lambda_i$(subnetworks)	.09	.78	8.33	15.0

The accuracy can be improved by including the ε-terms and still keeping the subnetworks decoupled. Let us remind the reader that we have examined the asymptotic behavior of (18) as $\varepsilon \to 0$ by introducing $r = \varepsilon R$.

For connoisseurs of singular perturbations it may be of interest to notice that, had we instead substituted $R = \frac{r}{\varepsilon}$ in (18), we would have obtained a singular singularly perturbed system. Following a different route, such as in Campbell (1979), we would have arrived at the same standard form (29).

A GROUPING ALGORITHM BASED ON "COHERENCY"

Although this may not be obvious, the problem we have been discussing is a disguised version of a well known problem in power system analysis. Most power engineers are familiar with the problem of grouping of synchronous machines into coherent areas. The literature on this subject is rich and will not be quoted here. Only a recently developed algorithm (Avramovic, 1980; Avramovic et al. 1980; Winkelman et al. 1980) will be outlined because of its direct connection with the preceding two sections. At present Avramovic's algorithm is being applied to conservative systems of the form

$$\ddot{x} = (A_0 + \varepsilon A_1)x = A_\varepsilon x \tag{30}$$

such as electromechanical models of power systems, mass-spring models of flexible structures etc. In power systems x represents the rotor angles.

The so called slow coherency problem is to find the groups of machines which "swing together" with respect to N slowest modes of A_ε. The machines i and j for which the difference $x_i - x_j$ contains a negligible contribution of slow modes, are grouped in the same coherent area. If V_ε is a basis n×N matrix for the selected N slow modes of A_ε, then the entries of the ith row of V_ε are the weights with which the modes appear in the state x_i. If the rows i and j are identical, the machines i and j are coherent, that is $x_i - x_j$ contains only fast modes. Avramovic's algorithm permutes the ordering of the states, that is the ordering of the rows of V_ε, until the first N rows become as linearly independent as possible. The N machines corresponding to these rows definitely are not coherent and each can be used as a reference machine for a distinct area. To associate other machines with these reference machines denote the first N rows of V_ε by V_1 and the remaining n-N rows by V_2. Now, when V_ε is postmultiplied by V_1^{-1} a matrix $L = V_2 V_1^{-1}$ appears

$$V_\varepsilon V_1^{-1} = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} V_1^{-1} = \begin{bmatrix} I \\ L \end{bmatrix} \tag{31}$$

which is the main tool of the grouping algorithm. The fast variables are then defined as

$$z = x^2 - Lx^1 \tag{32}$$

where x^1 is the subvector of the angles of all the N reference machines and x^2 of the remaining n-N machines. The slow coherence is approximately achieved if each row of L contains only one entry close to one, and all other entries close to zero.

By replacing the near-one entries by ones, and the near-zero entries by zeros, a "grouping matrix" L_g is formed. When used to replace L in (32) this matrix L_g defines the components of x^2 whose differences with the components of x^1 in (32) are predominantly fast. They then belong to the area whose reference is the component of x^1 appearing in the difference.

If we were to apply Avramovic's algorithm to our network in Fig. 2a, we would find that x_2 and x_4 can be used as "reference currents." They will be the components of x^1 and x_1 and x_3 the components of x^2. Then (32) with L_g replacing L, would become

$$z = \begin{bmatrix} x_1 \\ x_3 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_2 \\ x_4 \end{bmatrix} \tag{33}$$

which is in agreement with (24).

What distinguishes Avramovic's algorithm from the discussion in the preceding sections, is that it eliminates the need for an explicit separation of A_ϵ into A_0 and ϵA_1. Instead of a grouping based on the null-space of a heuristically or empirically constructed matrix A_0, this algorithm computes a basis V_ϵ for the actual slow eigenspace of A_ϵ and then finds $L = V_2 V_1^{-1}$ by a Gaussian elimination procedure. For our system (18) a basis V_ϵ for $\lambda_1 = -.08$ and $\lambda_2 = -.77$ is

$$V_\epsilon = \begin{bmatrix} 0.33 & 0.66 \\ 0.36 & 0.67 \\ 0.61 & 0.21 \\ 0.62 & 0.25 \end{bmatrix} \tag{34}$$

which, after a permutation to place the rows 2 and 4 as the first two rows, gives

$$V_1 = \begin{bmatrix} 0.36 & 0.67 \\ 0.62 & 0.25 \end{bmatrix}, V_2 = \begin{bmatrix} 0.33 & 0.66 \\ 0.61 & 0.21 \end{bmatrix}, V_2 V_1^{-1} = \begin{bmatrix} 1.007 & 0.05 \\ -0.07 & 1.02 \end{bmatrix} \tag{35}$$

The final step is to approximate $V_2 V_1^{-1}$ by the grouping matrix in (33).

A common feature of our two examples and Avramovic's algorithm is a search for a set of physically meaningful state variables to exhibit the subsystems and their time scales. Of course, the time scales of linear time invariant systems can be exhibited by modal transformations, but the meaning of the modal variables may be far from the meaning of the original state variables. The original state variables obtained from physical laws or experiments contain a wealth of intuitive and empirical information which a rational modeler wants to preserve. Our examples illustrate how this can be accomplished by a constructive use of the decompositions such as (7) and (21), which to an algebraist are mere projections. With physically meaningful subsystems not only linearized, but also nonlinear analysis will be simplified. In a recent application (Winkelman et al. 1980) of Avramovic's algorithm to a nonlinear 48 machine model of the Eastern United States power system, nonlinear analogs of the subsystems identified on a linearized model were used. Simulation results confirmed the validity of the subsystem models and time scales.

MULTIMODELING

Thus far we have not considered the presence of control inputs. How are they to be allocated among the subsystems? Political, geographic, and other issues may interfere with our strictly dynamic criteria. A transformation of an original model into a model exhibiting time scales and subsystems will also transform the control matrix B. When transformed, at least one control variable should be allocated to each fast subsystem while the presence of other fast subsystem controls should be weak. If this is not the case with the original control variables, voluntary grouping of controls and mutual release of control authority is required. After this has been accomplished, the linearized model of a large scale system exhibiting one slow and N fast subsystems can be written as

$$\dot{y} = A_c y + \sum_{j=1}^{N} A_{cj} z_j + \sum_{j=1}^{N} B_{cj} u_j \tag{36}$$

$$\epsilon_i \dot{z}_i = A_{ic} y + A_{ii} z_i + \Sigma\, \epsilon_{ij} A_{ij} z_j + B_{ii} u_i \tag{37}$$

where we have allowed each fast subsystem to have a different small parameter ϵ_i and to be weakly coupled to other fast subsystems through ϵ_{ij}. The fast subsystem i is controlled by its own input u_i. The slow subsystem (36) is the common slow "core" and, in general, will have the input from several or all fast subsystem controls.

In a situation like this it is rational for a fast subsystem controller to neglect all other fast subsystems and to concentrate on its own subsystem, plus, of course, the slow interaction with others through the "core." For the ith controller "to neglect all other subsystems" simply means to set all ϵ parameters equal to zero except for ϵ_i which is to be kept at its true value. The ith controller's simplified model is then

$$\dot{y}^i = A_i y^i + A_{ci} z_i + B_{ci} u_i + \sum_{j \neq i} B_{ij} u_j \tag{38}$$

$$\epsilon_i \dot{z}_i = A_{ic} y^i + A_{ii} z_i + B_{ii} u_i \tag{39}$$

where

$$A_i = A_c - \sum_{j \neq i} A_{cj} A_{jj}^{-1} A_{jc}, \quad B_{ij} = B_{cj} - A_{cj} A_{jj}^{-1} B_{jj}. \tag{40}$$

We denote y^i with a superscript rather than a subscript to stress the fact that y^i is not a component of y, but the ith controller's view of y. In reality, the model (38),(39) is often all the ith controller knows about the whole system. The kth controller, on the other hand, has a different kth model of the same large scale system. This situation, called multimodeling, has been formulated and investigated in Khalil and Kokotovic (1978, 1979a, 1979b, 1980) and, more broadly, in Volume II of the U.S. Department of Energy report (Fink and Trygar, 1979).

What remains of (36),(37) when all ϵ parameters are neglected is the slow core, which is , in general, a strongly coupled subsystem. Decentralized, team, and game approaches to the design of control strategies have been considered for this subsystem. Control u_i can be divided into a slow part, which contributes to the control of the core, and a fast part controlling only its own fast subsystem. Sometimes the total authority for the slow subsystem can be delegated to a single controller-coordinator. In this case the control hierarchy would naturally match the hierarchy of the time scales.

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Singular Perturbation Analysis of Systems with Sustained High Frequency Oscillations*†

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Singular perturbation techniques, extended to treat systems with slightly damped high frequency modes, provide better understanding of the system's structural properties, and they yield computational advantages since the resulting subsystems are analyzed in separate time scales.

Key Word Index—Perturbation techniques; eigenvalues; two time scales; system order reduction; approximation theory; modeling; power system control.

Summary—Using singular perturbation techniques, a system with high frequency oscillations is decomposed into two lower order subsystems, one containing only the slowly varying part and the other containing only the fast oscillatory part. Eigenvalue and state approximations achieved by the subsystems are given. A mass-spring-damper example shows that a stiff spring can be regarded as a perturbation of a rigid rod and an interconnected power system example illustrates the occurrence of coherency and inter-machine oscillations.

1. INTRODUCTION

MECHANICAL and electromechanical systems often have slightly damped modes oscillating at frequencies much higher than the rest of the system. Well known examples are spring-mass suspension systems and multi-machine power systems. In linearized models of such systems some eigenvalues have small real parts and large imaginary parts. Typically they are due to either strong coupling, or small masses and inertias, or both. Synchronous machines connected through a low impedance can serve as an illustration.

In properly designed systems the amplitudes of high frequency oscillations are small and their effect negligible. However, the analysis and design methods must take these potentially troublesome modes into account. This leads to numerically stiff problems requiring expensive integration routines. A way out of this difficulty is to treat systems with oscillatory modes as singularly perturbed systems and analyze their slow and fast parts in different time scales. Presently available singular perturbation methods[1] assume that the fast modes decay in the fast time scale during a boundary layer interval. Thus they do not incorporate the case of slightly damped or purely oscillatory modes. This paper extends the singular perturbation approach to systems with fast oscillatory modes.

Our approach is to decompose a system with high frequency oscillations into two separate subsystems, one containing the slowly varying dynamics and the other containing the oscillatory modes. We show that the decomposition in [2, 3] is also applicable to systems whose slightly damped large eigenvalues result in sustained high frequency oscillations. The slowly varying dynamics can be approximately analyzed by averaging methods[4–6]. However for the linear time-invariant case considered here, our algebraic decomposition is more direct and yields estimates of the eigenvalues and states of the original high frequency oscillatory system. This procedure requires only the verification of an assumption given in the next section and the computation of a matrix inverse. Furthermore our decomposition retains the meaning of the physical variables.

Illustrating the decomposition procedure by a simple mass-spring-damper system in which one of the springs is stiff, we show that the slow motion of the masses can be obtained by approximating the stiff spring as a rigid rod. The high frequency oscillations between the masses are then analyzed using a fast time scale. An interesting application of this procedure is in the transient stability studies of interconnected power systems. If several machines belong to a 'coherent' group, they are usually represented by an 'equivalent' machine[7–9]. Our procedure gives a perturbational interpretation of the coherency approach. Moreover, it reintroduces the intermachine high frequency oscillations by representing them separately by an oscillatory subsystem. Hence this procedure is applicable when the intermachine oscillations are not negligible.

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The organization of the paper is as follows. Section 2 outlines the modeling aspect of high frequency oscillatory systems. The technique of averaging is used in Section 3 to obtain the slowly varying part of the oscillatory states. Section 4 contains our main results on eigenvalue and state approximations of the subsystems. Sections 5 and 6 are devoted to the examples.

2. MODELING OF SYSTEMS WITH HIGH FREQUENCY OSCILLATIONS

Systems governed by physical laws such as Newton's law and Kirchhoff's law can be modeled as second order matrix differential equations

$$\ddot{s} + P\dot{s} + Qs = 0, \quad \dot{s}(t_0) = \dot{s}_0, \quad s(t_0) = s_0 \qquad (1)$$

where $s \in R^r$ and P, Q are $r \times r$ matrices. We assume that system (1) is in the form

$$s = \begin{bmatrix} s_1 \\ s_2 \end{bmatrix}, \quad P = \begin{bmatrix} P_1 & P_2 \\ P_3 & P_4 \end{bmatrix}, \quad Q = \begin{bmatrix} Q_1 & Q_2/\mu^2 \\ Q_3 & Q_4/\mu^2 \end{bmatrix} \qquad (2)$$

where μ is a small positive parameter which arises due to the presence of stiff springs or small masses and is responsible for the high frequency oscillations in (1). Then (1), (2) can be rewritten as a singularly perturbed system of first order differential equations

$$\dot{x} = Ax + Bz, \quad x(t_0) = x_0 \qquad (3a)$$

$$\mu\dot{z} = Cx + Dz, \quad z(t_0) = z_0 \qquad (3b)$$

where

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} s_1 \\ \dot{s}_1 \end{bmatrix}, \quad z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} s_2/\mu^2 \\ \dot{s}_2/\mu \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & I \\ -Q_1 & -P_1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ -Q_2 & -\mu P_2 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 0 \\ -Q_3 & -P_3 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & I \\ -Q_4 & -\mu P_4 \end{bmatrix}. \qquad (4)^*$$

Our analysis of (3) does not require the matrices A, B, C, D to be in the special form (4). The only assumptions that system (3) has to satisfy are the following:

I. The norms of A, B, C, D are bounded about $\mu = 0$ and the state z is of even dimension, that is, $z \in R^{2m}$.

II. The matrix D is in the form

$$D = \begin{bmatrix} \mu D_1 & D_2 \\ D_3 & \mu D_4 \end{bmatrix} \qquad (5)$$

where D_2, D_3 are $m \times m$ nonsingular matrices and the matrix $D_2 D_3$ has simple and negative eigenvalues $-\omega_i^2$, $i = 1, 2, \ldots, m$.

There is no restriction on the dimension n of the state $x \in R^n$. Assumption II guarantees that high frequency oscillations will occur in (3).

As an example of a system in the form of (3), we consider a mass-spring-damper system shown in Fig. 1 where the spring k_2 is stiff. A set of convenient

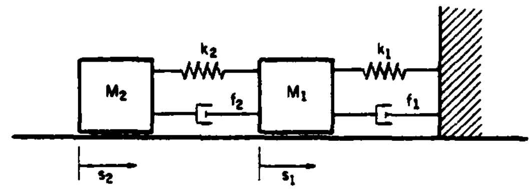

FIG. 1. A mass-spring-damper system.

state variables for this system is the position of the center of mass

$$s_c = (M_1 s_1 + M_2 s_2)/M, \quad M = M_1 + M_2 \qquad (6)$$

and the relative displacement between the masses

$$s_d = s_1 - s_2 \qquad (7)$$

where s_1, s_2 are the positions of the masses M_1, M_2. The equation of motion for this system is

$$\ddot{s}_c + \frac{f_1}{M}\dot{s}_c + \frac{f_1 M_2}{M^2}\dot{s}_d + \frac{k_1}{M}s_c + \frac{k_1 M_2}{M^2}s_d = 0$$

$$s_c(t_0) = s_{co}, \quad \dot{s}_c(t_0) = v_c(t_0) = v_{co}$$

$$\ddot{s}_d + \frac{f_1}{M_1}\dot{s}_c + \left(\frac{f_1 M_2}{M_1 M} + \frac{f_2 M}{M_1 M_2}\right)\dot{s}_d$$

$$+ \frac{k_1}{M_1}s_c + \frac{k_2 M}{M_1 M_2}\left(1 + \frac{k_1 M_2^2}{k_2 M^2}\right)s_d = 0$$

$$s_d(t_0) = s_{do}, \quad \dot{s}_d(t_0) = v_d(t_0) = v_{do}. \qquad (8)$$

Since the spring k_2 is stiff, we define

$$\frac{1}{\mu^2} = \frac{k_2 M}{M_1 M_2} \qquad (9)$$

such that μ is small. In the state variables

$$x_1 = s_c, \quad x_2 = \dot{s}_c = v_c, \quad z_1 = s_d/\mu^2,$$

$$z_2 = \dot{s}_d/\mu = v_d/\mu \qquad (10)$$

*The matrix I denotes an identity matrix of an appropriate dimension.

(8) becomes

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -\frac{k_1}{M}x_1 - \frac{f_1}{M}x_2 - \mu^2\frac{k_1 M_2}{M^2}z_1 - \mu\frac{f_1 M_2}{M^2}z_2$$

$$\mu\dot{z}_1 = z_2$$

$$\mu\dot{z}_2 = -\frac{k_1}{M_1}x_1 - \frac{f_1}{M_1}x_2 - \left(1+\mu^2\frac{k_1 M_2}{MM_1}\right)z_1 - \mu\left(\frac{f_1 M_2}{M_1 M} + \frac{f_2 M}{M_1 M_2}\right)z_2 \qquad (11)$$

which is in the form (3) and satisfies Assumptions I and II with

$$\omega_1^2 = 1 + \mu^2\frac{k_1 M_2}{M_1 M}.$$

3. AVERAGING OF OSCILLATORY STATES

Before analyzing (3), we investigate the behavior of the system

$$\mu\dot{w} = Dw + u \qquad (12)$$

where D satisfies Assumption II. The characteristic polynomial of D/μ is

$$\varphi(\lambda) = \det\begin{bmatrix}\lambda I - D_1 & -D_2/\mu \\ -D_3/\mu & \lambda I - D_4\end{bmatrix}$$

$$= \det\begin{bmatrix}0 & -D_2/\mu + \mu(\lambda I - D_4)D_3^{-1}(\lambda I - D_1) \\ -D_3/\mu & \lambda I - D_4\end{bmatrix}$$

$$= -(-1)^m \det[\lambda^2 I - \lambda(D_4 + D_3^{-1}D_1 D_3) - (D_2 D_3 - \mu^2 D_4 D_3^{-1} D_1 D_3)/\mu^2]. \qquad (13)$$

Let T diagonalize $D_2 D_3$ such that

$$TD_2 D_3 T^{-1} = \Lambda = \mathrm{diag}(-\omega_1^2, -\omega_2^2, \ldots, -\omega_m^2) \qquad (14)$$

and rewrite the characteristic polynomial as

$$\varphi(\lambda) = -(-1)^m \det[\lambda^2 I - R_1\lambda - (\Lambda + \mu^2 R_2)/\mu^2] \qquad (15)$$

where

$$R_1 = T(D_4 + D_3^{-1}D_1 D_3)T^{-1}$$

and

$$R_2 = TD_4 D_3^{-1} D_1 D_3 T^{-1}.$$

Expanding the determinant in (15), it is readily shown that the coefficients of λ^i, $i = 0, 1, \ldots, 2m$, are of the form $a_i(\mu)/\mu^{2m-i}$ for i even and $a_i(\mu)/\mu^{2m-i-1}$ for i odd, where $a_i(\mu)$ is bounded about $\mu = 0$. Neglecting $\mu^2 R_2$ and the off-diagonal elements in R_1, (15) becomes

$$\tilde{\varphi}(\lambda) = -(-1)^m \prod_{i=1}^{m}(\lambda^2 - 2\sigma_i\lambda + \omega_i^2/\mu^2) \qquad (16)$$

where $2\sigma_i$ is the ith diagonal element of R_1. The coefficients of λ^i of (16) are also of the form $b_i(\mu)/\mu^{2m-i}$ for i even and $b_i(\mu)/\mu^{2m-i-1}$ for i odd, and furthermore, $b_i(\mu)$ approximates $a_i(\mu)$ to $O(\mu^2)$. Instead of (13) $\varphi(\lambda)$ can be expressed as

$$\varphi(\lambda) = -(-1)^m \det[\lambda^2 I - \lambda(D_1 + D_2^{-1}D_4 D_2) - (D_3 D_2 - \mu^2 D_1 D_2^{-1} D_4 D_2)/\mu^2]. \qquad (17)$$

Letting $S = \Gamma T D_3^{-1}$ where Γ is any nonsingular diagonal matrix, we obtain $SD_3 D_2 S^{-1} = \Lambda$. Then the diagonal elements of

$$S(D_1 + D_2^{-1}D_4 D_2)S^{-1}$$

are identical to those of R_1 and (17) can also be approximated by (16). To analyze the roots of $\varphi(\lambda)$ we use the following lemma.

Lemma 1. If D satisfies Assumption II, then, as $\mu \to 0^+$, the eigenvalues of D/μ approach infinity as

$$\sigma_i \pm j\omega_i/\mu, \quad i = 1, 2, \ldots, m. \qquad (18)$$

By Lemma 1, as $\mu \to 0^+$, the eigenvalues of (12) approach infinity along asymptotes parallel to the imaginary axis. Note that the large imaginary parts of (18) are the consequence of solving for λ of the quadratic equations in (16). If some of the eigenvalues of $D_2 D_3$ are either positive or not simple, then in general some of the eigenvalues of D/μ may be positive and $O(1/\mu)$. This case of fast instability is less realistic and will not be considered here.

Due to the eigenvalues with large imaginary parts, the response $w(t)$ of (12) will in general consist of high frequency oscillations superimposed on slowly varying dynamics. Our purpose is to compute this slowly varying response due to the input $u(t)$.

Lemma 2. If D satisfies Assumption II and if $u(t) = \bar{u}(t) + \tilde{u}(t)$ is an input where $\bar{u}(t)$ is the slowly varying part with $|\bar{u}| \leq c_1$ and $|\dot{\bar{u}}| \leq c_2$ for some fixed c_1 and c_2 and $\tilde{u}(t)$ is the oscillatory part, then there exists a finite $T(\mu)$ such that the slowly varying part $\bar{w}(t)$ of $w(t)$ of (12) for $t_0 \leq t \leq T$ is

$$\bar{w}(t) = -\begin{bmatrix}0 & D_3^{-1} \\ D_2^{-1} & 0\end{bmatrix}\bar{u}(t) + O(\mu). \qquad (19)$$

Proof. Integrating the variation of constants formula

$$w(t)=\Phi(t,t_0)w(t_0)+\frac{1}{\mu}\int_{t_0}^{t}\Phi(t,\tau)u(\tau)\,d\tau \quad (20)$$

where

$$\Phi(t,\tau)=\exp\{D(t-\tau)/\mu\},$$

by parts, we obtain

$$\begin{aligned} w(t)= & -D^{-1}\bar{u}(t)+\Phi(t,t_0)w(t_0) \\ & +D^{-1}\Phi(t,t_0)\bar{u}(t_0) \\ & +D^{-1}\int_{t_0}^{t}\Phi(t,\tau)\dot{\bar{u}}(\tau)\,d\tau \\ & +\frac{1}{\mu}\int_{t_0}^{t}\Phi(t,\tau)\tilde{u}(\tau)\,d\tau. \end{aligned} \quad (21)$$

But the first integral term in (21) is $O(\mu)$ since a further integration by parts reveals that for $t_0 \leqq t \leqq T$,

$$\begin{aligned} \left|\int_{t_0}^{t}\Phi(t,\tau)\dot{\bar{u}}(\tau)\,d\tau\right| \leqq \mu |D^{-1}| \{ & c_1(1+|\Phi(t,t_0)|) \\ & +c_2\int_{t_0}^{t}|\Phi(t,\tau)|\,d\tau\}. \end{aligned} \quad (22)$$

We also note that $\tilde{u}(\tau)$ generates high frequency terms and the terms contributed by $\Phi(t,t_0)$ are approximately of the type

$$\exp\{\sigma_i(t-t_0)\}\sin(\omega_i(t-t_0)/\mu)$$

and

$$\exp\{\sigma_i(t-t_0)\}\cos(\omega_i(t-t_0)/\mu), \quad i=1,2,\ldots,m.$$

Since

$$D^{-1}=\begin{bmatrix} \mu X_1 & D_3^{-1}+\mu^2 X_2 \\ D_2^{-1}+\mu^2 X_3 & \mu X_4 \end{bmatrix} \quad (23)$$

where

$$\begin{aligned} X_1 &= -(D_3-\mu^2 D_4 D_2^{-1} D_1)^{-1} D_4 D_2^{-1} \\ X_2 &= -D_3^{-1} D_4 X_4 \\ X_3 &= -D_2^{-1} D_1 X_1 \\ X_4 &= -(D_2-\mu^2 D_1 D_3^{-1} D_4)^{-1} D_1 D_3^{-1}, \end{aligned} \quad (24)$$

the only significant slowly varying part $-D^{-1}\bar{u}(t)$ of $w(t)$ in (12) is approximated to $O(\mu)$ by $-\hat{D}^{-1}\bar{u}(t)$, where

$$\hat{D}=\begin{bmatrix} 0 & D_2 \\ D_3 & 0 \end{bmatrix} \quad (25)$$

implying (19).

This analysis justifies a simple method to obtain $\bar{w}(t)$, which is to set $\mu=0$ in (12), as is usually done in singular perturbations. However, the meaning of setting $\mu=0$ here is different. Considering $u=\bar{u}$ as the input and w as the output, the input-output behavior of system (12) is that of a lowpass wideband filter. Then $\bar{w}(t)$ is the dominant part of the filter output which shows the relationship with the usual assumption in the technique of averaging [4–6]. Thus $\bar{w}(t)$ approximates $w(t)$ closely if the high frequency component of w is negligible or if $w(t)$ is used as an input to a slow filter.

4. EIGENVALUE AND STATE APPROXIMATIONS

Letting $\bar{x}$ be the slowly varying part of x and either applying Lemma 2 to (3b) or setting $\mu=0$, we obtain the slowly varying part $\bar{z}$ of z as

$$\begin{aligned} \bar{z} &= -D^{-1}C\bar{x}+O(\mu) \\ &= -\hat{D}^{-1}C\bar{x}+O(\mu). \end{aligned} \quad (26)$$

To separate completely the slowly varying part $\bar{z}$ from z, we introduce the change of variables

$$\eta=z+D^{-1}Cx+\mu Gx\equiv z+Lx \quad (27)$$

and determine G such that (3) is transformed into

$$\dot{x}=(A_0-\mu BG)x+B\eta \quad (28a)$$

$$\mu\dot{\eta}=(D+\mu LB)\eta \quad (28b)$$

where

$$A_0=A-BD^{-1}C. \quad (29)$$

Thus G is required to satisfy

$$-DG+(D^{-1}C+\mu G)(A_0-\mu BG)=0. \quad (30)$$

By the implicit function theorem, the solution of (30) is

$$\begin{aligned} G &= D^{-2}CA_0+O(\mu) \\ &= \hat{D}^{-2}C\hat{A}_0+O(\mu) \end{aligned} \quad (31)$$

where

$$\hat{A}_0=A-B\hat{D}^{-1}C. \quad (32)$$

Let

$$\begin{aligned} D+\mu LB &= \begin{bmatrix} \mu D_1 & D_2 \\ D_3 & \mu D_4 \end{bmatrix} + \mu \begin{bmatrix} \mu D_1 & D_2 \\ D_3 & \mu D_4 \end{bmatrix}^{-1} CB + O(\mu^2) \\ &= \begin{bmatrix} \mu(D_1+D_3^{-1}C_2B_1) & D_2+\mu D_3^{-1}C_2B_2 \\ D_3+\mu D_2^{-1}C_1B_1 & \mu(D_4+D_2^{-1}C_1B_2) \end{bmatrix} + O(\mu^2) \\ &\equiv \begin{bmatrix} \mu\hat{D}_1 & \hat{D}_2 \\ \hat{D}_3 & \mu\hat{D}_4 \end{bmatrix} + O(\mu^2) \equiv \tilde{D}+O(\mu^2) \end{aligned} \quad (33)$$

where

$$B=[B_1 \quad B_2], \quad C=\begin{bmatrix} C_1 \\ C_2 \end{bmatrix}. \quad (34)$$

Then the upper block triangular form (28) exhibits the eigenvalues of (3).

Lemma 3. If Assumptions I and II are satisfied, then the eigenvalues of $\bar{A}_0$ and $\tilde{D}/\mu$ are an $O(\mu)$ approximation to the eigenvalues of the original system (3). Furthermore as $\mu \to 0^+$, the eigenvalues of $\tilde{D}/\mu$ approach infinity as

$$\rho_i \pm j\omega_i/\mu, \quad i = 1, 2, \ldots, m \tag{35}$$

where $2\rho_i$ is the ith diagonal element of the matrix

$$T(\tilde{D}_4 + D_3^{-1}\tilde{D}_1 D_3)T^{-1}.$$

The second statement of Lemma 3 follows from Lemma 1. The meaning of Lemma 3 is that n eigenvalues of system (3) are small. They are responsible for the slowly varying dynamics of the system. The large imaginary parts of the other $2m$ eigenvalues are responsible for the high frequency oscillations while the real parts modulate the envelope of these high frequency oscillations.

The approximation in Lemma 3 is purely algebraic and does not require the eigenvalues of system (3) to be stable. However, it can be used to guarantee the stability of system (3) as the following observation shows.

Corollary 1. Under the assumptions of Lemma 3, if $\bar{A}_0$ is Hurwitz and ρ_i, $i = 1, 2, \ldots, m$, are negative, then there exists a $\mu^* > 0$ such that system (3) is asymptotically stable for all $\mu \in (0, \mu^*]$.

This corollary is of interest when feedback control is implemented in system (3) and can be used to separately stabilize the slowly varying and the fast oscillatory subsystems. Such control laws can be designed using an extension of the methodology described in [10], as it will be explored in a forthcoming paper.

To separate the slowly varying part in x, we introduce

$$\xi = x - \mu(CD^{-1} + \mu N)\eta \equiv x - \mu H\eta \tag{36}$$

and choose N such that

$$B + \mu(A_0 - \mu BG)H - H(D + \mu LB) = 0. \tag{37}$$

By the implicit function theorem,

$$\begin{aligned} N &= A_0 BD^{-2} - BD^{-2}CBD^{-1} + O(\mu) \\ &= \bar{A}_0 B\tilde{D}^{-2} - B\tilde{D}^{-2}CB\tilde{D}^{-1} + O(\mu). \end{aligned} \tag{38}$$

This completes the transformation (27), (36) which becomes

$$\begin{bmatrix} \xi \\ \eta \end{bmatrix} = \begin{bmatrix} I - \mu HL & -\mu H \\ L & I \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} \tag{39}$$

and its inverse is

$$\begin{bmatrix} x \\ z \end{bmatrix} = \begin{bmatrix} I & \mu H \\ -L & I - \mu LH \end{bmatrix} \begin{bmatrix} \xi \\ \eta \end{bmatrix}. \tag{40}$$

The original system (3) rewritten in the state variables ξ, η is completely decomposed into the fast and slow subsystems

$$\dot{\xi} = \mathscr{A}\xi \tag{41}$$

$$\mu\dot{\eta} = \mathscr{D}\eta \tag{42}$$

where $\mathscr{A} = A_0 - \mu BG$, $\mathscr{D} = D + \mu LB$.

The decomposition (39), (40) is an exact block diagonalization transformation. Neglecting the $O(\mu)$ term in (41), we define the slowly varying subsystem of (3) as

$$\dot{\bar{x}} = \bar{A}_0\bar{x}, \quad \bar{x}(t_0) = x_0 \tag{43a}$$

$$\bar{z} = -\tilde{D}^{-1}C\bar{x}. \tag{43b}$$

The oscillatory subsystem

$$\mu\dot{\tilde{z}} = \tilde{D}\tilde{z}, \quad \tilde{z}(t_0) = z_0 + \tilde{D}^{-1}Cx_0 \tag{44}$$

is obtained from (42) by neglecting the $O(\mu^2)$ terms in $\mathscr{D}$.

The state approximations achieved by the subsystems (43), (44) are stated as follows.

Theorem 1. If the original system (3) satisfies Assumptions I and II, then there exists a finite $T(\mu)$ such that the states of (3) are approximated to $O(\mu)$ by the subsystems (43), (44) for $t_0 \leqq t \leqq T$, that is,

$$x(t) = \bar{x}(t) + O(\mu) \tag{45a}$$

$$z(t) = \bar{z}(t) + \tilde{z}(t) + O(\mu). \tag{45b}$$

The result of Theorem 1 implies that if the initial condition $|\tilde{z}(t_0)|$ is much smaller than $|\bar{x}(t_0)|$, then the high frequency oscillation can be neglected and the original system (3) is adequately modeled by its lower order slowly varying subsystem (43). Furthermore the subsystems (43), (44) can be used to simulate approximately the actual response of (3). Due to the presence of μ, the ill-conditioned $(n + 2m)$th order system (3) requires a prohibitively small integration stepsize. However, using the lower order subsystems, the small integration stepsize is necessary only for the $2m$th order fast oscillatory subsystem (44), while the integration of the slowly varying subsystem (43) can be computed with a much larger stepsize, resulting in savings of computing time. In the case when the high frequency oscillations are negligible, only the integration of the slowly varying subsystem is required.

5. MASS-SPRING-DAMPER SYSTEM

We illustrate the subsystem decomposition procedure with the mass-spring-damper system (11). Neglecting the μ terms, the slowly varying subsystem (43) of (11) in the original state variables is

$$
\begin{aligned}
\dot{\bar{s}}_c &= \bar{v}_c, \quad \bar{s}_c(t_0) = s_{co} \\
\dot{\bar{v}}_c &= -\frac{k_1}{M}\bar{s}_c - \frac{f_1}{M}\bar{v}_c, \quad \bar{v}_c(t_0) = v_{co} \\
\bar{s}_d &= 0 \\
\bar{v}_d &= 0.
\end{aligned} \tag{46}
$$

Subsystem (46) represents the motion of the center of mass as if M_1 and M_2 are connected by a rigid rod and are moving together. Intuitively this can be explained by assuming that the spring restoring force $k_2 s_d$ remains finite in the limit as $k_2 \to \infty$. The displacement s_d becomes negligible, that is the spring becomes a rigid rod.

To reintroduce the high frequency oscillations due to the fact that k_2 is finite, we consider the fast oscillatory subsystem (44)

$$
\begin{aligned}
\mu\dot{\tilde{z}}_1 &= \tilde{z}_2, & \tilde{z}_1(t_0) &= s_{do}/\mu^2 \\
\mu\dot{\tilde{z}}_2 &= -\tilde{z}_1 - \mu\left(\frac{f_1 M_2}{M_1 M} + \frac{f_2 M}{M_1 M_2}\right)\tilde{z}_2, & \tilde{z}_2(t_0) &= v_{do}/\mu.
\end{aligned} \tag{47}
$$

Since the spring k_2 is stiff, the initial displacement s_{do} is small. In the spring and rod analogy, the rod is now allowed to be slightly elastic. Assuming that forces are finite, $z_1 = s_d/\mu^2$ is not large and is actually properly scaled. The same property holds for $z_2 = v_d/\mu$ as $|s_d| = \mu|v_d|$ due to the high frequency oscillations in s_d. We rewrite (47) in the original variable $\tilde{s}_d = \mu^2\tilde{z}_1$ as a second order equation

$$
\ddot{\tilde{s}}_d + \left(\frac{f_1 M_2}{M_1 M} + \frac{f_2 M}{M_1 M_2}\right)\dot{\tilde{s}}_d + \frac{1}{\mu^2}\tilde{s}_d = 0 \tag{48}
$$

that is,

$$
\frac{M_1 M_2}{M}\ddot{\tilde{s}}_d + \left(f_2 + \frac{f_1 M_2^2}{M^2}\right)\dot{\tilde{s}}_d + k_2\tilde{s}_d = 0. \tag{49}
$$

Equation (49) describes the motion of the masses M_1 and M_2 connected by a spring k_2 and a damper $f_2 + f_1 M_2^2/M^2$. Thus our decomposition procedure shows that in analyzing the high frequency modes, the spring k_1 can be neglected while the damper f_1 is reflected through the connections and increases the effective damping.

Thus concluding from Theorem 1, if the initial conditions s_{do} and v_{do} are of $O(\mu)$, we obtain

$$
\begin{aligned}
s_c &= \bar{s}_c + O(\mu) & v_c &= \bar{v}_c + O(\mu) \\
s_d &= O(\mu), & v_d &= O(\mu).
\end{aligned} \tag{50}
$$

6. POWER SYSTEM EXAMPLE

A potentially important field for the application of this methodology is in power systems. In transient stability studies of interconnected power systems[8, 9], coherent machines* are usually modeled as a single unit to reduce the dimension of the problem. We now interpret this coherency idea by applying our decomposition procedure to a three machine system shown in Fig. 2.

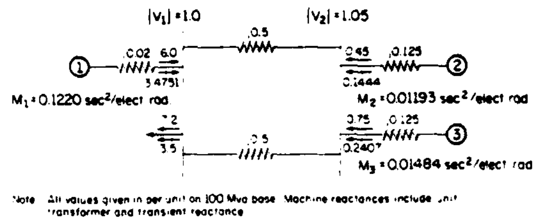

FIG. 2. Three machine power system.

The opening of one transmission line from bus 1 to bus 2 causes the system to oscillate. The following post-disturbance differential equations for the machine rotor angles may be written[11]

$$
\begin{aligned}
M_1\ddot{\delta}_1 &= P_{in_1} - V_1'^2 Y_{11}\cos\theta_{11} - V_1' V_2' Y_{12} \\
&\quad \times \cos(\theta_{12} + \delta_2 - \delta_1) \\
&\quad - V_1' V_3' Y_{13}\cos(\theta_{13} + \delta_3 - \delta_1) \\
M_2\ddot{\delta}_2 &= P_{in_2} - V_2'^2 Y_{22}\cos\theta_{22} - V_1' V_2' Y_{12} \\
&\quad \times \cos(\theta_{12} + \delta_1 - \delta_2) \\
&\quad - V_2' V_3' Y_{23}\cos(\theta_{23} + \delta_3 - \delta_2) \\
M_3\ddot{\delta}_3 &= P_{in_3} - V_3'^2 Y_{33}\cos\theta_{33} - V_1' V_3' Y_{13} \\
&\quad \times \cos(\theta_{13} + \delta_1 - \delta_3) \\
&\quad - V_2' V_3' Y_{23}\cos(\theta_{23} + \delta_2 - \delta_3).
\end{aligned} \tag{51}
$$

The notation for this and other equations of this example is given in the Appendix.

If Y_{23} is large compared to Y_{12} and Y_{13}, then machines 2 and 3 will be strongly coupled. In this case it is convenient to rewrite (51) in terms of the variables

$$
\delta_{c1} = \frac{M_2\delta_2 + M_3\delta_3}{M_2 + M_3} - \delta_1, \quad \delta_{23} = \delta_2 - \delta_3 \tag{52}
$$

---

*Two machines are defined to be coherent if the difference between their angles is sufficiently small[8, 9].

as a fourth order system

$$\ddot{\delta}_{c1} = \frac{P_{in_2} + P_{in_3} - V_2'^2 Y_{22}\cos\theta_{22} - V_3'^2 Y_{33}\cos\theta_{33}}{M}$$
$$-\frac{P_{in_1} - V_1'^2 Y_{11}\cos\theta_{11}}{M_1}$$
$$-\frac{V_1'V_2'Y_{12}}{M_{12}}\cos\left(\Psi_{12} - \frac{M_3}{M}\delta_{23} - \delta_{c1}\right)$$
$$-\frac{V_1'V_3'Y_{13}}{M_{13}}\cos\left(\Psi_{13} + \frac{M_2}{M}\delta_{23} - \delta_{c1}\right)$$
$$-\frac{2V_2'V_3'}{M} Y_{23}\cos\theta_{23}\cos\delta_{23} \qquad (53)$$

$$\ddot{\delta}_{23} = \frac{P_{in_2} - V_2'^2 Y_{22}\cos\theta_{22}}{M_2} - \frac{P_{in_3} - V_3'^2 Y_{33}\cos\theta_{33}}{M_3}$$
$$-\frac{V_1'V_2'Y_{12}}{M_2}\cos\left(\theta_{12} - \delta_{c1} - \frac{M_3}{M}\delta_{23}\right)$$
$$+\frac{V_1'V_3'Y_{13}}{M_3}\cos\left(\theta_{13} - \delta_{c1} + \frac{M_2}{M}\delta_{23}\right)$$
$$-\frac{V_2'V_3'Y_{23}}{M_{23}}\cos(\Psi_{23} - \delta_{23})$$

where δ_1 is used as the reference.

In order to apply the decomposition procedure, we linearize (53) about the equilibrium point δ_{c1}^o and δ_{23}^o. The linearization yields the following differential equations for the perturbations $\Delta\delta_{c1}$ and $\Delta\delta_{23}$

$$\begin{aligned} \Delta\ddot{\delta}_{c1} &= -a_{11}\Delta\delta_{c1} + a_{12}\Delta\delta_{23} \\ \Delta\ddot{\delta}_{23} &= a_{21}\Delta\delta_{c1} - a_{22}\Delta\delta_{23}. \end{aligned} \qquad (54)$$

In the case of strong coupling between machines 2 and 3, a_{22} is much larger than a_{11}, a_{12} and a_{21}. Hence, let

$$\frac{1}{\mu^2} = a_{22}. \qquad (55)$$

Defining

$$x_1 = \Delta\delta_{c1}, x_2 = \Delta\dot{\delta}_{c1}, z_1 = \Delta\delta_{23}/\mu^2, z_2 = \Delta\dot{\delta}_{23}/\mu \qquad (56)$$

(54) becomes

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -a_{11}x_1 + \mu^2 a_{12} z_1 \\ \mu\dot{z}_1 &= z_2 \\ \mu\dot{z}_2 &= a_{21}x_1 - z_1. \end{aligned} \qquad (57)$$

Equation (57) is in the form of (3). Setting $\mu = 0$ in (57) gives the following for the slowly varying subsystem

$$\left.\begin{aligned} \dot{\bar{x}}_1 &= \bar{x}_2, & \bar{x}_1(t_0) &= x_1(t_0) \\ \dot{\bar{x}}_2 &= -a_{11}\bar{x}_1, & \bar{x}_2(t_0) &= x_2(t_0) \end{aligned}\right\} \qquad (58a)$$

$$\left.\begin{aligned} \bar{z}_1 &= a_{21}\bar{x}_1 \\ \bar{z}_2 &= 0. \end{aligned}\right\} \qquad (58b)$$

Equation (58a) describes the oscillation of the center of inertia of machines 2 and 3 with respect to machine 1 and is identical to the linearized swing equation obtained by regarding machines 2 and 3 as an equivalent unit[8, 9]. Since machines 2 and 3 are relatively weakly tied to machine 1, this oscillation is of a relatively low frequency. Thus assuming that the initially $\Delta\delta_{23}$ is small, we show that the fast oscillations are negligible when only the slow dynamics are of interest.

To recover the intermachine oscillations, we obtain the fast oscillatory subsystem (44) as

$$\begin{aligned} \mu\dot{\tilde{z}}_1 &= \tilde{z}_2, & \tilde{z}_1(t_0) &= z_1(t_0) - a_{21}x_1(t_0) \\ \mu\dot{\tilde{z}}_2 &= -\tilde{z}_1, & \tilde{z}_2(t_0) &= z_2(t_0). \end{aligned} \qquad (59)$$

Equation (59) describes the oscillation of machine 3 with respect to machine 2. Since the connection between machines 2 and 3 is relatively strong, compared to their respective connections to machine 1, this oscillation is of a higher frequency than the oscillation of the center of inertia of machines 2 and 3 with respect to machine 1. Equation (59) will be useful when the intermachine oscillations become significant.

We may readily solve (58) and (59) by hand. Expressing the solutions in terms of the original variables gives

$$\begin{aligned} \Delta\bar{\delta}_{c1}(t) &= a\cos\sqrt{a_{11}}\,t + b\sin\sqrt{a_{11}}\,t \\ \Delta\tilde{\delta}_{23}(t) &= c\cos\frac{1}{\mu}t + d\sin\frac{1}{\mu}t. \end{aligned} \qquad (60)$$

The initial conditions are

$$\begin{aligned} \Delta\bar{\delta}_{c1}(0) &= \Delta\delta_{c1}(0) \\ \Delta\dot{\bar{\delta}}_{c1}(0) &= 0 \\ \Delta\tilde{\delta}_{23}(0) &= \Delta\delta_{23}(0) - \mu^2 a_{21}\Delta\delta_{c1}(0) \\ \Delta\dot{\tilde{\delta}}_{23}(0) &= 0 \end{aligned} \qquad (61)$$

where the μ^2 term is retained for improved accuracy. Using (61) in (58) and (59) gives

$$\Delta\bar{\delta}_{c1}(t) = \Delta\delta_{c1}(0)\cos\sqrt{a_{11}}\,t \qquad (62)$$

$$\Delta\tilde{\delta}_{23}(t) = [\Delta\delta_{23}(0) - \mu^2 a_{21}\Delta\delta_{c1}(0)]\cos\frac{1}{\mu}t.$$

Now by applying (45) we can write

$$\begin{aligned}\Delta\delta_{c1}(t) &\cong \Delta\bar{\delta}_{c1}(t)\\ \Delta\delta_{23}(t) &\cong \Delta\bar{\delta}_{23}(t)+\Delta\tilde{\delta}_{23}(t).\end{aligned} \tag{63}$$

Substituting (62) into (63) and using the fact that

$$\Delta\bar{\delta}_{23}=\mu^2 a_{21}\Delta\bar{\delta}_{c1}$$

yield

$$\begin{aligned}\Delta\delta_{c1}(t) &\cong \Delta\delta_{c1}(0)\cos\sqrt{a_{11}}\,t\\ \Delta\delta_{23}(t) &\cong \mu^2 a_{21}\,\Delta\delta_{c1}(0)\cos\sqrt{a_{11}}\,t\\ &\quad+[\Delta\delta_{23}(0)\\ &\quad-\mu^2 a_{21}\,\Delta\delta_{c1}(0)]\cos\frac{1}{\mu}t.\end{aligned} \tag{64}$$

Finally, we recall that

$$\begin{aligned}\delta_{c1}(t) &= \delta^o_{cd}+\Delta\delta_{c1}(t)\\ \delta_{23}(t) &= \delta_{23}+\Delta\delta_{23}(t).\end{aligned} \tag{65}$$

Hence we may write the following solutions for the angles

$$\begin{aligned}\delta_{c1}(t) &\cong \delta^o_{c1}+[\delta_{c1}(0)-\delta^o_{c1}]\cos\sqrt{a_{11}}\,t\\ \delta_{23}(t) &\cong \delta^o_{23}+\mu^2 a_{21}[\delta_{c1}(0)-\delta^o_{c1}]\cos\sqrt{a_{11}}\,t\\ &\quad+\{[\delta_{23}(0)-\delta^o_{23}]-\mu^2 a_{21}[\delta_{c1}(0)\\ &\quad-\delta^o_{c1}]\}\cos\frac{1}{\mu}t.\end{aligned} \tag{66}$$

From the numerical values given in Fig. 2, the following expressions may be obtained

$$\begin{aligned}\delta_{c1}(t) &\cong 32.86^\circ-18.75^\circ\cos 8.130t\\ \delta_{23}(t) &\cong -1.88^\circ+0.3435^\circ\cos 8.130t\\ &\quad-0.3235^\circ\cos 26.02t.\end{aligned} \tag{67}$$

Note that (67) is expressed in electrical degrees since this unit is in more common use than electrical radians.

Figure 3 shows a plot of $\delta_{c1}(t)$ as obtained from the nonlinear system (53) and from the analytic solution (67) for a period of 1 second following the opening of the line from bus 1 to bus 2. Figure 4 shows a similar plot for $\delta_{23}(t)$ from (53) and (67). We note the excellent agreement. Of course this result depends on the fact that the disturbance applied to the system caused only small oscillations of the machines. The accuracy of the time scale decomposition is much better than it would appear in these curves. The error indicated is mainly due to linearization. Had the solution of the full linearized system (57) been compared to the time decomposition approximation (67), the curves would have been indistinguishable.

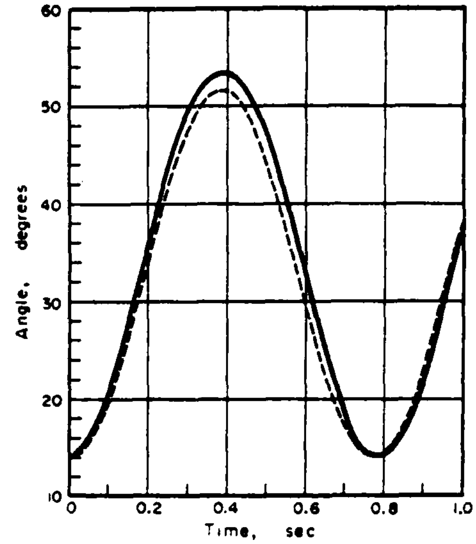

FIG. 3. Plot of accurate and approximate δ_{c1}. δ_{c1}; ---- $\bar{\delta}_{c1}$.

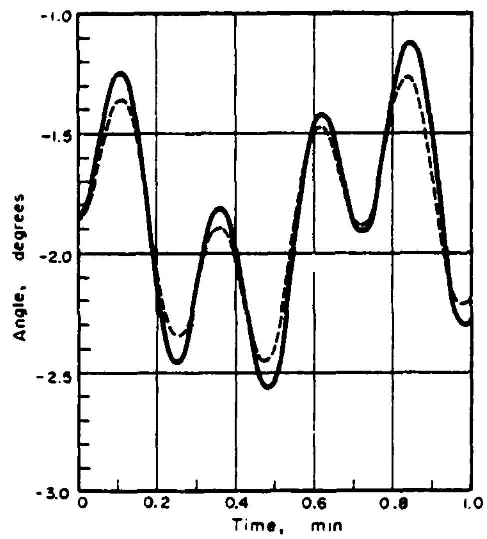

FIG. 4. Plot of accurate and approximate δ_{23}. δ_{23}; ---- $\bar{\delta}_{23}+\tilde{\delta}_{23}$.

6. CONCLUSION

It has been shown that singular perturbation techniques are applicable to systems which possess slightly damped modes oscillating at high frequencies. Our analysis procedures consist of first identifying the small parameter μ and expressing the system in the form (3). Then the original system is decomposed into a slowly varying subsystem and a fast oscillatory subsystem. Using these subsystems, we obtain $O(\mu)$ approximations of both the eigenvalues and the states of the original system (3). Beside the computational advantages of dealing

with the lower order subsystems, the concept of subsystems contributes to the understanding of structural properties of physical systems. The limitation of this decomposition procedure is that we require a sufficient separation between the frequencies of the slowly varying dynamics and the fast oscillatory modes. A mass-spring damper example shows that a stiff spring can be regarded as a perturbation of a rigid rod, the imperfection resulting in high frequency oscillations between the masses. In an inter-connected power system, neglecting the intermachine oscillations, the power angles of the tightly connected machines are shown to be coherent.

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APPENDIX

Notation used in (51)

δ_i: rotor angle of machine i in electrical radians.

M_i: inertia constant of machine i in sec^2 elect. rad.

P_{in_i}: input power to machine i in per unit

V_i': voltage behind transient reactance of machine i in per unit.

Y_{ij}: per unit magnitude of the ijth element of the reduced network admittance matrix.

θ_{ij}: angle in radians of the ijth element of the reduced network admittance matrix.

Notation used in (53)

$$M = M_2 + M_3$$

$$\left(\frac{1}{M_{12}}\right)^2 = \left(\frac{1}{M} - \frac{1}{M_1}\right)^2 \cos^2\theta_{12}$$

$$+\left(\frac{1}{M} + \frac{1}{M_1}\right)^2 \sin^2\theta_{12}, \tan\Psi_{12} = \frac{\left(\frac{1}{M} + \frac{1}{M_1}\right)}{\left(\frac{1}{M} - \frac{1}{M_1}\right)} \tan\theta_{12}$$

$$\left(\frac{1}{M_{13}}\right)^2 = \left(\frac{1}{M} - \frac{1}{M_1}\right)^2 \cos^2\theta_{13} +$$

$$+\left(\frac{1}{M} + \frac{1}{M_1}\right)^2 \sin^2\theta_{13}, \tan\Psi_{13} = \frac{\left(\frac{1}{M} + \frac{1}{M_1}\right)}{\left(\frac{1}{M} - \frac{1}{M_1}\right)} \tan\theta_{13}$$

$$\left(\frac{1}{M_{23}}\right)^2 = \left(\frac{1}{M_2} - \frac{1}{M_3}\right)^2 \cos^2\theta_{23}$$

$$+\left(\frac{1}{M_2} + \frac{1}{M_3}\right)^2 \sin^2\theta_{23}, \tan\Psi_{23} = \frac{\left(\frac{1}{M_2} + \frac{1}{M_3}\right)}{\left(\frac{1}{M_2} - \frac{1}{M_3}\right)} \tan\theta_{23}.$$

Notation used in (54)

$$a_{11} = \frac{V_1'V_2'Y_{12}}{M_{12}} \sin\left(\Psi_{12} - \frac{M_3}{M}\delta_{23}^o - \delta_{c1}^o\right)$$

$$+ \frac{V_1'V_3'Y_{13}}{M_{13}} \sin\left(\Psi_{13} + \frac{M_2}{M}\delta_{23}^o - \delta_{c1}^o\right)$$

$$a_{12} = -\left[\frac{V_1'V_2'Y_{12}}{M_{12}} \frac{M_3}{M} \sin\left(\Psi_{12} - \frac{M_3}{M}\delta_{23}^o - \delta_{c1}^o\right)\right.$$

$$- \frac{V_1'V_3'Y_{13}}{M_{13}} \frac{M_2}{M} \sin\left(\Psi_{13} + \frac{M_2}{M}\delta_{23}^o - \delta_{c1}^o\right)$$

$$\left. - \frac{2V_2'V_3'}{M} Y_{23} \cos\theta_{23} \sin\delta_{23}^o\right]$$

$$a_{21} = -\left[\frac{V_1'V_2'Y_{12}}{M_2} \sin\left(\theta_{12} - \delta_{c1}^o - \frac{M_3}{M}\delta_{23}^o\right)\right.$$

$$\left. - \frac{V_1'V_3'Y_{13}}{M_3} \sin\left(\theta_{13} - \delta_{c1}^o + \frac{M_2}{M}\delta_{23}^o\right)\right]$$

$$a_{22} = \frac{V_1'V_2'Y_{12}}{M_2} \frac{M_3}{M} \sin\left(\theta_{12} - \delta_{c1}^o - \frac{M_3}{M}\delta_{23}^o\right)$$

$$+ \frac{V_1'V_3'Y_{13}}{M_3} \frac{M_2}{M} \sin\left(\theta_{13} - \delta_{c1}^o + \frac{M_2}{M}\delta_{23}^o\right)$$

$$+ \frac{V_2'V_3'Y_{23}}{M_{23}} \sin(\Psi_{23} - \delta_{23}^o).$$

SECTION 2

TWO-TIME-SCALE SYSTEM PROPERTIES

Controllability and Time-Optimal Control of Systems with Slow and Fast Modes

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Abstract—The controllability of linear systems with large and small time constants (singularly perturbed systems) is established. The time-optimal control of such systems is shown to be separable into two time scales related to the slow and fast modes of the system. An approximate design method of the time-optimal control is proposed, which is based on the separability of the fast and slow controls.

Introduction

Control systems with large and small time constants, or with slow and fast modes, are frequent in applications. A model of such systems is

$$\dot{x} = A_{11}x + A_{12}z + B_1u \tag{1}$$

$$\mu\dot{z} = A_{21}x + A_{22}z + B_2u \tag{2}$$

where x, z, and u are n, m, and r vectors, respectively, and the scalar parameter μ represents small time constants. If μ is neglected and (2) is replaced by

$$0 = A_{21}\bar{x} + A_{22}\bar{z} + B_2\bar{u}, \tag{3}$$

and if A_{22}^{-1} exists, then the substitution of $\bar{z}$ into (1) results in a reduced order system:

$$\dot{\bar{x}} = A_0\bar{x} + B_0\bar{u} \tag{4}$$

where

$$A_0 = A_{11} - A_{12}A_{22}^{-1}A_{21}, \qquad B_0 = B_1 - A_{12}A_{22}^{-1}B_2. \tag{5}$$

In the design of time-optimal controls, this order reduction is motivated by the well-known difficulties with high-order systems which are considerable even in the linear time-invariant problems. An approach to simplified design of time-optimal controls is the quasi-optimum technique by Friedland [1], [2]. Also related is an averaging approach by Gerashchenko *et al.* [3]. The only explicit treatment of the time-optimal control for a singularly perturbed system of the type (1), (2) is a recent study by Collins [4], whose results we generalize in several directions. Collins uses the phase-canonic form of (1), (2) and restricts his derivations to single-input systems.

We first establish that the controllability properties of the system (1), (2) are determined by the controllability properties of the slow and fast subsystems. We then demonstrate how the fast–slow separation can be accomplished in a general formulation of the time-optimal control problem.

Controllability of Slow and Fast Modes

An interpretation of (3) is that for $u = 0$, the slowly varying "steady state" of z is $\bar{z} = -A_{22}^{-1}A_{21}\bar{x}$. To separate $\bar{z}$ from the fast transient of z, a change of variables is used:

$$\eta = z + A_{22}^{-1}A_{21}x + \mu Gx \equiv z + Lx, \tag{6}$$

transforming (1), (2) into

$$\dot{x} = (A_0 - \mu A_{12}G)x + A_{12}\eta + B_1u \tag{7}$$

$$\mu\dot{\eta} = Fx + (A_{22} + \mu LA_{12})\eta + (B_2 + \mu LB_1)u \tag{8}$$

where $F = -A_{22}G + (A_{22}^{-1}A_{21} + \mu G)(A_0 - \mu A_{12}G)$. By the implicit function theorem, the solution of $F = 0$ is

$$G = A_{22}^{-2}A_{21}A_0 + 0(\mu). \tag{9}$$ [1]

With $F = 0$, system (7), (8) is block upper triangular, and hence we have the following result.

Lemma 1: Suppose that A_{22}^{-1} exists. Then, as $\mu \to 0$, the first n eigenvalues of the original system (1), (2) tend to the eigenvalues of the reduced system (4), while the remaining m eigenvalues tend to infinity as the eigenvalues of $(1/\mu)A_{22}$.

To separate the slow modes, we introduce

$$\xi = x - \mu(A_{12}A_{22}^{-1} + \mu M)\eta \equiv x - \mu H\eta \tag{10}$$

and choose M such that

$$A_{12} + \mu(A_0 - \mu A_{12}G)H - H(A_{22} + \mu LA_{12}) = 0. \tag{11}$$

By the implicit function theorem,

$$M = A_0A_{12}A_{22}^{-2} - A_{12}A_{22}^{-2}A_{21}A_{12}A_{22}^{-1} + 0(\mu). \tag{12}$$

The transformation (6), (10) can be written as

$$\begin{bmatrix} \xi \\ \eta \end{bmatrix} = \begin{bmatrix} I_n - \mu HL & -\mu H \\ L & I_m \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} \tag{13}$$

and its inverse is

$$\begin{bmatrix} x \\ z \end{bmatrix} = \begin{bmatrix} I_n & \mu H \\ -L & I_m - \mu LH \end{bmatrix} \begin{bmatrix} \xi \\ \eta \end{bmatrix} \tag{14}$$

where I_n and I_m are the $n \times n$ and $m \times m$ identity matrices, respectively. We note that (13) is a special case of a transformation due to Chang [5]. The original system (1), (2) is finally transformed into

$$\dot{\xi} = \mathcal{A}_0\xi + \mathcal{B}_0u \tag{15}$$

$$\mu\dot{\eta} = \mathcal{A}_2\eta + \mathcal{B}_2u \tag{16}$$

where $\mathcal{A}_0 = A_0 - \mu A_{12}G$, $\mathcal{B}_0 = B_0 - \mu\,(HLB_1 + MB_2)$, $\mathcal{A}_2 = A_{22} - \mu LA_{12}$, $\mathcal{B}_2 = B_2 + \mu LB_1$. The controllability properties of (1), (2) are the same as the properties of (15), (16). The subsystem (15) is a regular perturbation of the reduced system (4). The subsystems (15) and (16) are connected through u, but have different eigenvalues. These facts prove the following theorem.

Theorem 1: If A_{22}^{-1} exists and if

$$\operatorname{rank}[B_0, A_0B_0, \cdots, A_0^{n-1}B_0] = n \tag{17}$$

$$\operatorname{rank}[B_2, A_{22}B_2, \cdots, A_{22}^{m-1}B_2] = m, \tag{18}$$

then there exists $\mu^* > 0$ such that the system (1), (2) is controllable for all $\mu \in (0, \mu^*]$.

It should be pointed out that, in view of (18), a matrix K exists such that $A_{22} + B_2K$ is nonsingular. Also, the controllability of (1), (2) is not influenced by $u = Kz + v$. Thus, even if A_{22}^{-1} does not exist, Theorem 1 still holds, but with the matrix $A_{22} + B_2K$ replacing A_{22} in the definition (5) of A_0 and B_0.

Control of Slow Modes

The problem in this section is to steer the variable x of the system (1), (2) from an initial value x^0 at $t = 0$ to zero in minimum time subject to

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[1] A function $f(\mu)$ is $0(\mu)$ if there exist positive constants μ^* and c such that the norm $|f|$ satisfies $|f| < c\mu$ for all $\mu \in [0, \mu^*]$.

$|u_k| < 1$, $k = 1, \cdots, r$. No specific requirements are imposed on the variable z. Its behavior is of interest only insofar as it can impede or improve the control of x. The minimum principle for (1), (2) or, equivalently, for (15), (16) yields

$$u = -\operatorname{sgn}\left\{ \mathcal{B}_0' e^{\mathcal{A}_0'(T-t)} p + \frac{\mathcal{B}_2'}{\mu} e^{\mathcal{A}_2'(T-t/\mu)} q \right\} \tag{19}$$

where $[p', q']'$ is a constant $(n+m)$ vector. If (16) is asymptotically stable, η is bounded, ξ is $0(\mu)$ at $t = T$, the term depending on $(T - t/\mu)$ in (19) rapidly decays away from T, and hence the control in (19) can be approximated by a "slow-mode control"

$$\bar{u} = -\operatorname{sgn}\left\{ \mathcal{B}_0' e^{\mathcal{A}_0'(T-t)} \bar{p} \right\}, \tag{20}$$

which is a time-optimal control for the slow subsystem (15). When $\mu \to 0$, then $\mathcal{A}_0 \to A_0$, $\mathcal{B}_0 \to B_0$, and $\bar{u}$ becomes a "reduced control" $\bar{u}_0$, that is, a time-optimal control for the reduced system (4). In applications, z is often a "parasitic" variable, that is, A_{12}, A_{22}, A_{21}, B_2 are not known and the only data available about the system (1), (2) are the matrices A_0, B_0. Then, instead of $\bar{u}$, the reduced control $\bar{u}_0$ is applied to (1), (2). To summarize this discussion, we define systems (1), (2) in which A_{22} has all the eigenvalues with negative real parts as "robust" systems.

Lemma 2: Suppose: a) the system (1), (2) is robust; b) the time-optimal control problem for the reduced model (4) is normal; c) $x = 0$ is reachable from x^0. Then there exists $\mu^* > 0$ such that for all $\mu \in [0, \mu^*]$, the slow-mode control $\bar{u}$ is a near-optimal control for the system (1), (2) in the sense that it steers x to a $0(\mu)$ neighborhood of zero in the slow-mode minimum time $\bar{T}$. An analogous statement applies to the reduced control $\bar{u}_0$ and T_0, the minimum time for (4).

Control of Fast and Slow Modes

The problem is now to steer x^0, z^0 of (1), (2) or, equivalently, ξ^0, η^0 of (15), (16) to zero in minimum time. In addition to the minimum principle condition (19), we now also use the fact that the Hamiltonian is zero for all t. It follows that $(q/\mu) = \hat{q}$ remains finite as $\mu \to 0$. After substituting $q = \mu\hat{q}$, we rewrite (19) more compactly as

$$u = -\operatorname{sgn}\left\{ s(T-t) + f\left(\frac{T-t}{\mu}\right) \right\}. \tag{21}$$

For s we assume that it has N_s zeros, that they are distinct, and that they are located in a subinterval $[0, t_s]$ of the interval $[0, T]$. Then $s + 0(\mu)$ also has N_s zeros and they all lie in $0(\mu)$ neighborhoods of the corresponding s zeros. We recall that in a robust system, f exhibits a rapid exponential decay. Thus, $f < 0(\mu)$ in $[0, t_s]$. Hence, $g = s + f = s + 0(\mu)$ has N_s zeros in $[0, t_s]$ and they lie in $0(\mu)$ neighborhoods of the corresponding zeros of s. These N_s zeros determine the "slow" switchings of the control (21). At the end of the interval $[0, T]$, there is a subinterval $[t_f, T]$ in which f is not negligible and g may have N_f additional zeros. In view of the rapid decay of f, intervals $[0, t_s]$ and $[t_f, T]$ are disjoint, $t_s < t_f$, and $T - t_f = 0(\mu)$. The additional N_f zeros define the "fast" switchings of the control (21) and are to be determined in a fast time scale $\sigma = (T - t/\mu)$. We assume that the zeros of $s(0) + f(\sigma)$ are distinct. Since $s(\mu\sigma) + f(\sigma) = s(0) + f(\sigma) + 0(\mu)$, we see that if σ' is a zero of $s(0) + f(\sigma)$, then $t' = T - \mu[\sigma' + 0(\mu)]$ is a zero of $s(T-t) + f(T - t/\mu)$. These facts are summarized as follows.

Theorem 2: Let (1), (2) be a robust system and $x = 0$, $z = 0$ be reachable from x^0, z^0. Suppose that the zeros of $s(T-t)$ as well as the zeros of $s(0) + f(\sigma)$ are distinct. Then there exist $\mu^* > 0$ and $\theta > 0$ such that for all $\mu \in (0, \mu^*]$, the time-optimal control (21) is separable in the following sense:

$$u = \begin{cases} u_s(t), & 0 < t < T - \mu\theta \\ u_f(\tau), & 0 < \tau < \theta \end{cases} \tag{22}$$

where $\tau = \theta - \sigma$ and the switchings of $u_s(t)$ are in $0(\mu)$ neighborhoods of the zeros of $s(T-t)$ and the switchings of $u_f(\tau)$ are in $0(\mu)$ neighborhoods of the zeros of $s(0) + f(\theta - \tau)$.

To interpret and exploit this important separation property, we express $\xi(T)$ and $\eta(T)$ in terms of $u_s(t)$ and $u_f(\tau)$. We have

$$\xi(T) = e^{\mu\mathcal{A}_0\theta}\left\{ e^{\mathcal{A}_0\hat{t}}\xi^0 + \int_0^{\hat{t}} e^{\mathcal{A}_0(\hat{t}-t)}\mathcal{B}_0 u_s(t)\,dt \right\} + \mu\int_0^{\theta} e^{\mu\mathcal{A}_0(\theta-\tau)}\mathcal{B}_0 u_f(\tau)\,d\tau \tag{23}$$

where $\hat{t} = T - \mu\theta$ and

$$\eta(T) = e^{\mathcal{A}_2\theta}\eta^* + \int_0^{\theta} e^{\mathcal{A}_2(\theta-\tau)}\mathcal{B}_2 u_f(\tau)\,d\tau \tag{24}$$

where

$$\eta^* = -\mathcal{A}_2^{-1}\mathcal{B}_2 u_s(\hat{t}^-) + \Delta \tag{25}$$

is the value to which $u_s(t)$ steers η after its last switching; see (16). In view of the rapid decay of the η transient, Δ is an exponentially small term and will be neglected henceforth. From (24) and (25), an interpretation of $u_f(\tau)$ is that it steers η^* to zero in minimum time $\tau = \theta$. On the other hand, it is clear from (23) that the contribution of $u_f(\tau)$ to $\xi(T)$ is $0(\mu)$. Thus $u_s(t)$ steers ξ to a $0(\mu)$ neighborhood of zero. This means that $u_s(t)$ can be approximated by the slow-mode control $\bar{u}$ or, even simpler, by the reduced control $\bar{u}_0$ (see Lemma 2). From Theorem 2 we know that $u_f(\tau)$ can be approximated by

$$\bar{u}_f(\tau) = -\operatorname{sgn}\{ \mathcal{B}_2' e^{\mathcal{A}_2'(\theta-\tau)}\hat{q} + s(0) \} \tag{26}$$

where $\hat{q}$ is to be determined such that η^* for (16) is steered to zero. As a further simplification, $\mathcal{A}_2$, $\mathcal{B}_2$ in (26) can be replaced by A_{22}, B_2, respectively.

In conclusion, to obtain a near-optimum control for (1), (2), we first calculate the time-optimal control $\bar{u}_0(t)$ or $\bar{u}(t)$ for the reduced order model (4) or for (15). Then we use η^* as the initial condition for the fast subsystem (16) and evaluate the fast control $\bar{u}_f(\tau)$ from (26) to steer the state of this subsystem to zero. This control is near optimum in the sense that the error in $\xi(T)$, $\eta(T)$, and hence in $x(T)$, $z(T)$, is $0(\mu)$.

It is possible to further correct the slow control by selecting $u_s(t)$ as the time-optimal control for steering ξ from ξ^0 to $\xi(t) = -\mu\int_0^{\theta} e^{-\mu\mathcal{A}_0\tau}\mathcal{B}_0\bar{u}_f(\tau)\,d\tau \equiv -\mu\xi^*$, as seen from (23). To obtain a correction

$$u_s(t) = \bar{u}(t) + \delta u(t) \tag{27}$$

let $\bar{t}_{ij}$ and t_{ij} be switching times of $\bar{u}_j$ and $(u_s)_j$, respectively, and let $\hat{t}$ and p be the corresponding final time and co-state vector for the corrected control. Then we may write

$$t_{ij} = \bar{t}_{ij} + \mu t_{ij}'$$

$$t = \bar{T} + \mu T'$$

$$p = \bar{p} + \mu\tilde{p}; \tag{28}$$

then

$$\delta u_j = \begin{cases} -2\bar{u}_j(\bar{t}_{ij}+), & 0 < (t - \bar{t}_{ij}) < \mu t_{ij}', \quad \text{for } t_{ij}' > 0 \\ 2\bar{u}_j(\bar{t}_{ij}+), & \mu t_{ij}' < (t - \bar{t}_{ij}) < 0, \quad \text{for } t_{ij}' < 0. \end{cases} \tag{29}$$

The corrections in the switching times become approximately

$$t_{ij}' = T' + \tilde{p}'c_j(\bar{t}_{ij})\left[p'\mathcal{A}_0 c_j(\bar{t}_{ij})\right]^{-1} \tag{30}$$

where c_j is the jth column of $e^{\mathcal{A}_0(T-\bar{t}_{ij})}\mathcal{B}_0$. The substitution of (28), (29), and (30) in (23) results in the following set of linear equations for $\tilde{p}$ and T':

$$-\xi^* = T'\mathcal{B}_0\bar{u}(\bar{T}) - 2\sum_i\sum_j \bar{u}_j(\bar{t}_{ij}+)c_j(\bar{t}_{ij})\left[T' + \frac{c_j'(\bar{t}_{ij})\tilde{p}}{p'\mathcal{A}_0 c_j(\bar{t}_{ij})}\right]. \tag{31}$$

The solution of (31) together with (30) and (29) yields the corrected control $u_s(t)$. The resulting errors in $\xi(T)$, and hence in $x(T)$, become

$O(\mu^2)$. Since no correction is applied to $\bar{u}_f$, the error in η and z remain $O(\mu)$; however, it decays to zero very rapidly.

EXAMPLE

In most dc motors, the mechanical time constant is large compared to the electrical time constant. Let ω and i be the speed and armature current, respectively; then

$$\frac{d\omega}{dt} = -\frac{B}{J}\omega + \frac{K_T}{J}i \tag{32}$$

$$\frac{L}{R}\frac{di}{dt} = -\frac{K_e}{R}\omega - i + \frac{u}{R} \tag{33}$$

where the armature voltage u is the control. We introduce $\tau_m = [(B/J) + (K_T K_e/JR)]^{-1}$ and $\mu = (L/R\tau_m)$. Then (32), (33) in the new time scale $t' = t/\tau_m$ become

$$\dot{x} = -\frac{1}{1+k}x + \frac{k_1}{1+k}z \tag{34}$$

$$\mu\dot{z} = -\frac{k}{k_1}x - z + bu \tag{35}$$

where $x \equiv \omega, z \equiv i, k_1 = (K_T/B), k = k_1(K_e/R), b = (1/R)$. We seek the time-optimal control to steer both speed and current to zero, that is, $\xi(T) = \eta(T) = 0$. The control of the slow mode is

$$\bar{u}(t) = \bar{u}_s(t) = -\operatorname{sgn}\xi^0, \qquad 0 < t < \bar{T} = \frac{1}{a_0}\ln\left[1 + a_0\frac{|\xi^0|}{b_0}\right]. \tag{36}$$

The error in $\xi(\bar{T})$ is obviously zero, but the final value of $\eta(\bar{T})$ and the miss distance for $x(\bar{T})$ are given by

$$\eta(\bar{T}) = \left(\eta^0 + \frac{b}{a_2}\operatorname{sgn}\xi^0\right)e^{-a_2\bar{T}/\mu} - \frac{b}{a_2}\operatorname{sgn}\xi^0 \cong -\frac{b}{a_2}\operatorname{sgn}\xi^0 \tag{37}$$

$$x(\bar{T}) = -\mu k_1\frac{(1-c)}{\Delta}\eta(\bar{T}) = \mu k_1\frac{(1-c)}{\Delta}\frac{b}{a_2}\operatorname{sgn}\xi^0 \tag{38}$$

where

$$a_0 = (1-\mu\lambda)^{-1},\ a_2 = (1-\mu\lambda),\ b_0 = \frac{bk_1}{1+k}\left[1 + \mu\frac{(1+c)}{\Delta}\right]$$

$$c = \lambda(1-\mu\lambda),\ \Delta = [(1-\mu\lambda)^2 - \mu]$$

$$\lambda = \left(1 - \frac{\mu}{1+k}\right)\left\{\frac{1}{2\mu}\left[1 - \sqrt{1 - \frac{4\mu k}{(1+k)\left(1-\frac{\mu}{1+k}\right)^2}}\right]\right\}$$

$$= \frac{k}{1+k} + O(\mu).$$

The fast variable may now be controlled by using the control $\bar{u}_f(\tau) = \operatorname{sgn}\xi^0$, $0 < \tau < \theta = (1/a_2)\ln 2$. The exact miss distance in $\xi(T)$ at $T = \bar{T} + \mu\theta$:

$$\xi(T) = [1 - e^{-a_0\mu\theta}]\frac{b_0}{a_0}\operatorname{sgn}\xi^0 = O(\mu), \tag{39}$$

which is the same as the miss distance in $x(T)$ since $\eta(T)$ is negligible. The slow control may now be corrected for this miss distance.

CONCLUSION

The controllability properties of the original system are determined by a separate analysis of the slow and fast subsystems. An application of this approach to the time-optimal control problem has revealed similar time-scale separation properties of linear time-optimal controls.

The time-scale separation property is shown to be particularly useful in a near-optimum design procedure, which can be divided into two lower order phases, corresponding to the design of a slow mode, or a reduced control, and, in a different time scale, a fast control.

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A Riccati Equation for Block-Diagonalization of Ill-Conditioned Systems

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Abstract—A simple transformation, originally introduced for singularly perturbed systems, is now applicable to a larger class of time-invariant systems.

INTRODUCTION

In this note we discuss a method to transform

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}A_{11} & A_{12}\\ A_{21} & A_{22}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} \tag{1}$$

into

$$\begin{bmatrix}\dot{x}_1\\ \dot{y}_2\end{bmatrix} = \begin{bmatrix}B_1 & A_{12}\\ 0 & B_2\end{bmatrix}\begin{bmatrix}x_1\\ y_2\end{bmatrix} \tag{2}$$

and into

$$\begin{bmatrix}\dot{y}_1\\ \dot{y}_2\end{bmatrix} = \begin{bmatrix}\tilde{B}_1 & 0\\ 0 & B_2\end{bmatrix}\begin{bmatrix}y_1\\ y_2\end{bmatrix} \tag{3}$$

where x_1 and y_1 are n_1-vectors, and x_2 and y_2 are n_2-vectors. The transformation is particularly convenient for systems possessing n_1 small and n_2 large eigenvalues, when (2) and (3) can be used to separate the

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"slow" and "fast" subsystems of (1). In this context the transformation has been introduced by Chang [1], [2] to study singularly perturbed systems and it was also applied to an optimal control problem [3]. The conditions obtained here for time-invariant systems are less restrictive than those in [1], [2] and the system (1) need not be singularly perturbed. Instead, it is sufficient that the norms of its submatrices satisfy an inequality. In the course of the proof of this result a convergent iterative procedure is derived for calculation of the transformation matrices.

The Riccati Equation

To transform (1) into (2) we use

$$y_2 = x_2 + Lx_1 \tag{4}$$

and require that the $n_2 \times n_1$ matrix L be a real root of

$$A_{22}L - LA_{11} + LA_{12}L - A_{21} = 0. \tag{5}$$

If L exists then the substitution of (4) into (1) yields the block-triangular system (2) where

$$B_1 = A_{11} - A_{12}L \tag{6}$$

$$B_2 = A_{22} + LA_{12}. \tag{7}$$

Assuming that A_{22} is nonsingular we introduce

$$L_0 = A_{22}^{-1}A_{21}, \quad A_0 = A_{11} - A_{12}L_0 \tag{8}$$

and seek L in the form

$$L = L_0 + D \tag{9}$$

where D is a real root of

$$DA_0 - (A_{22} + L_0A_{12})D - DA_{12}D + L_0A_0 = 0. \tag{10}$$

The following lemma gives a sufficient condition for the existence and uniqueness of a real root D and establishes a bound for its norm $\|D\|$. It also formulates a convergent procedure for iterative calculation of D.

Lemma 1: If A_{22} is nonsingular and if

$$\|A_{22}^{-1}\| < \frac{1}{3}(\|A_0\| + \|A_{12}\|\,\|L_0\|)^{-1} \tag{11}$$

then a unique real root of (10) exists satisfying

$$0 < \|D\| < \frac{2\|A_0\|\,\|L_0\|}{\|A_0\| + \|A_{12}\|\,\|L_0\|}. \tag{12}$$

This root is an asymptotically stable equilibrium of the difference equation

$$D_{k+1} = A_{22}^{-1}(L_0A_0 + D_kA_0 - L_0A_{12}D_k - D_kA_{12}D_k) \equiv f(D_k) \tag{13}$$

and its domain of attraction encompasses the set of matrices defined by (12).

Proof: For $a = \|A_0\|$, $b = \|A_{12}\|\,\|L_0\|$, $c = \|A_{22}^{-1}\|$ and

$$d_k = \frac{\|D_k\|}{\|A_0\|\,\|L_0\|} \tag{14}$$

we obtain from (13)

$$d_{k+1} < c[1 + (a+b)d_k + abd_k^2] < c\left(\frac{a+b}{2}d_k + 1\right)^2 \tag{15}$$

and we analyze the upper bound of d_k defined by

$$\bar{d}_{k+1} = c\left(\frac{a+b}{2}\bar{d}_k + 1\right)^2. \tag{16}$$

Obviously $d_k < \bar{d}_k$ for all k. When $c < \frac{1}{4}(a+b)^{-1}$ then the scalar difference equation (16) has two real equilibrium points d' and $d'' > d'$ and d' lies in the interval $[0, 2(a+b)^{-1}]$. For all $\bar{d}_k \neq d'$ in this interval we have $|\bar{d}_{k+1} - d'| < |\bar{d}_k - d'|$, which proves that D_k is bounded, that is

$$0 < \|d_k\| < 2(a+b)^{-1} \tag{17}$$

holds for all $k = 1, 2, \cdots$, if it holds for $k = 0$. Substituting $\delta D_k = D_k - D$ into (13) we get

$$\delta D_{k+1} = A_{22}^{-1}[\delta D_k(A_0 - A_{12}D) - (L_0 + D)A_{12}\delta D_k - \delta D_kA_{12}\delta D_k] \tag{18}$$

which, using $\|\delta D_k\| \equiv v_k < \|D_k\| + \|D\|$, implies

$$v_{k+1} < c[a + b + 3\|A_{12}\|\,\|D\| + \|A_{12}\|\,\|D_k\|]v_k. \tag{19}$$

When both $\|D_k\|$ and $\|D\|$ satisfy (17), then (19) yields

$$v_{k+1} < c\left(a + b + \frac{8ab}{a+b}\right)v_k < 3c(a+b)v_k \tag{20}$$

and, hence, if $c < \frac{1}{3}(a+b)^{-1}$, as required by (11), then $f(D_k)$ in (13) is a contraction mapping and D is its fixed point. We complete the proof by noting that v_k is a Lyapunov function and $v_{k+1} - v_k < 0$ for all D_k satisfying (17).

Using (13) with the initial condition $D_0 = 0$ we can calculate D iteratively. By (11) and (20) after k iterations the relative error is

$$\frac{\|D_k - D\|}{\|D\|} < [3\|A_{22}^{-1}\|(\|A_0\| + \|A_{12}\|\,\|L_0\|)]^k \tag{21}$$

and it decreases as $\|A_{22}^{-1}\|$ and $\|A_0\|$ decrease, that is as the ill-conditioning of the system (1) increases.

Block-Diagonalization

To further transform (2) into (3) we use

$$y_1 = x_1 - My_2 \tag{22}$$

where the $n_1 \times n_2$ matrix M is a real root of

$$B_1M - MB_2 + A_{12} = 0. \tag{23}$$

The following lemma formulates a convergent iterative method for solving (23).

Lemma 2: Under the conditions of Lemma 1 the solution M of (23) is the asymptotically stable equilibrium of the linear difference equation

$$M_{k+1} = [(A_{11} - A_{12}L)M_k - M_kLA_{12}]A_{22}^{-1} + A_{12}A_{22}^{-1}. \tag{24}$$

Proof: For $m_k = \|M_k - M\|$ we obtain from (23) and (24)

$$m_{k+1} < c[a + b + 2\|A_{12}\|\,\|D\|]m_k \tag{25}$$

and, by virtue of (12),

$$m_{k+1} < c\left[a + b + \frac{4ab}{a+b}\right]m_k. \tag{26}$$

Thus, $m_{k+1} < m_k$ if $c < \frac{1}{3}(a+b)^{-1}$ which is satisfied by (11).

To summarize, the transformation of (1) into (3) is

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} I_1 & M \\ -L & I_2 - LM \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \tag{27}$$

where I_1 and I_2 are the n_1- and n_2-dimensional identity matrices, respectively. The transformation (27) is performed in two stages, (4) and (22).

Concluding Remarks

In applications L can be calculated from (9) and the iterative scheme (13). It is somewhat simpler to program the equivalent iterative scheme for (5) with the initial value L_0 as in (8). In several tests with matrices

satisfying (11) an accuracy of four significant digits was achieved after only two iterations, indicating that (11) is a conservative condition. An example when (11) is not satisfied is a power system model developed in [4, pp. 104–106] whose matrix

$$A=\begin{bmatrix} -0.11 & 0.02 & 0.03 & 0.00 & 0.02 \\ 0.00 & -0.17 & 0.00 & 0.00 & 0.17 \\ 0.00 & 2.00 & -4.00 & 0.00 & 0.00 \\ -4.00 & 0.00 & 0.00 & -2.00 & 0.00 \\ 0.00 & 0.00 & 0.00 & 4.75 & -5.00 \end{bmatrix} \tag{28}$$

with $n_1=2$, $n_2=3$ and the absolute value norm yields $a=0.676$, $b=0.968$, and $c=1.425$, seven times larger than in (11). After four iterations the accuracy better than four digits has been achieved and the eigenvalues $-0.15538\pm j0.11466$ of B_1, and -5.0273, -3.9938, -1.9482 of B_2, are within ± 0.00005 of the eigenvalues of A. Thus, the method can be used even when Lemma 1 is violated. When (11) is satisfied rapid convergence can be expected.

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Eigensensitivities in Reduced Order Modeling

J. J. ALLEMONG AND P. V. KOKOTOVIC

Abstract—Expressions for eigenvalue and eigenvector sensitivities are derived with respect to the singular perturbation parameter whose variation changes the system order. They are illustrated by the sensitivity of a power system model with respect to its neglected fast part.

INTRODUCTION

For almost two decades eigenvalue and eigenvector sensitivities have been among the most common tools in circuit and system analysis [1]–[5]. They are used to determine influence of system components and parameters on individual modes.

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This correspondence extends the notion of sensitivity to the case of a parameter whose variation changes the system order. Using the expressions derived here it is possible to analyze modes' sensitivities with respect to neglected parasitics and to the fast modes eliminated from a reduced order model.

The change of system order is parametrized by writing the system equation in a singularly perturbed form [7], that is,

$$\dot{x}=Ax+Bz \tag{1}$$

$$\epsilon\dot{z}=Cx+Dz \tag{2}$$

where the small positive scalar ϵ is of the order of the ratio of speeds of slow and fast modes. When ϵ is set equal to 0, that is when the transients of the fast modes are assumed to be instantaneous, then the substitution of z from the quasi-steady-state expression

$$0=Cx+Dz \tag{3}$$

into (1) yields

$$\dot{x}=(A-BD^{-1}C)x \triangleq A_0x. \tag{4}$$

This is the reduced order model frequently used as an approximation of (1) and (2). Considering $\epsilon=0$ corresponding to the simplified model (4) as the "nominal" value of ϵ, we now investigate how the eigenvalues and eigenvectors change when, instead of 0, ϵ has a small positive value corresponding to the exact model (1) and (2).

EIGENVALUE SENSITIVITIES

It is shown in [6], [7] that the models (1) and (2) can be transformed into the block diagonal form

$$\dot{x}_d=\mathcal{A}(\epsilon)x_d \tag{5}$$

$$\epsilon\dot{z}_d=\mathcal{D}(\epsilon)z_d \tag{6}$$

where

$$\mathcal{A}(\epsilon)=A_0-\epsilon BD^{-2}CA_0+O(\epsilon^2) \tag{7}$$

$$\mathcal{D}(\epsilon)=D+\epsilon D^{-1}CB+O(\epsilon^2). \tag{8}$$

We first consider the eigenvalues λ_i of $\mathcal{A}(\epsilon)$ and denote by u_i and v_i, such that $v_i'u_i=1$, the eigenvectors of $\mathcal{A}(\epsilon)$ and of its transpose $\mathcal{A}'(\epsilon)$, respectively. Using the well-known [1], [3] eigenvalue sensitivity expression

$$\frac{\partial\lambda_i}{\partial\epsilon}=v_i'\frac{\partial\mathcal{A}}{\partial\epsilon}u_i \tag{9}$$

and evaluating $\partial\mathcal{A}/\partial\epsilon$ from (7) we obtain

$$\frac{\partial\lambda_i}{\partial\epsilon}=-\lambda_i v_i'BD^{-2}Cu_i. \tag{10}$$

Analogously, for the eigenvalues μ_j of $\mathcal{D}(\epsilon)$, with q_j and p_j, $p_j'q_j=1$, being the corresponding eigenvectors of $\mathcal{D}(\epsilon)$ and $\mathcal{D}'(\epsilon)$, we obtain, in view of (8),

$$\frac{\partial\mu_j}{\partial\epsilon}=\frac{1}{\mu_j}p_j'CBq_j. \tag{11}$$

The expressions (10) are the actual sensitivities of those eigenvalues of (1) and (2) which remain finite as $\epsilon\to 0$. The remaining eigenvalues μ_j/ϵ tend to infinity as $\epsilon\to 0$ and expressions (11) are the sensitivities of their asymptotes.

EIGENVECTOR SENSITIVITIES

Differentiating $\mathcal{A}u_i=\lambda_iu_i$ and evaluating the derivatives at $\epsilon=0$ we obtain

$$(A_0-\lambda_iI)\frac{\partial u_i}{\partial\epsilon}=\frac{\partial\lambda_i}{\partial\epsilon}u_i+\lambda_iBD^{-2}Cu_i. \tag{12}$$

Following [3] and assuming that the eigenvalues are distinct this expres-

TABLE I
EIGENVALUES AND SENSITIVITIES OF (17)

Exact	Uncorrected	Corrected	Sensitivity
$-0.362 \pm j.56$	$-0.4 \pm j.58$	$-0.355 \pm j.56$	$0.044 \pm j.02$
-3.93	-3.72	-3.95	-0.23
$-8.53 \pm j8.22$	$-8.29 \pm j7.95$	$-8.50 \pm j8.23$	$-0.21 \pm j.28$
$-0.86 \pm j8.37$	$-0.78 \pm j8.39$	$-0.85 \pm j8.38$	$-0.07 \pm j.01$

sion can be reduced to

$$\frac{\partial u_i}{\partial \epsilon} = \sum_k \phi_{ik} u_k, \qquad k \neq i \tag{13}$$

where

$$\phi_{ik} = \frac{\lambda_i}{\lambda_k - \lambda_i} v_k' BD^{-2} C u_i \tag{14}$$

and $\phi_{ii} = 0$. Similarly,

$$\frac{\partial q_j}{\partial \epsilon} = \sum_k \Psi_{jk} q_k, \qquad k \neq j \tag{15}$$

where

$$\Psi_{jk} = \frac{1}{\lambda_k(\lambda_k - \lambda_j)} p_k' CB q_j \tag{16}$$

and $\Psi_{jj} = 0$.

EXAMPLE

The matrix appearing in a seventh-order model of a synchronous machine connected to an infinite bus [6], [7] is

$$\begin{bmatrix} A & B \\ \frac{1}{\epsilon} C & \frac{1}{\epsilon} D \end{bmatrix}$$

$$= \begin{bmatrix} -0.58 & 0 & 0 & -0.27 & 0 & 0.2 & 0 \\ 0 & -1.0 & 0 & 0 & 0 & 1.0 & 0 \\ 0 & 0 & -5.0 & 2.1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 377 & 0 & 0 \\ -0.14 & 0 & 0.14 & -0.2 & -0.28 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.08 & 2.0 \\ -173 & 66.7 & -116 & 40.9 & 0 & -66.7 & -16.7 \end{bmatrix} \tag{17}$$

where A is 2×2 and D is 5×5, that is, the simplified model (4) is of second order. In this case we scale the fast modes by $\epsilon = 0.1$. Note however that the approximations such as

$$\lambda_i(\epsilon) = \lambda_i(0) + \epsilon \frac{\partial \lambda_i}{\partial \epsilon} \tag{18}$$

do not depend on this scale factor since the sensitivities are scaled correspondingly. In Table I we give the approximations of the type (18) for all the eigenvalues of (17). The columns from left to right are the exact eigenvalues, the uncorrected eigenvalues of A_0 and $1/\epsilon D$, the corrected eigenvalues of (17) and their sensitivities. It should be noted that the errors of 10 percent or more have been reduced to less than 2 percent.

CONCLUSION

When the change of system order is parameterized using singular perturbations the eigensensitivities with respect to this change can be evaluated from the expressions analogous to the usual expressions for sensitivities with respect to parameters.

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SUBSPACE ITERATIONS APPROACH TO THE TIME SCALE SEPARATION*

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ABSTRACT

Some properties of the time scale separation are reviewed and a new globally convergent algorithm is proposed. A connection between the new algorithm and the existing Riccati-like algorithms is established.

BASIS FREE TIME SCALE SEPARATION

After introducing necessary notation we will state basic properties of the time scale separation. We use $\lambda_i(A)$ to denote i-th eigenvalue of A and we assume that

$$|\lambda_i(A)| \geq |\lambda_{i+1}(A)|, \quad i=1,2,\dots,n-1. \tag{1}$$

The spectrum of A is

$$\sigma(A) = \{|\lambda_i(A)|, \ i=1,2,\dots,n\}. \tag{2}$$

Then the problem of the time scale separation for a system $\dot{x} = Ax$, $x \in R^n$ partitioned with given $n_1 = \dim x_1$ as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{3}$$

is to find a transformation yielding

$$\begin{bmatrix} \dot{x}_1 \\ \dot{y}_2 \end{bmatrix} = \begin{bmatrix} B_1 & A_{12} \\ 0 & B_2 \end{bmatrix} \begin{bmatrix} x_1 \\ y_2 \end{bmatrix} \tag{4}$$

with the spectrum of A separated into $\sigma(B_1)$ and $\sigma(B_2)$ such that

$$\varepsilon \triangleq \frac{\sup \sigma(B_2)}{\inf \sigma(B_1)} = \frac{|\lambda_{n_1+1}(A)|}{|\lambda_{n_1}(A)|} < 1. \tag{5}$$

Every n_1 satisfying (5) will be called admissible. We remark that the formulation (5) is appropriate when the action of the subsystem (x_1, A_{11}) on (x_2, A_{22}) is weaker than the reverse action. An example of such systems are the power systems considered in [6]. An analogous problem formulation with

$$\sup \sigma(B_1) < \inf \sigma(B_2) \tag{6}$$

is also possible. Without the loss of generality for the rest of this paper we assume (5).

If an admissible n_1 is known, then it was shown in [2,8,10] that a numerically suitable transformation for the time scale separation is

$$y_2 = x_2 + Lx_1 \tag{7}$$

where L satisfies

$$R(L) = A_{22}L - LA_{11} + LA_{12}L - A_{21} = 0 \tag{8}$$

and the matrices in (4) are

$$\begin{aligned} B_1 &= A_{11} - A_{12}L \\ B_2 &= A_{22} + LA_{12} \end{aligned} \tag{9}$$

Conditions on existence of L and its explicit form are given in the following standard result [see e.g., 7, 11], rephrased here in a basis free form.

__Lemma 1__: Given (1) and an admissible n_1, let $\sigma_D = \{|\lambda_i(A)|, \ i=1,2,\dots,n_1\}$ be the dominant spectrum and let $\mathcal{S}$ be the dominant eigenspace of A. Further, let the columns of a matrix $\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \in R^{n \times n_1}$ be a basis for $\mathcal{S}$. Then, provided V_1 is nonsingular, the solution of (7) satisfying

$$\sigma(B_1) = \sigma_D \tag{10}$$

is

$$L = -V_2V_1^{-1}. \tag{11}$$

__Proof__: Note first that L is invariant to the bais of $\mathcal{S}$, that is there always exists a nonsingular matrix K such that if $M = VK$, then

$$V_2V_1^{-1} = M_2M_1^{-1}. \tag{12}$$

In particular let the columns of $M = \begin{bmatrix} M_1 \\ M_2 \end{bmatrix}$ be the eigenvectors (and the generalized eigenvectors) of A spanning $\mathcal{S}$,

$$A \begin{bmatrix} M_1 \\ M_2 \end{bmatrix} = \begin{bmatrix} M_1 \\ M_2 \end{bmatrix} J_1 \tag{13}$$

and $\sigma(J_1) = \sigma_D$. Now if (7) is rewritten as

$$R(L) = -[L \quad I]A \begin{bmatrix} I \\ -L \end{bmatrix} \tag{14}$$

then it simply follows from (13) and (14) that

$$R(-M_2M_1^{-1}) = R(-V_2V_1^{-1}) = 0. \tag{15}$$

To prove (10) we use (13) in the expanded form

$$M_1^{-1}(A_{11} + A_{12}M_2M_1^{-1})M_1 = J_1 \tag{16}$$

which, due to (11) and (13), proves that $B_1 = A_{11} - A_{12}L$ has as its spectrum the dominant spectrum σ_D.

__Discussion__: The three important facts about this lemma should be noticed. First, that L can be expressed in a basis free form. Second, that n_1 is admissible and third, that V_1 is nonsingular.

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The freedom in expressing L is an advantage explored more in the next section. The need to specify n_1 is a disadvantage since an arbitrary n_1 may not be admissible and/or V_1 may be singular. Therefore it is desirable that an iterative algorithm has the capability to detect if n_1 is not admissible and to redefine it.

For an admissible n_1, as shown in [4], there always exists an ordering of variables within a state vector x for which V_1 is nonsingular. It is desirable to find this ordering before the block triangularization is attempted. For this purpose it is useful to note [11] that the existence of L, that is the nonsingularity of V_1 is related to the controllability and observability properties of the pairs (A_{11},A_{12}) and (A_{22},A_{12}). If B_1 is viewed as a regulator system matrix with L as a feedback assigning the dominant spectrum, then for V_1 to be nonsingular all the eigenvalues of A_{11} uncontrollable through A_{12} must be in the dominant spectrum. Since the same L is an observer matrix assigning the nondominant spectrum to B_2, all the eigenvalues of A_{22} unobservable through A_{12} must be in the nondominant spectrum.

These conditions are likely to be met if an ordering of states is such that the norm of A_{22} is the smallest possible meaning that x_2 should contain slow variables only. Such an ordering usually results in the reduction of the norm of either A_{12} or A_{21} and hence $\sigma(A_{11})$ becomes close to $\sigma(B_1)$ and $\sigma(A_{22})$ close to $\sigma(B_2)$. For the class of systems implied by the formulation (5) this further means that the solution L of (8) will have small norm.

The following example illustrates some of these observations. The system

$$\dot{x} = \begin{bmatrix} 5 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix} x, \quad n_1 = 2 \tag{17}$$

is already in a block triangular form but with $\sigma(B_1) \neq \sigma_D$. For this system there is no L of (6) which achieves time scale separation (4) and (5). The reasons are the misplaced eigenvalues $\lambda = 4$, which is unobservable, and $\lambda = 2$, which is uncontrollable. It is obvious that the same system with reordered states $\tilde{x} = (x_1,x_4,x_2,x_3)^T$ has V_1 of (11) nonsingular and $L = 0$ satisfies (7). Although $\lambda = 2$ is still unobservable and $\lambda = 4$ is uncontrollable, they both belong to the appropriate spectrum and do not have to be moved.

THE SUBSPACE METHOD

The well documented globally convergent simultaneous iteration method for computing a basis of the dominant eigenspace [1] is now applied to solve for L in (11). It consists of a simple iterative scheme

$$V^{k+1} = AV^k. \tag{18}$$

Through iterations (18) initially given column vectors of V rotate until they form a basis of $\mathscr{W}$. As an initial guess almost any full rank matrix $V^0 \in R^{n \times n_1}$ can be used. The only restriction considered as mild [1], is that no column of V^0 is orthogonal to any of the left eigenvectors of A corresponding to σ_D. A matrix V^0 generated by the random number generator will almost always be admissible. However such a matrix may need many iterations (18) before it makes a basis for $\mathscr{W}$. It was found experimentally that with the ordering of states according to the preceding discussion a better initial condition is given by

$$V^0 = \begin{bmatrix} A_{11} \\ A_{21} \end{bmatrix}. \tag{19}$$

For numerical reasons at each iteration columns of V are scaled to have unit norms. Occasionally, an orthonormalization of V is also performed in order to retain a good basis for an n_1-dimensional subspace. Further discussion of the numerical aspects of (18) is contained in [1].

It has been shown in [1] that the subspace iterations have linear rate of convergence with the slowest converging column of V^k differing after k iterations from the corresponding vector in $\mathscr{W}$ by $O(\varepsilon^k)$. Thus by observing the speed of convergence of the columns one can decide whether to reduce the assumed n_1. After n_1 is reduced the iteration process continues with the remaining columns in V^k.

The convergence behavior of (18) is best monitored through the *error matrix*

$$E = AQ - QT. \tag{20}$$

The matrix Q in (20) is

$$Q = V^k \cdot U \tag{21}$$

where V^k is an orthonormal matrix from (18) and U is an orthonormal matrix transforming n_1-dimensional matrix $(V^k)^T AV^k$ to a quasi-upper-triangular* T,

$$U^T((V^k)^T AV^k)U = T. \tag{22}$$

Note that as V^k approaches a basis for $\mathscr{W}$, Q approaches the Schrer vector basis [1] for the same space and r_1 tends to zero. Furthermore if in each step diagonal blocks of T are ordered in descending order of eigenvalue magnitudes, then the first columns in Q tend faster to the basis vectors and correspondingly the first columns in E_1 tend faster to zero.

The computational load of the convergence monitoring is contained in computing the unitary transformation of the low order matrix of (22). Alternatively, when it is known beforehand that n_1 is admissible a convergence test using $\|R(L)\|$ computed from (8) can be used.

Due to linear convergence of (18), predicted by [1], the two consecutive tests of column norms in E can be used for several purposes.

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*Diagonal elements of T are 2x2 blocks containing complex eigenvalues and 1x1 blocks containing real eigenvalues of $(V^k)^T AV^k$.

First, to predict the number of iterations before each of the columns in Q falls below the specified tolerance. Second, to reduce n_1 if the slow convergence is predicted for some columns. Third, to remove from V^k the columns that satisfy (22). They are reconsidered again only during the orthonormalization.

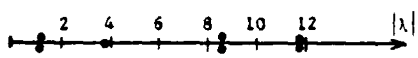
<u>Example</u>: In the following 7-th order power system example [6], we illustrate the speed of convergence of the subspace iterations by showing norm of columns in E as a function of iterations and eigenvalue separation ε. In (18) we use A^{-1} instead of A and hence problem formulation (6). From Figures 1b and 1c it is clear that smaller ε results in faster convergence. Figure 1d shows that for $\varepsilon = 1$ there is no convergence for some columns. The convergence behavior of Figure 1d would suggest to reduce n_1 from 4 to 3. Eigenvalues of A are given in figure 1a.

With V obtained from (18) the transformation

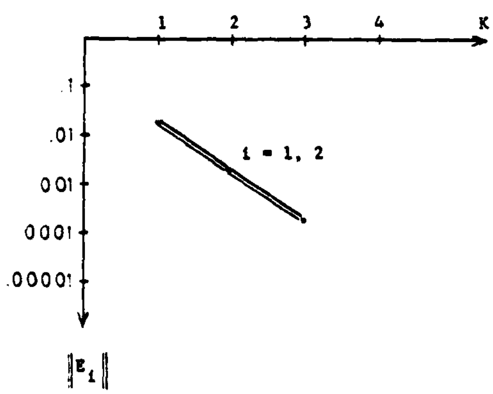
$$y_2 = x_2 - V_2 V_1^{-1} x_1 \tag{23}$$

due to Lemma 1, yields the time scale separation (4), (5). In order to have a well conditioned V_1 before the inversion an ordering of states (equivalent to row permutations of V) can be performed to make a norm of V_1 as large as possible. Then, as discussed earlier, resulting L will have a small norm.

The subspace iterations are particularly useful for large systems with sparse A. An advantage of sparsity can be taken while carrying out the recursion (18) and while storing A.

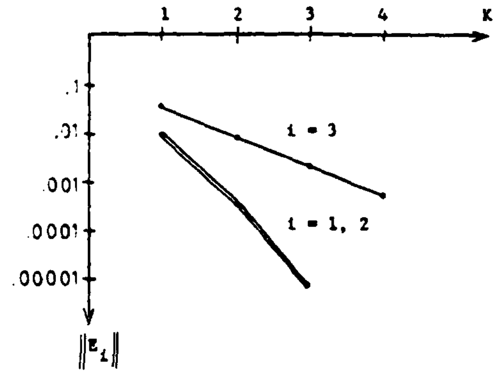


1a Eigenvalue Magnitudes of A



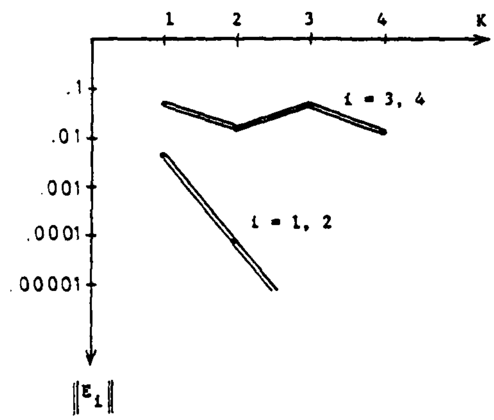
$n_1 = 2$, $\epsilon = 0.17$

(b)



$n_1 = 3$, $\epsilon = 0.47$

(c)



$n_1 = 4$, $\epsilon = 1$

(d)

Figures b, c, d: Convergence behavior of the subspace iterations.

<u>RELATION BETWEEN THE SUBSPACE AND THE RICCATI ITERATIONS</u>

We now establish a relation between the subspace iterations (18) and the Riccati iterations

$$L_{k+1} = L_k - R(L_k)(A_{11} - A_{12}L_k)^{-1} \tag{24}$$

where $R(L_k)$ is defined by (8). If the iteration (24) converge we will show that they yield the spectrum separation (5). Similar iterations were proposed in [2,3] for obtaining the separation (6). They are claimed to be locally convergent. Here we show that (24) is globally convergent. By using a similar approach the same can be proved for the algorithms in [2,3].

<u>Lemma 2</u>: Provided all indicated inverses exist, the sequences $-(V_2^k(V_1^k)^{-1})$ of the subspace algorithm (18) and L_k of the Riccati algorithm (24) are identical

$$L_k = -V_2^k(V_1^k)^{-1}. \tag{25}$$

Substituting $V_2^k = -P^k V_1^k$ into the partitioned form of (18) one gets, after some simple manipulations

$$P^{k+1}(A_{11} - A_{12}P_k^k) = A_{22}P^k - A_{21}. \tag{26}$$

Using $P^k = L_k$, where L_k satisfies (24), the equation (26) is identically satisfied for every k.

This lemma shows that both algorithms have the same speed of convergence. Their differences are in numerical conditioning, memory requirements, and ability to redefine n_1.

CONCLUDING REMARKS

We have considered the application of the subspace method and Riccati iterations for the time scale decomposition. It has been shown that both algorithms have global convergency property. They converge linearly with the corresponding error after k iterations being $O(\epsilon^k)$, where ϵ is a measure of spectrum separation (5). In light of the established connection between the two algorithms, the stringent requirements on the initial condition of Riccati iterations (required earlier) are replaced by the mild restrictions of the subspace iterations. Subspace method provides an opportunity to redefine n_1 when necessary by observing the convergence rate.

When initial ordering of state vector variables is such that x_1 contains physically fast variables and x_2 physically slow variables then an initial guess for V^o given by (19) is preferable over randomly generated one.

The subspace iterations are particularly useful for the time scale decomposition of large dynamical systems resulting in sparse system matrices A. In the case that the dimension of the slow subsystem is much smaller than the dimension of the fast subsystem, a modification of the subspace method is suggested:

- use A^{-1} instead of A in (18)
- order states so that x_1 contains slow variables and x_2 the fast ones.

These changes amount to using problem formulation (6) instead of (5) and result in less computer work.

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Preservation of controllability in linear time-invariant perturbed systems†

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The controllability of systems with weak connections is studied. A necessary and sufficient condition for a singularly perturbed system to be strongly controllable is obtained. The controllability invariance of the slow subsystem of a singularly perturbed system due to a class of fast feedback controls is shown.

1. Introduction

Systems with small parameters are common in control problems. These small parameters, with values proportional to a small positive number μ, represent weak connections or parasitics (Desoer and Shensa 1970). In networks, for example, they are the stray capacitances and lead inductances, which induce high and low frequency behaviour. In this paper the dependence of the controllability on μ is discussed for regularly perturbed systems (O'Malley 1974) where the system matrices are bounded for $\mu = 0$, and for singularly perturbed systems (O'Malley 1974, Kokotovic *et al.* 1976) where the system orders are reduced as $\mu \to 0$. It is shown that these systems may lose their controllability without weak connections and the loss of controllability is investigated by using Jordan forms. A necessary and sufficient condition for a singularly perturbed system to be 'strongly controllable' is obtained. Furthermore, the controllability of the slow subsystem of a singularly perturbed system is shown to be invariant to a class of fast feedback controls, and hence we can neglect the fast subsystem if it is stable. These results clarify those obtained by Kokotovic and Haddad (1975), Kokotovic and Yackel (1972) and Chow and Kokotovic (1976 b). The presentation in this paper is aimed at giving a structural interpretation of the controllability (Lin 1974) of perturbed systems.

2. Weakly and strongly controllable systems

Consider a linear time-invariant perturbed system

$$\dot{x} = A(\mu)x + B(\mu)u \tag{1}$$

where the state x is an n-vector, the control u an m-vector, μ a small positive parameter and $A(\mu)$, $B(\mu)$ are matrix polynomials in μ which are bounded at

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$\mu = 0$. System (1) is regularly perturbed (O'Malley 1974) and letting $\mu = 0$, that is, eliminating the weak connections, it becomes the unperturbed system

$$\dot{\bar{x}} = A\bar{x} + Bu \tag{2}$$

where $A = A(0)$, $B = B(0)$.

It is known (Lee and Markus 1967) that the set of all controllable pairs (A, B) of system (2) is open and dense, that is, if the pair (A, B) is controllable, then there exists a positive μ^* such that the pair $(A(\mu), B(\mu))$ is controllable for all $\mu \in [0, \mu^*)$. Here we show that controllability of the perturbed system (1) for $\mu > 0$ does not guarantee the controllability of the unperturbed system (2). A counter example is the system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} -3 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1-\mu & 0 \\ 0 & 0 & 0 & -1-2\mu \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} \mu \\ 1 \\ 1 \\ 1 \end{bmatrix} u \tag{3}$$

which is controllable for $\mu \in (0, 2)$, but is uncontrollable at $\mu = 0$.

Using a Jordan form transformation $\bar{x} = S(\mu)x$ with $S(\mu)$ nonsingular and bounded for μ small and non-negative, the perturbed system (1) becomes

$$\begin{aligned} \dot{\bar{x}} &= S(\mu)A(\mu)S^{-1}(\mu)\bar{x} + S(\mu)B(\mu)u \\ &= \begin{bmatrix} J_1(\mu) & & 0 \\ & J_2(\mu) & \\ 0 & & \ddots \; J_k(\mu) \end{bmatrix} \bar{x} + \begin{bmatrix} G_1(\mu) \\ G_2(\mu) \\ \vdots \\ G_k(\mu) \end{bmatrix} u \end{aligned} \tag{4}$$

where $J_i(\mu)$, $i = 1, 2, \ldots, k$, are Jordan blocks. Note that the unperturbed system of (4) is also the Jordan form of the unperturbed system (2) because $S(\mu)$ is continuous with respect to μ. Eliminating the weak connections in (4), the $0(\mu)$ elements in the matrices $G_i(\mu)$ will become zero, and if the eigenvalues of the Jordan blocks $J_p(\mu)$ and $J_q(\mu)$ differ only by $0(\mu)$, $J_p(0)$ and $J_q(0)$ will have the same eigenvalue. If the last rows of the matrices $G_i(0)$ corresponding to $J_i(0)$ having the same eigenvalue are linearly dependent, then the unperturbed system of (4) is not controllable (Chen 1970). For example in system (3), we have $G_1(0) = [0]$ and hence the state x_1 is uncontrollable. Furthermore, $J_i(0)$, $i = 2, 3, 4$ have the same eigenvalue and since $G_i(0)$, $i = 2, 3, 4$, are linearly dependent, the controllable subspace of the states x_2, x_3, x_4 is only the subspace $x_2 = x_3 = x_4$.

We define the perturbed system (1) to be 'weakly controllable' if it loses its controllability when the weak connections or parasitics are removed. In the terminology of Lin (1974), the unperturbed system (2) of a weakly controllable system (1) can be structurally controllable as it regains its controllability by a slight perturbation. On the other hand, we define system (1)

to be 'strongly controllable' if its controllability is maintained at $\mu = 0$. From this definition and that of Lee and Markus (1967), we conclude that the perturbed system (1) is strongly controllable if and only if the unperturbed system (2) is controllable.

A property of weakly controllable systems (1) is that controls with gains of the order of $1/\mu$ or higher are required for the placement of the weakly controllable eigenvalues. For example, placing the pole -3 of system (3) at -6, the control $u = -(1/\mu)x_1$ is required, while placing the poles -1, $-1-\mu$, $-1-2\mu$ at -2, -3, -4, the control

$$u = -\frac{3}{\mu^2}x_2 + \frac{(6 - 11\mu + 6\mu^2 - \mu^3)}{\mu^2}x_3 - \frac{(3 - 11\mu + 12\mu^2 - 4\mu^3)}{\mu^2}x_4 \tag{5}$$

is required (Mayne and Murdoch 1970). It is of practical importance that these undesirable situations involving large gains can be revealed by investigating the unperturbed system (2).

3. Singularly perturbed systems

In this section the controllability of a singularly perturbed system as $\mu \to 0$ is discussed with respect to its slow and fast subsystems (Chow and Kokotovic 1976 b) because its system matrix is unbounded at $\mu = 0$. A necessary and sufficient condition for a singularly perturbed system to be strongly controllable is obtained.

A linear time-invariant singularly perturbed system is modelled as

$$\dot{y} = A_{11}(\mu)y + A_{12}(\mu)z + B_1(\mu)u \tag{6 a}$$

$$\mu\dot{z} = A_{21}(\mu)y + A_{22}(\mu)z + B_2(\mu)u \tag{6 b}$$

where the states y, z are n_1, n_2 vectors, the control u an m-vector and μ a small positive parameter. $A_{ij}(\mu)$, $B_i(\mu)$, $i = 1, 2$, $j = 1, 2$, are matrix polynomials in μ where $A_{ij}(0)$, $B_i(0)$, which are denoted by A_{ij}, B_i, are bounded and A_{22} is non-singular.

We first define the slow and fast subsystems of the singularly perturbed system (6). It is known (Kokotovic and Haddad 1975) that system (6) possesses slow modes with n_1 small eigenvalues of magnitude $0(1)$ and fast modes with n_2 large eigenvalues of magnitude $0(1/\mu)$. Assuming that the transient of the fast modes is instantaneous, that is, letting $\mu = 0$, we obtain from (6) the reduced order system

$$\dot{y}_s = A_{11}y_s + A_{12}z_s + B_1u_s \tag{7 a}$$

$$0 = A_{21}y_s + A_{22}z_s + B_2u_s \tag{7 b}$$

where the subscript s denotes the slow modes. z_s can be solved from (7 b) and its substitution into (7 a) yields the slow subsystem

$$\dot{s} = A_0 s + B_0 u_s \tag{8}$$

where $s = y_s$, u_s is a control of slow variables only and

$$A_0 = A_{11} - A_{12}A_{22}^{-1}A_{21} \tag{9 a}$$

$$B_0 = B_1 - A_{12}A_{22}^{-1}B_2. \tag{9 b}$$

On the other hand, assuming that the slow modes are constant during the fast transient period and the perturbations in $A_{22}(\mu)$ and $B_2(\mu)$ are small, the fast subsystem is obtained from (7 b) as

$$\mu \dot{f} = A_{22} f + B_2 u_f \tag{10}$$

where f is the fast part in z and u_f is a control of fast variables only. For $\mu \neq 0$, the fast sysbsystem (10) is controllable if and only if the pair (A_{22}, B_2) is controllable.

Since the eigenvalues of A_0 and A_{22}/μ are far apart for μ sufficiently small, the following lemma holds.

Lemma 1

If A_{22} is non-singular and if the subsystems (8), (10) are controllable, then there exists a $\mu^* > 0$ such that the singularly perturbed system (6) is controllable for all $\mu \in (0, \mu^*]$.

Lemma 1 is known from the work of Kokotovic and Haddad (1975). Here we show that the controllability of the singularly perturbed system (6) for $\mu > 0$ does not necessarily require the controllability of the subsystems (8), (10). To illustrate this possibility, consider the system

$$\begin{bmatrix} \dot{y} \\ \mu \dot{z} \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \end{bmatrix} u \tag{11}$$

(shown in Fig. 1) which is controllable for $\mu \in (0, 1)$. For $\mu > 0$, the state z is not equal to the control u but is tracking it with a small time delay, and hence the state y is controllable. But letting $\mu \to 0$, the control for the state y is eliminated and the slow subsystem $\dot{s} = -s$ is uncontrollable. Note that the fast subsystem is controllable because there is a strong dynamic connection between the control u and the state z. This strong dynamic connection is different from the weak static connection in the perturbed system (1).

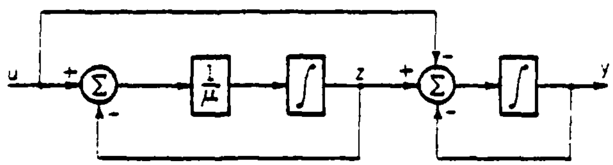

Figure 1. Block diagram of system (11).

Another example is the control of the fast subsystem through the slow subsystem. The system

$$\begin{bmatrix} \dot{y} \\ \mu \dot{z} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u \tag{12}$$

(shown in Fig. 2) is controllable for $\mu > 0$ but its fast subsystem $\mu \dot{f} = -f$ is uncontrollable. In the complete system (12), the connection between the control u and the state y is a strong connection by itself, but it acts as a slow filter such that the fast transient in the state z is weakly affected by the control u.

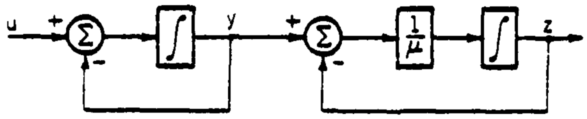

Figure 2. Block diagram of system (12).

Similar to the perturbed system (1), weak connections may also cause a singularly perturbed system (6) to lose its controllability as $\mu \to 0$. The system

$$\begin{bmatrix} \dot{y} \\ \mu\dot{z}_1 \\ \mu\dot{z}_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1-\mu \end{bmatrix} \begin{bmatrix} y \\ z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} u \tag{13}$$

is controllable for $\mu \in (0, 1)$. However, the state z_2 is connected to $\dot{z}_2$ through a gain of $-1/\mu - 1$ and the gain -1 is a weak connection because it is small compared to the gain $-1/\mu$ for μ close to zero. Hence the fast subsystem

$$\begin{bmatrix} \mu\dot{f}_1 \\ \mu\dot{f}_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u \tag{14}$$

is not controllable.

Although the strong dynamic connections in systems (11) and (12) are not weak connections by themselves, they create weak connections when combined with other parts of the systems, which can be revealed by a proper transformation of the singularly perturbed system (6). Since A_{22} is non-singular, there exists a transformation

$$\begin{bmatrix} \xi \\ \eta \end{bmatrix} = T(\mu) \begin{bmatrix} y \\ z \end{bmatrix} \tag{15}$$

with $T(\mu)$ bounded and non-singular for μ small, such that system (6) becomes

$$\begin{bmatrix} \dot{\xi} \\ \mu\dot{\eta} \end{bmatrix} = \begin{bmatrix} J_s(\mu) & 0 \\ 0 & J_f(\mu) \end{bmatrix} \begin{bmatrix} \xi \\ \eta \end{bmatrix} + \begin{bmatrix} G_s(\mu) \\ G_f(\mu) \end{bmatrix} u \tag{16}$$

where $J_s(\mu)$, $J_f(\mu)$ are composed of Jordan blocks and $J_s(0)$, $J_f(0)$, $G_s(0)$, $G_f(0)$ are bounded (Kokotovic and Haddad 1975). Note that system (16) is not the Jordan form of the singularly perturbed system (6) except for $\mu = 1$. Since $\|T(\mu)\| = O(1)$, the pairs $(J_s(0), G_s(0))$, $(J_f(0), G_f(0))$ can be obtained through transformations of the pairs (A_0, B_0), (A_{22}, B_2), respectively. Thus if the pairs $(J_s(\mu), G_s(\mu))$, $(J_f(\mu), G_f(\mu))$ maintain their controllability without weak connections, then the subsystems (3), (10) are controllable. Hence the study of the controllability of a singularly perturbed system for $\mu \to 0$ reduces to the study of weak connections in the pairs $(J_s(\mu), G_s(\mu))$ and $(J_f(\mu), G_f(\mu))$.

Consider the systems (11)–(13) in the form of (16). For system (11), $G_s(\mu) = [\mu/(1-\mu)]$ and hence its slow subsystem is uncontrollable as $G_s(0) = [0]$. For system (12), $G_f(\mu) = [-\mu/(1-\mu)]$ and hence its fast subsystem is uncontrollable as $G_f(0) = [0]$. Since the eigenvalues are the same in

$$J_f(\mu) = \begin{bmatrix} -1 & 0 \\ 0 & -1-\mu \end{bmatrix} \tag{17}$$

at $\mu = 0$, the controllable subspace of the single-input fast subsystem of (13) is only the subspace $f_1 = f_2$.

Similar to a perturbed system (1), we define the singularly perturbed system (6) to be weakly controllable if it loses its controllability as $\mu \to 0$, and strongly controllable if it maintains its controllability as $\mu \to 0$. Combining this definition and Lemma 1, the following theorem is immediate.

Theorem 1

The singularly perturbed system (6) is strongly controllable if and only if its subsystems (8), (10) are controllable.

Note that the placement of weakly controllable eigenvalues in singularly perturbed systems (6) requires controls with gains of the order of $1/\mu$ or higher. For example, placing the pole -1 of system (11) at -2, the control

$$u = (1 - 1/\mu)y - z \tag{18}$$

is required. Hence our definitions of weak controllability for the perturbed system (1) and the singularly perturbed system (6) are consistent in this sense.

The above discussion of weak and strong controllability of singularly perturbed systems (6) complete the results presented by Kokotovic and Haddad (1975). It is important to note that the separation of designs proposed by Chow and Kokotovic (1976a, 1976b) for the approximate designs of singularly perturbed systems (6) is applicable only when they are strongly controllable. In addition to saving numerical computation, this method avoids dealing with the ill-conditioned system matrices of (6). For weakly controllable singularly perturbed systems, exact designs are possible, but their computations are often numerically unstable. Hence in practice, these systems are undesirable.

4. Invariance of slow subsystem controllability

Examining the singularly perturbed system (6) and its slow and fast subsystems (8), (10), we observe that to change the dynamics of the subsystem (10), we need the fast feedback control

$$u = v + Fz \tag{19}$$

where $\|F\| = 0(1)$ and $(A_{22} + B_2F)$ is non-singular. System (6) controlled by (19) becomes

$$\dot{y} = A_{11}(\mu)y + [A_{12}(\mu) + B_1(\mu)F]z + B_1(\mu)v \tag{20 a}$$

$$\mu\dot{z} = A_{21}(\mu)y + [A_{22}(\mu) + B_2(\mu)F]z + B_2(\mu)v \tag{20 b}$$

and its slow and fast subsystems are

$$\dot{s} = A_0^* s + B_0^* v_s \tag{21}$$

$$\mu \dot{f} = (A_{22} + B_2 F) f + B_2 v_f \tag{22}$$

respectively, where

$$A_0^* = A_{11} - (A_{12} + B_1 F)(A_{22} + B_2 F)^{-1} A_{21} \tag{23 a}$$

$$B_0^* = B_1 - (A_{12} + B_1 F)(A_{22} + B_2 F)^{-1} B_2. \tag{23 b}$$

Since the state z contains a slow part as well as a fast part, the fast control (19) also changes the slow subsystem from (8) to (21) whose dynamics is very much different from that of (8). However, the controlabillity of the slow subsystem of (6) is preserved with the fast control (19).

Theorem 2

If A_{22} and $(A_{22} + B_2 F)$ are non-singular, then system (21) is controllable if and only if system (8) is controllable. Furthermore, if $\|F\| = 0(1)$, the slow subsystem controllability of the singularly perturbed system (6) is invariant to the class of fast controls (19).

Proof

For system (8) we construct a non-singular transformation M of the control u_s, such that the new control is $w = M^{-1} u_s$, and then introduce a partial feedback $w = v_s + Ns$, such that system (8) becomes

$$\dot{s} = (A_0 + B_0 M N) s + B_0 M v_s. \tag{24}$$

Let $M = I - F(A_{22} + B_2 F)^{-1} B_2$ such that $M^{-1} = I + F A_{22}^{-1} B_2$. Then using the identity

$$A_{22}^{-1}[I - B_2 F(A_{22} + B_2 F)^{-1}] = (A_{22} + B_2 F)^{-1} \tag{25}$$

we obtain

$$\begin{aligned} B_0 M &= (B_1 - A_{12} A_{22}^{-1} B_2)[I - F(A_{22} + B_2 F)^{-1} B_2] \\ &= B_1 - A_{12} A_{22}^{-1}[I - B_2 F(A_{22} + B_2 F)^{-1}] B_2 - B_1 F(A_{22} + B_2 F)^{-1} B_2 \\ &= B_0^*. \end{aligned} \tag{26}$$

Let $N = -F A_{22}^{-1} A_{21}$. Then

$$\begin{aligned} A_0 + B_0 M N &= A_0 + B_0^* N \\ &= A_{11} - (A_{12} + B_1 F) A_{22}^{-1} A_{21} + (A_{12} + B_1 F) \\ &\qquad \times (A_{22} + B_2 F)^{-1} B_2 F A_{22}^{-1} A_{21} \\ &= A_{11} - (A_{12} + B_1 F)[I - (A_{22} + B_2 F)^{-1} B_2 F] A_{22}^{-1} A_{21} \\ &= A_0^*. \end{aligned} \tag{27}$$

Hence system (24) reduces to system (21). Since the transformation $w = M^{-1} u_s$ and the partial feedback control $w = v_s + Ns$ do not change the controllable subspace of system (8), the columns of $(B_0, A_0 B_0, \ldots, A_0^{n_1 - 1} B_0)$ and $(B_0^*, A_0^* B_0^*, \ldots, A_0^{*n_1 - 1} B_0^*)$ span the same subspace, and the first part of the theorem is proved. In addition, if $\|F\| = 0(1)$, then system (21) is the slow subsystem of the singularly perturbed system (20), and hence the controllability of the slow subsystem of (6) with feedback (19) is preserved.

Theorem 2 shows that to determine whether the slow subsystem of (6) is controllable or not, we only have to test the controllability of the pair (A_0^*, B_0^*) for a single value of F. In addition, we can arbitrarily place the fast poles $(A_{22}+B_2F)/\mu$ without affecting the controllability of the slow subsystem. If $(A_{22}+B_2F)$ is stable, we can neglect the fast subsystem because we are able to control the slow subsystem for any disturbances of the type (19).

Theorem 2 also clarifies the issue that different sufficient conditions are obtained by Kokotovic and Yackel (1972) and Chow and Kokotovic (1976 b) for the existence of the near-optimal solution to a singularly perturbed regulator which is decomposed into a slow and a fast regulator. In the paper by Kokotovic and Yackel (1972) the fast regulator problem is solved at the first stage and then the slow regulator problem is solved at the second stage which requires the controllability of the pair (A_0^*, B_0^*). In the work of Chow and Kokotovic (1976 b), due to the separation of designs, the solution of the slow regulator problem requires only the controllability of the pair (A_0, B_0). By Theorem 2, these conditions are equivalent.

5. Conclusion

It is shown that a perturbed system will lose its controllability without weak connections if it is weakly controllable. Pole placement of such weakly controllable systems requires controls with large gains. A necessary and sufficient condition for a singularly perturbed system to be strongly controllable is the controllability of its slow and fast subsystems. In addition, the controllability of its slow subsystem is invariant to a class of fast feedback controls.

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*Uniform Asymptotic Stability of Linear Time-Varying Singularly Perturbed Systems**

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ABSTRACT: *An upper bound for the singular perturbation parameter is found for the uniform asymptotic stability of singularly perturbed linear time-varying systems.*

I. Introduction

System (1)

$$\begin{aligned} \dot{x} &= A_{11}(t)x + A_{12}(t)z \quad x(t_0) = x_0 \\ \mu\dot{z} &= A_{21}(t)x + A_{22}(t)z \quad z(t_0) = z_0 \end{aligned} \tag{1}$$

where x and z are n- and m-dimensional vectors respectively, μ is a small positive scalar and t_0 is any initial time, is referred to as a singularly perturbed system. The "reduced" system

$$\dot{\bar{x}} = (A_{11}(t) - A_{12}(t)A_{22}(t)^{-1}A_{21}(t))\bar{x} \triangleq A_1(t)\bar{x} \quad \bar{x}(t_0) = x_0 \tag{2}$$

is a singular perturbation of (1) resulting from setting $\mu = 0$ and is here assumed to be uniformly asymptotically stable.

We make the following hypotheses concerning (1).

(H1) The matrices $A_{ij}(t)i$, $j = 1, 2$ are bounded and have bounded first derivatives for all t.

(H2) The eigenvalues of $A_{22}(t)$ satisfy $\mathrm{Re}(\lambda_i(t)) < -\gamma < 0$ for all t where γ is a constant.

It has been shown (1), (2) that under H1 and H2 where system (2) is assumed uniformly asymptotically stable, system (1) is uniformly asymptotically stable for μ sufficiently small. Under these hypotheses this paper finds a bound μ^* such that for $\mu \in (0, \mu^*)$, system (1) is uniformly asymptotically stable.

It is well known (3) that for μ sufficiently small, a condition for the motion $\bar{x}(t)$ of (2) to be an $0(\mu)$ approximation of $x(t)$ of (1), is the uniform asymptotic stability of the "fast" subsystem

$$\mu\dot{\bar{z}} = A_{22}(t)\bar{z}. \tag{3}$$

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Thus it is of interest to be able to determine the stability of (3). Clearly H1 and H2 alone are not enough for (3) to be stable. The existence of a μ_0 such that when H1 and H2 are satisfied and $\mu \in (0, \mu_0)$, system (3) is uniformly asymptotically stable has been shown (4). From Theorem 12 of [(6), p. 117] a bound for the stability of (3) can be obtained. Another bound is found here in Theorem I which is less conservative for a wide class of systems.

Before stating the main results of this paper several facts should be presented. First, under H1,

$$|A_{22}(t) - A_{22}(t_0)| \le \beta(t - t_0) \tag{4}$$

where β is a positive constant equal to the maximum of $\dot{A}_{22}(t)$ for all t by the mean value theorem. Also for $t \ge t_0$ there exists a K such that

$$\left|\exp\left(A_{22}(t_0)\frac{t-t_0}{\mu}\right)\right| \le K \exp\left(-\gamma\left(\frac{t-t_0}{\mu}\right)\right) \tag{5}$$

when H2 is satisfied (7).

Let $\Phi_{22}(t, t_0)$ be the state transition matrix of (3) and define $\varphi(t, t_0)$

$$\varphi(t, t_0) \triangleq \Phi_{22}(t, t_0) - \exp\left(A_{22}(t_0)\left(\frac{t-t_0}{\mu}\right)\right). \tag{6}$$

Lemma 1. Assume H1 and H2 are satisfied and $\hat{\mu} = \alpha^2/\beta K$ where $0 < \alpha < \gamma$. Then for $\mu \in (o, \hat{\mu})$, $\varphi(t, t_0)$ possesses the properties

$$\varphi(t_0, t_0) = 0 \tag{7}$$

$$|\varphi(t, t_0)| \le 2\frac{\mu K^2\beta}{e^2(\alpha^2 - \mu K\beta)} \exp\left(-\sigma\left(\frac{t-t_0}{\mu}\right)\right) \tag{8}$$

where $\sigma = \gamma - \alpha > 0$.

This lemma which is proved in the next section gives an estimate for the error $|\varphi(t, t_0)|$ which results from using $\exp(A_{22}(t_0)((t-t_0)/\mu))$ to approximate $\Phi_{22}(t, t_0)$. Thus for $\mu \in (0, \hat{\mu})$ system (3) is uniformly asymptotically stable. That is, as $\mu \to 0$, $\Phi_{22}(t, t_0) \to \exp(A_{22}(t_0)((t-t_0)/\mu))$ and we may approximate the solution of (3) by the solution to the time-invariant system

$$\mu\dot{\hat{z}} = A_{22}(t_0)\hat{z} \quad \hat{z}(t_0) = z_0 \tag{9}$$

obtained from (3) by fixing $A_{22}(t)$ at t_0. Since $A_{22}(t_0)$ is a constant matrix we can always solve for $\exp(A_{22}(t_0)((t-t_0)/\mu))$ whereas it is often difficult to find a closed form solution for $\Phi_{22}(t, t_0)$.

The upper bound on α in Lemma 1 is γ and consequently we state Theorem I which is proved in the next section.

Theorem I

Assume H1 and H2 are satisfied and $\mu_0 = \gamma^2/\beta K$. Then for $\mu \in (0, \mu_0)$ system (3) is uniformly asymptotically stable.

If we set

$$\bar{\gamma} = \gamma/\mu, \quad \bar{\beta} = \beta/\mu \tag{10}$$

and

$$\bar{A}_{22}(t) = \frac{1}{\mu}A_{22}(t) \tag{11}$$

we obtain the time-varying system

$$\dot{w} = \bar{A}_{22}(t)w \quad w(t_0) = z_0 \tag{12}$$

from (3). From Theorem I for $\bar{\beta} < \bar{\gamma}^2/K$, (12) is uniformly asymptotically stable. The bound obtained for (12) in [(6), p. 117] is $\bar{\beta} < \gamma^{-2}/K \ln K$. Thus for systems where $K > e$ the bound obtained in Theorem I is less conservative. It is interesting to note the correspondence between small μ in (3) and slow-varying matrices $\bar{A}_{22}(t)$ in (12).

In the proof of the next theorem we treat the transformed system

$$\begin{aligned} \dot{x} &= A_1(t)x + A_{12}(t)\eta \\ \mu\dot{\eta} &= \mu(\dot{L}(t) + L(t)A_1(t))x + A_{22}(t)\eta + \mu L(t)A_{12}(t)\eta \end{aligned} \tag{13}$$

which is a result of applying the transformation

$$\eta = z + A_{22}(t)^{-1}A_{21}(t)x \triangleq z + L(t) \tag{14}$$

to (1). Here A_1 is as defined in (2).

Theorem II

Let (2) and (3) be uniformly asymptotically stable systems so that their state transition matrices satisfy (15) and (16) respectively

$$\begin{gathered} |\Phi_1(t, t_0)| \le K_1 \exp(-\sigma_1(t - t_0)) \\ Vt_0,\ t \ge t_0. \end{gathered} \tag{15}$$

$$|\Phi_{22}(t, t_0)| \le K_2 \exp\left(-\sigma_2\left(\frac{t - t_0}{\mu}\right)\right). \tag{16}$$

If constants M_1, M_2, and M_3 exist such that for all t

$$|A_{12}(t)| \le M_1, \quad |L(t)A_{12}(t)| \le M_2, \quad |\dot{L}(t) + L(t)A_1(t)| \le M_3, \tag{17}$$

then for all $\mu \in (0, \mu_1)$, where $\mu_1 = \sigma_1\sigma_2/(\sigma_1 K_2 M_2 + K_1 M_1 K_2 M_3)$, system (1) is uniformly asymptotically stable.

The new result in this theorem is the explicit expression for μ_1. For linear time-invariant systems (5, 8, 10), Zien (10) obtains a bound for μ which when integrated with (15), (16), and (17) is μ_1.

The next corollary follows immediately from Lemma 1 and Theorem II.

Corollary 1. Let $\mu^* = \min(\bar{\mu}, \mu_1)$. If system (2) is uniformly asymptotically stable, then H1 and H2 guarantee that for $\mu \in (0, \mu^*)$, system (1) is uniformly asymptotically stable.

In (1) and (2) the existence of μ^* is shown via Lyapunov functions.

The new results of this paper are the explicit bounds μ_0, μ_1, and μ^* and the expression bounding $|\varphi(t, t_0)|$. Section II contains the proofs of Lemma 1 and Theorems I and II, and Section III gives an example.

II. Proofs

In the proofs we will use the following lemma.

Gronwall's Lemma (6): Let $\lambda(t)$ be a real continuous function and $\gamma(t)$ a non-negative continuous function on the interval $[t_0, t_1]$. If a continuous function $y(t)$ has the property that

$$y(t) \le \lambda(t) + \int_{t_0}^{t} \gamma(s) y(s)\, \mathrm{d}s \tag{18}$$

for $t_0 \le t \le t_1$, then on the same interval

$$y(t) \le \lambda(t) + \int_{t_0}^{t} \lambda(s)\gamma(s) \exp\left(\int_{s}^{t} \gamma(t)\, \mathrm{d}t\right) \mathrm{d}s. \tag{19}$$

Proof of Lemma 1:

The definition of $\varphi(t, t_0)$ implies (20) and (21),

$$\varphi(t_0, t_0) = 0 \tag{20}$$

$$\dot{\varphi}(t, t_0) = \frac{A_{22}(t)}{\mu}\varphi(t, t_0) + \frac{A_{22}(t) - A_{22}(t_0)}{\mu} \exp\left(A_{22}(t_0)\left(\frac{t - t_0}{\mu}\right)\right). \tag{21}$$

Applying the variation-of-constants formula to (21), we obtain

$$\begin{aligned}\varphi(t, t_0) = \frac{1}{\mu}\int_{t_0}^{t} \exp\left(A_{22}(t_0)\left(\frac{t-\tau}{\mu}\right)\right)(A_{22}(\tau) \\ - A_{22}(t_0)) \exp\left(A_{22}(t_0)\left(\frac{\tau - t_0}{\mu}\right)\right) \mathrm{d}\tau \\ + \frac{1}{\mu}\int_{t_0}^{t} \varphi(t, \tau)(A_{22}(\tau) - A_{22}(t_0)) \exp\left(A_{22}(t_0)\left(\frac{\tau - t_0}{\mu}\right)\right) \mathrm{d}\tau.\end{aligned} \tag{22}$$

We let $\gamma = \alpha + \sigma$, multiply (22) through by $\exp(\sigma((t - t_0)/\mu))$ and let $\eta(t, t_0) = \exp(\sigma((t - t_0)/\mu))\, \varphi(t, t_0)$ to yield:

$$\begin{aligned}\eta(t, t_0) = \frac{1}{\mu}\int_{t_0}^{t} \exp\left(\sigma\left(\frac{t - t_0}{\mu}\right)\right) \exp\left(A_{22}(t_0)\left(\frac{t-\tau}{\mu}\right)\right)(A_{22}(\tau) \\ - A_{22}(t_0)) \exp\left(A_{22}(t_0)\left(\frac{\tau - t_0}{\mu}\right)\right) \mathrm{d}\tau - \frac{1}{\mu}\int_{t_0}^{t} \eta(t, \tau) \exp\left(\sigma\left(\frac{\tau - t_0}{\mu}\right)\right) \\ \times (A_{22}(\tau) - A_{22}(t_0)) \exp\left(A_{22}(t_0)\left(\frac{\tau - t_0}{\mu}\right)\right) \mathrm{d}\tau.\end{aligned} \tag{23}$$

We next construct the successive approximation

$$\eta^{(k+1)}(t, t_0) = \frac{1}{\mu}\int_{t_0}^{t} \exp\left(\sigma\left(\frac{t-t_0}{\mu}\right) \exp\left(A_{22}(t_0)\left(\frac{t-\tau}{\mu}\right)\right)(A_{22}(\tau)\right.$$

$$- A_{22}(t_0)) \exp\left(A_{22}(t_0)\left(\frac{\tau-t_0}{\mu}\right)\right) \mathrm{d}\tau + \frac{1}{\mu}\int_{t_0}^{t} \eta^{(k)}(t, \tau) \exp\left(\sigma\left(\frac{\tau-t_0}{\beta}\right)\right)$$

$$\times (A_{22}(\tau) - A_{22}(t_0)) \exp\left(A_{22}(t_0)\left(\frac{\tau-t_0}{\mu}\right)\right) \mathrm{d}\tau \quad (24)$$

with initial guess $\eta^{(0)}(t, t_0) = 0$. The initial guess corresponds to the assumption that for μ small (3) has a solution near to that of the time-invariant system (9). Substituting (4) and (5) into (24) and integrating for $\eta^{(1)}(t, t_0)$, we obtain

$$|\eta^{(1)}(t, t_0)| \le \frac{\mu K^2 \beta}{2} \exp\left(-\sigma\left(\frac{t-t_0}{\mu}\right)\right)\left(\frac{t-t_0}{\mu}\right)^2 \le 2\frac{\mu K^2\beta}{\alpha^2 e^2} \quad (25)$$

for all t, t_0, $t \ge t_0$. Taking the difference between two successive terms for η, we obtain

$$\eta^{(k+1)}(t, t_0) - \eta^{(k)}(t, t_0) = \frac{1}{\mu}\int_{t_0}^{t} |\eta^{(k)}(t, \tau) - \eta^{(k-1)}(t, \tau)|$$

$$\times \exp\left(\sigma\left(\frac{\tau-t_0}{\mu}\right)\right)(A_{22}(\tau) - A_{22}(t_0)) \exp\left(A_{22}(t_0)\left(\frac{\tau-t_0}{\mu}\right)\right) \mathrm{d}\tau. \quad (26)$$

Substituting in (4) and (5), yields

$$|\eta^{(k+1)}(t, t_0) - \eta^{(k)}(t, t_0)|$$

$$\le \int_{t_0}^{t} |\eta^{(k)}(t, \tau) - \eta^{(k-1)}(t, \tau)|\, \beta K\left(\frac{\tau-t_0}{\mu}\right) \exp\left(-\alpha\left(\frac{\tau-t_0}{\mu}\right)\right) \mathrm{d}\tau. \quad (27)$$

Suppose for $k \le p$

$$|\eta^{(k)}(t, \tau) - \eta^{(k-1)}(t, \tau)| \le C^{(k)} \quad (28)$$

where $C^{(k)}$ are constants. Then by (27)

$$|\eta^{(k+1)}(t, t_0) - \eta^{(k)}(t, t_0)| \le \frac{\mu\beta K}{\alpha^2} C^{(k)} \quad (29)$$

for all t, t_0 and $t \ge t_0$. Since for $k = 1$

$$|\eta^{(k)}(t, \tau) - \eta^{(k-1)}(t, \tau)| \le 2\frac{\mu K^2\beta}{\alpha^2 e^2}, \quad (30)$$

we have by induction

$$|\eta^{(k+1)}(t, t_0) - \eta^{(k)}(t, t_0)| \le 2\left(\frac{\mu\beta K}{\alpha^2}\right)^k \left(\frac{\mu K^2\beta}{\alpha^2 e^2}\right). \quad (31)$$

Define $\rho = \mu\beta K/\alpha^2$. Since

$$|\eta^{(k)}(t,t_0) - \eta^{(k-1)}(t,t_0)| \le \sum_{j=1}^{k} |\eta^{(j)}(t,t_0) - \eta^{(j)}(t,t_0)|$$

$$\le 2(\rho^{k-1} + \ldots + \rho + 1)\left(\frac{\mu K^2\beta}{\alpha^2 e^2}\right) = 2\left(\frac{1-\rho^k}{1-\rho}\right)\left(\frac{\mu K^2\beta}{\alpha^2 e^2}\right) \quad (32)$$

then for $\rho < 1$ or $\mu < \alpha^2/\beta K$

$$\lim_{k\to\infty} |\eta^{(k)}(t,t_0)| = \lim_{k\to\infty} |\eta^{(k)}(t,t_0) - \eta^{(0)}(t,t_0)| \le \frac{2}{1-\rho}\left(\frac{\mu K^2\beta}{\alpha^2 e^2}\right) = 2\frac{\mu K^2\beta}{e^2(\alpha^2 - \mu\beta K)}. \quad (33)$$

Thus for $\mu < \alpha^2/\beta K$ the successive approximation (24) converges to a solution which satisfies

$$|\eta(t,t_0)| \le 2\frac{\mu K^2\beta}{e^2(\alpha^2 - \mu\beta K)}. \quad (34)$$

Now $\exp \sigma(t - t_0/\mu))\ \varphi(t,t_0) = \eta(t,t_0)$ and therefore

$$|\varphi(t,t_0)| \le \frac{\mu 2K^2\beta}{e^2(\alpha^2 - \mu\beta K)} \exp\left(-\sigma\left(\frac{t-t_0}{\mu}\right)\right). \quad (35)$$

This completes the proof of Lemma 1.

Proof of Theorem I: Define $\mu_0 = \gamma^2/\beta K$ and $\alpha = \frac{1}{2}\sqrt{(\mu\beta K)} + \gamma/2$ where $\mu \in (0, \mu_0)$. Then $\sigma = \gamma - \alpha > 0$ and for $\mu \in (0, \mu_0)$ equation (35) implies that (3) is uniformly asymptotically stable since the definition of α implies that $\alpha^2 - \mu K\beta$ is never equal to zero.

Proof of Theorem II: Applying the variation-of-constants formula to (13), we obtain

$$x(t) = \Phi_1(t,t_0)x_0 + \int_{t_0}^{t} \Phi_1(t,\tau)A_{12}(\tau)\eta(\tau)\,d\tau$$

$$\eta(t) = \Phi_{22}(t,t_0)\eta_0 + \int_{t_0}^{t} \Phi_{22}(t,\tau)L(\tau)A_{12}(\tau)\eta(\tau)\,d\tau \quad (36)$$

$$+ \int_{t_0}^{t} \Phi_{22}(t,\tau)(L(\tau) + L(\tau)A_1(\tau))x(\tau)\,d\tau$$

where $\eta_0 = z_0 + A_{22}^{-1}(t_0)A_{21}(t_0)x_0$. The bounds of Eqs. (15), (16), and (17) imply

$$|x(t)| \le K_1 \exp(-\sigma_1(t-t_0))|x_0| + \int_{t_0}^{t} K_1 \exp(-\sigma_1(t-\tau))M_1|\eta(\tau)|\,d\tau \quad (37)$$

$$|\eta(t)| \leq K_2 \exp\left(-\sigma_2\left(\frac{t-t_0}{\mu}\right)\right)|\eta_0| + \int_{t_0}^{t} K_2 \exp\left(-\sigma_2\left(\frac{t-\tau}{\mu}\right)\right)M_2\,|\eta(\tau)|\,\mathrm{d}\tau$$

$$+ \int_{t_0}^{t} K_\epsilon \exp\left(-\sigma_2\left(\frac{t-\tau}{\mu}\right)\right)M_3\,|x(\tau)|\,\mathrm{d}\tau. \quad (38)$$

In this proof we apply Gronwall's Lemma to (37) and then to (36) to derive the upper bound μ_1 such that for $\mu \in (0, \mu_1)$, $|x(t)|$ and $|\eta(t)|$ are bounded by a decreasing exponential. Letting $w(t) = \exp(\sigma_2 t/\mu)\,|\eta(t)|$ in Eq. (38), yields

$$w(t) \leq K_2 \exp(\sigma_2 t_0/\mu)\,|\eta_0| + \int_{t_0}^{t} K_2 \exp(\sigma_2 \tau/\mu) M_3\,|x(\tau)|\,\mathrm{d}\tau + \int_{t_0}^{t} K_2 M_2 w(\tau)\,\mathrm{d}\tau. \quad (39)$$

Applying Gronwall's Lemma and integrating, we obtain

$$w(t) \leq K_2 \exp(\sigma_2 t_0/\mu)\,|\eta_0| \exp(K_2 M_2(t-t_0))$$

$$+ \int_{t_0}^{t} \exp(K_2 M_2(t-\tau)) K_2 \exp(\sigma_2 \tau/\mu) M_3\,|x(\tau)|\,\mathrm{d}\tau \quad (40)$$

which yields

$$|\eta(t)| \leq K_2 \exp(-\sigma_3(t-t_0))\,|\eta_0| + \int_{t_0}^{t} K_2 M_3 \exp(-\sigma_3(t-\tau))\,|x(\tau)|\,\mathrm{d}\tau \quad (41)$$

where $\sigma_3 = \sigma_2/\mu - K_2 M_2$. In the following we will need $\sigma_3 > 0$.

Substituting (41) into (37), yields

$$|x(t)| \leq K_1 \exp(-\sigma_1(t-t_0))\,|x_0|$$

$$+ \int_{t_0}^{t} K_1 \exp(-\sigma_1(t-\tau)) M_1 K_2\,|\eta_0| \exp(-\sigma_3(\tau - t_0))\,\mathrm{d}\tau$$

$$+ \int_{t_0}^{t} K_1 \exp(-\sigma_1(t-\tau)) M_1 \left(\int_{t_0}^{t} K_2 M_3 \exp(-\sigma_3(t-s))\,|x(s)|\,\mathrm{d}s\right)\mathrm{d}\tau \quad (42)$$

which implies

$$|x(t)| \leq K_1\,|x_0| - \frac{K_1 M_1 K_2\,|\eta_0|}{\sigma_3 - \sigma_1} \exp(-\sigma_1(t-t_0)) + \frac{K_1 M_1 K_2 |\eta_0|}{\sigma_3 - \sigma_1}$$

See corrections on the next page

$$\times \exp(-\sigma_3(t-t_0)) + \frac{K_1 M_1 K_2 M_3}{\sigma_1} \int_{t_0}^{t} \exp(-\sigma_3(t-\tau))\,|x(\tau)|\,\mathrm{d}\tau. \quad (43)$$

Let $y(t) = \exp \sigma_3 t\,|x(t)|$, apply Gronwall's Lemma and integrate to obtain

$$
\begin{aligned}
|x(t)| \leq & \frac{K_1 M_1 K_2 |\eta_0|}{\sigma_3 - \sigma_1}\left(1 - \frac{\sigma_1}{K_1 M_1 K_2 M_3}\right) \exp\left(-\sigma_3(t - t_0)\right) \\
& + \left(K_1 |x_0| - \frac{K_1 M_1 K_2 |\eta_0|}{\sigma_3 - \sigma_1}\right)\left(1 + \frac{1}{\sigma}\right) \exp\left(-\sigma_1(t - t_0)\right) + \left(\frac{\sigma_1 |\eta_0|}{M_3(\sigma_3 - \sigma_1)}\right. \\
& \left. - \frac{1}{\sigma}\left(K_1 |\eta_0| - \frac{K_1 M_1 K_2 |\eta_0|}{\sigma_3 - \sigma_1}\right)\right) \exp\left(\frac{K_1 M_1 K_2 M_2}{\sigma_1} - \sigma_3(t - t_0)\right) \qquad (44)
\end{aligned}
$$

where

$$\sigma = \sigma_3 - \sigma_1 - \frac{K_1 M_1 K_2 M_3}{\sigma_1}. \qquad (45)$$

Thus for (13) to be uniformly asymptotically stable, we need inequalities (46) and (47) to be satisfied, i.e.

$$\sigma_3 = \frac{\sigma_2}{\mu} - K_2 M_2 > 0 \qquad (46)$$

$$\sigma_3 - \frac{K_1 M_1 K_2 M_3}{\sigma_1} > 0. \qquad (47)$$

Let

$$\mu_1 = \frac{\sigma_1 \sigma_2}{\sigma_1 K_2 M_2 + K_1 M_1 K_2 M_3}.$$

If $\mu \in (0, \mu_1)$ inequalities (46) and (47) are satisfied and therefore (13) is uniformly asymptotically stabie which implies that (1) is uniformly asymptotically stable.

III. Example

The system

$$
\begin{bmatrix} \dot{x} \\ \mu \dot{z}_1 \\ \mu \dot{z}_2 \end{bmatrix} = \begin{bmatrix} -4 + \cos t & 1 & 0 \\ 1 & -1 + 1.1\cos^2 t & 1 - 1.1 \sin t \cos t \\ 0 & -1 - 1.1 \sin t \cos t & -1 + 1.1 \sin^2 t \end{bmatrix} \begin{bmatrix} x \\ z_1 \\ z_2 \end{bmatrix} \qquad (48)
$$

has the reduced system

$$\dot{\bar{x}} = (-3.474 + \cos t - 1.[illegible] \sin^2 t)\bar{x} \qquad (49)$$

and fast subsystem

$$\mu \dot{\eta} = \begin{bmatrix} -1 + 1.1\cos^2 t & 1 - 1.1 \sin t \cos t \\ -1 - 1.1 \sin t \sin t \cos t & -1 + 1.1 \sin^2 t \end{bmatrix} \eta. \qquad (50)$$

When $\mu = 1$ an unstable fundamental solution of (50) is

$$\Phi_2(t, 0) = \begin{bmatrix} e^{0.1t} \cos t & e^{-t} \sin t \\ -e^{0.1t} \sin t & e^{-t} \cos t \end{bmatrix} \qquad (51)$$

CORRECTIONS OF EXPRESSIONS (42) through (47)

In (42) variable τ was mistakenly taken as t. Instead, (42) should read

$$|x(t)| \le K_1 e^{-\sigma_1(t-t_o)}|x_o| + \int_{t_o}^{t} K_1 e^{-\sigma_1(t-\tau)} M_1 K_2 |\eta_o| e^{-\sigma_3(\tau-t_o)} d\tau$$

$$+\int_{t_o}^{t} K_1 e^{-\sigma_1(t-\tau)} M_1 \left(\int_{t_o}^{\tau} K_2 M_3 e^{-\sigma_3(\tau-s)} |x(s)| ds\right) d\tau.$$

Replace (43)-(47) by

$$|x(t)| \le \left\{K_1|x_o| - \frac{K_1M_1K_2|\eta_o|}{\sigma_3-\sigma_1}\right\} e^{-\sigma_1(t-t_o)} + \frac{K_1M_1K_2|\eta_o|}{\sigma_3-\sigma_1} e^{-\sigma_3(t-t_o)}$$

$$+ K_1K_2M_1M_3\, e^{-\sigma_1 t}\left[\int_{t_o}^{t} e^{(\sigma_1-\sigma_3)\tau}\left(\int_{t_o}^{\tau} e^{\sigma_3 s}|x(s)| ds\right) ds.\right. \tag{43}$$

Integrating by parts (47) becomes

$$|x(t)| \le P_1 e^{-\sigma_1(t-t_o)} + P_2 e^{-\sigma_3(t-t_o)} + P_3 \int_{t_o}^{t} \frac{[e^{-\sigma_3(t-\ )} - e^{-\sigma_1(t-\tau)}]}{\sigma_1-\sigma_3} |x(\tau)| d\tau \tag{44}$$

where

$$P_1 = K_1|x_o| - \frac{K_1M_1K_2|\eta_o|}{\sigma_3-\sigma_1} \qquad P_2 = \frac{K_1M_1K_2|\eta_o|}{\sigma_3-\sigma_1} \qquad P_3 = K_1K_2M_1M_3.$$

Take $\sigma_3 > \sigma_1$ (condition I) (44) can be written as

$$|x(t)| \le P_1 e^{-\sigma_1(t-t_o)} + P_2 e^{-\sigma_3(t-t_o)} + \frac{P_3}{\sigma_3-\sigma_1}\int_{t_o}^{t} e^{-\sigma_1(t-\tau)} |x(\tau)| d\tau. \tag{45}$$

Let $y(t) = e^{\sigma_1 t}|x(t)|$, apply Gronwall's lemma and integrate to obtain

$$|x(t)| \le P_1 e^{-\sigma_1(t-t_o)} + P_2 e^{-\sigma_3(t-t_o)} + P_1[e^{-(\sigma_1-P_4)(t-t_o)} - e^{-\sigma_1(t-t_o)}]$$

$$+ P_2P_4[e^{-\sigma_3(t-t_o)} - e^{-(\sigma_1-P_4)(t-t_o)}] \tag{46}$$

where

$$P_4 = \frac{P_3}{\sigma_3 - \sigma_1} .$$

Thus for (13) to be uniformly asymptotically stable we need the following inequalities to be satisfied

$$\sigma_1 > P_4 \quad \text{and} \quad \sigma_3 > \sigma_1$$

i.e.

$$\frac{\sigma_2}{\mu} - K_2M_2 > \sigma_1 \Rightarrow \mu < \frac{\sigma_2}{\sigma_1 + K_2M_2} \tag{46a}$$

$$\sigma_1 > \frac{K_1K_2M_1M_3}{\frac{\sigma_2}{\mu} - K_2M_2 - \sigma_1} \Rightarrow \mu < \frac{\sigma_2\sigma_1}{\sigma_1^2 + K_2M_2\sigma_1 + K_1K_2M_1M_3} . \tag{46b}$$

From (46a) and (46b)

$$\mu_1 = \frac{\sigma_1\sigma_2}{\sigma_1^2 + \sigma_1K_2M_2 + K_1M_1K_2M_3} \tag{47}$$

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even though the eigenvalues of $A_{22}(t)$ have real parts $= -0.45$ for all t [(9), p. 147]. Since system (48) satisfies H1 and H2 and system (49) is uniformly asymptotically stable, we know that for μ sufficiently small, both systems (49) and (48) are uniformly asymptotically stable. In this example we find bounds μ_0 and μ_1.

Fixing the coefficients of the "fast" subsystem at any $t = t_0$, we obtain the linear time-invariant system

$$\mu\dot{\eta} = \begin{bmatrix} -1+1.1\cos^2 t_0 & 1+1.1\sin t_0\cos t_0 \\ -1-1.1\sin t_0\cos t_0 & -1+1.1\sin^2 t_0 \end{bmatrix}\eta. \tag{52}$$

The state transition matrix for (52) is

$$\exp A_{22}(t_0)\tau = \begin{bmatrix} \alpha_{11}(t_0)\cos 0.835\tau - \delta_{11}(t_0)) & \alpha_{12}(t_0)\sin 0.835\tau) \\ \alpha_{21}(t_0)\sin 0.835\tau) & \alpha_{22}(t_0)\cos 0.835\tau - \delta_{22}(t_0)) \end{bmatrix} \tag{53}$$

where $\tau = (t-t_0)/\mu$,

$$\alpha_{11}(t_0) = (1.377 - 1.617\sin^2 t_0 + 1.734\sin^4 t_0)^{\frac{1}{2}},$$
$$\alpha_{12}(t_0) = (1.198 - 1.317\sin t_0\cos t_0),$$
$$\alpha_{21}(t_0) = (-1.198 - 1.317\sin t_0\cos t_0),$$
$$\alpha_{22}(t_0) = (1.377 - 1.617\cos^2 t_0 + 1.734\cos^4 t_0)^{\frac{1}{2}},$$
$$\delta_{11}(t_0) = \tan^{-1}(0.614 - 1.317\sin^2 t_0),$$
$$\delta_{22}(t_0) = \tan^{-1}(0.614 - 1.317\cos^2 t_0).$$

Using as a norm $(\sum\alpha_{ij}^2(t_0))^{\frac{1}{2}}$ we find that $K = 7.358$ and $\gamma = 0.45$. Correspondingly we find the max $\dot{A}_{22}(t) = 1.555 = \beta$. The values of β, K and γ and Theorem 1 imply that $\mu_0 = 0.0177$ and that for $0 < \mu < \mu_0$ System (3.3) is uniformly asymptotically stable. The bound obtained using Theorem 12 of [(6), p. 117] is 0.0089 which is approximately $\frac{1}{2}$ of μ_0.

We next find a bound for the stability of the full-order system (48). From Lemma 1 we obtain

$$|\Phi_{22}(t, t_0)| \le K\left(1 + 2\frac{\mu K\beta}{e^2(\alpha^2 - \mu K\beta)}\right)\exp(-\sigma t/\mu). \tag{54}$$

If we let $\alpha = \sigma = \gamma/2$ we obtain a value for $\hat{\mu}$ of 0.00442. For $\mu \in (0, \hat{\mu})$ we may use the bounds of (54) for $\Phi_{22}(t, t_0)$. This saves the work which would be necessary to derive $\Phi_{22}(t, t_0)$. Thus

$$K_2 = K\left(1 + 2\frac{\mu K\beta}{e^2(\sigma^2 - \mu K\beta)}\right)$$

and $\sigma_2 = \sigma = 0.225$. From Eq. (49)

$$|\bar{x}(t)| \le |x_0|\exp(-1.89(t-t_0))$$

which yields $K = 1$ and $\sigma_1 = 1.89$.

Values for M_1, M_2, and M_3 are 1, 1.956, and 7.09 respectively. Substituting these values into

$$\mu \leq \frac{\sigma_1\sigma_2}{\sigma_1 K_2 M_2 + K_1 M_1 K_2 M_3}$$

yields $\mu_1 = 0.00317$. Since $\mu_1 < \hat{\mu}$ we know from Corollary 1 that for $\mu \in (0, \mu_1)$ system (48) is uniformly asymptotically stable.

This example illustrates the use of Lemma 1, Theorems I and II and Corollary 1 in obtaining the stability bounds of μ in system (48). The bounds K_2 and σ_2 are direct results of Lemma 1, thus making it unnecessary to determine the state transition matrix for the fast subsystem directly.

IV. Conclusion

The main results of this paper are contained in Lemma 1 and Theorems I and II. These provide bounds for the stability of the "fast" subsystem (3) and the full-order system (1). These bounds are found through consideration of reduced-order systems of order m and n. Thus the uniform asymptotic stability of an $n+m$th order system can be shown while considering only the reduced-order system and fast subsystem. Furthermore, the state transition matrix of the "fast" subsystem can be approximated by the more easily determined $\exp(A_{22}(t_0)((t-t_0)/\mu))$ to within $\varphi(t, t_0)$ error. The bound on $\varphi(t, t_0)$ found in Theorem 1 is a byproduct of the derivation of the bound μ_0. The fact that $\varphi(t, t_0)$ is $0(\mu)$ and is exponentially decaying with an $0(\mu)$ time constant is also of use in proofs of various optimality results for time-varying singularly perturbed systems. For instance it may be used in extending results already proved for linear time-invariant systems to time-varying systems. Thus, this paper unifies the work of a number of authors and adds bounds for μ.

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Stability of Singularly Perturbed Systems and Networks with Parasitics

R. R. WILDE AND P. V. KOKOTOVIC

Abstract—**It is noted that some recent stability results for singular perturbation problems are special cases of earlier theorems by Klimushev and Krasovskii. A simplified proof of one of these theorems is given.**

In [1]–[3] Desoer and Shensa present a stability analysis of singularly perturbed time-invariant systems and apply it to networks with small and large parasitics. The purpose of this correspondence is to point out that Klimushev and Krasovskii [4] encompass the stability theorems in [1]–[3]. As an illustration of this we quote and prove a theorem for linear time-varying systems, which represents the stability part of [4, theorem 1]. It is hoped that the familiar style and notation of the proof given here will contribute to better understanding of the little-known results of [4].

The theorem that follows deals with uniform asymptotic stability of the $(n + m)$-dimensional system

$$\dot{x} = A_{11}(t)x + A_{12}(t)z$$
$$\mu\dot{z} = A_{21}(t)x + A_{22}(t)z \qquad (1)$$

where μ is a small positive scalar and a dot denotes derivative with respect to t. Stability properties of (1) for μ sufficiently small are deduced from stability properties of two auxiliary systems: the m-dimensional system

$$\dot{q} = A_{22}(\theta)q \qquad (2)$$

where $\theta \geq t_0$ is a fixed parameter, and the n-dimensional system

$$\dot{p} = [A_{11}(t) - A_{12}(t)A_{22}^{-1}(t)A_{21}(t)]p. \qquad (3)$$

Theorem: If

1) all the matrices $A_{ij}(t)$ in (1) and their derivatives are bounded and continuous functions of t for all $t \geq t_0$,
2) the real parts of all the eigenvalues of $A_{22}(\theta)$ are smaller than a fixed negative number for all $\theta \geq t_0$,
3) system (3) is uniformly asymptotically stable,

then there exists a $\mu^* > 0$ such that system (1) is uniformly asymptotically stable for all $\mu \in (0, \mu^*]$.

Proof: Define δx and δy using

$$x_2 = x_1 + \delta x$$
$$z_2 = z_1 + \delta y - A_{22}^{-1}A_{21}\delta x \qquad (4)$$

where (x_1, z_1) and (x_2, z_2) are solutions of (1) corresponding to two different initial conditions. (For brevity, argument t is dropped when no confusion results.) Upon substitution of (4) into (1),

$$\delta\dot{x} = R\delta x + A_{12}\delta y$$
$$\delta\dot{y} = \frac{1}{\mu}A_{22}\delta y + [(\dot{S} + SR)\delta x + SA_{12}\delta y] \qquad (5)$$

where $R = A_{11} - A_{12}A_{22}^{-1}A_{21}$, $S = A_{22}^{-1}A_{21}$. Clearly, when (5) is uniformly asymptotically stable, so is (1). Let $M(\theta)$ be the unique positive definite solution of

$$A_{22}'(\theta)M + MA_{22}(\theta) = -I_m \qquad (6)$$

for all $\theta \geq t_0$. Here and in (9) I_k denotes a $k \times k$ identity. From condition 2) it follows that $q'M(\theta)q$ is a Lyapunov function for (2). Let the function $p'N(t)p$, whose derivative for (3) is $-p'p$, be a Lyapunov function guaranteeing condition 3). This function exists by a well-known Lyapunov theorem, such as [5, theorem 3].

It is now shown that, for a sufficiently small positive μ, the function

$$w = \delta x'N(t)\delta x + \delta y'M(t)\delta y \qquad (7)$$

is a Lyapunov function for (5) satisfying the requirements for uniform asymptotic stability, such as the conditions of [5, theorem 1]. By definition of $M(t)$ and $N(t)$ there exist continuous nondecreasing functions α and β of the norm $\|\delta x, \delta y\|$ such that $\alpha(0) = 0, \beta(0) = 0$, and

$$0 < \alpha(\|\delta x, \delta y\|) \leq w \leq \beta(\|\delta x, \delta y\|) \qquad (8)$$

holds for all $t \geq t_0$ and all $\delta x \neq 0$, $\delta y \neq 0$. The derivative of w for (5) is

$$\dot{w} = \begin{bmatrix}\delta x\\ \delta y\end{bmatrix}' \begin{bmatrix} -I_n & NA_{12} + (\dot{S} + SR)'M \\ [NA_{12} + (\dot{S} + SR)'M]' & -(1/\mu)I_m + L \end{bmatrix} \begin{bmatrix}\delta x\\ \delta y\end{bmatrix} \qquad (9)$$

where $L = \dot{M} + (SA_{12})'M + M(SA_{12})$. After substitution of θ by t in (6) and differentiation with respect to t, it follows that

$$\dot{M}(t) = \int_{t_0}^{\infty} e^{A_{22}'(t)\sigma}[\dot{A}_{22}'(t)M(t) + M(t)\dot{A}_{22}(t)]e^{A_{22}(t)\sigma}\, d\sigma. \qquad (10)$$

Hence L is bounded for all $t \geq t_0$ and is dominated by $-(1/\mu)I_m$ when μ is sufficiently small. Inspection of leading principal minors of the symmetric matrix in (9) shows that there exists a positive μ^* such that for all $\mu \in (0, \mu^*]$, all $t \geq t_0$ and all $\delta x \neq 0$, $\delta y \neq 0$

$$\dot{w} \leq -\gamma(\|\delta x, \delta y\|) < 0 \qquad (11)$$

where γ is a nondecreasing function and $\gamma(0) = 0$. Properties (8) and (11) of w and $\dot{w}$ prove that (5) is a uniformly asymptotically stable system for $\mu \in (0, \mu^*]$.

The technique of this proof also gives bounds for perturbed solutions and can be extended to nonlinear systems [4]. Some assumptions made here and in [4] can be relaxed. Stability theorems in [1]–[3] proven by a different technique are special cases of the theorems in [4]. The results of [4] have remained unnoticed not only in [1]–[3], but also in the book [6] and the survey [7]. A more general result on asymptotic stability of singularly perturbed systems is found in [8].

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Reduced order modelling and control of two-time-scale discrete systems†

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A class of linear shift-invariant discrete systems satisfying a two-time-scale property is defined and a model satisfying this property is given. A pair of explicitly invertible block diagonalizing transformations are used to obtain reduced order fast and slow models analogous to the continuous singularly perturbed case. A deadbeat approximation to the fast modes results in a reduced order slow model, and a ' boundary layer ' error in the original fast states. For control law design, the dual nature of these block diagonalizing transformations allows partial or total eigenvalue placement for fast and/or slow modes based on feedback designs for the reduced order slow and fast models.

1. Introduction

Methods for approximate control of large scale systems have received a great deal of attention in recent works. Of these methods, aggregation and singular perturbations seem to be the most well known (Aoki 1978). The analysis and control design of continuous linear singularly perturbed systems has been well documented (Kokotovic *et al.* 1976, Chow and Kokotovic 1976 a, b). The multiple-time-scale property of these systems has been used in deriving reduced order models and control laws for high order ' stiff ' models. Until recently, all the work done on systems possessing a multiple-time-scale property has been on continuous systems. The area of discrete two-time-scale systems has received little attention.

In this paper a class of discrete systems satisfying a two time-scale property is introduced. A pair of block diagonalizing transformations are then derived based on the properties of the two-time-scale model. The appealing feature of these transformations is that they have an explicit inverse. This block diagonal form is then used to obtain reduced order models for both simulation and control design. A design procedure is given which allows all eigenvalues of the higher order model to be placed at desired locations based on control laws designed for the reduced order models. Finally, an eighth order power system example is given.

2. Basic definitions

Consider the completely state controllable linear shift-invariant discrete-time system

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u(k) \tag{1}$$

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where $x_1(k) \in R^{N_s}$, $x_2(k) \in R^{N_f}$, $u(k) \in R^M$.

There will exist a basis such that (1) takes the form

$$\begin{bmatrix} x_s(k+1) \\ x_f(k+1) \end{bmatrix} = \begin{bmatrix} A_s & 0 \\ 0 & A_f \end{bmatrix} \begin{bmatrix} x_s(k) \\ x_f(k) \end{bmatrix} + \begin{bmatrix} B_s \\ B_f \end{bmatrix} u(k) \tag{2}$$

such that, if

$$\lambda_f \triangleq \max_j |\lambda_j(A_f)|$$

$$\lambda_s \triangleq \min_i |\lambda_i(A_s)|$$

then

$$\lambda_f < \lambda_s$$

The system (2) is not necessarily in its modal form, however, multiple and complex conjugate eigenvalues are naturally grouped together in either A_s or A_f.

System (1) is thus said to possess a two-time-scale property if there is sufficient 'gap' between the eigenvalues of A_s and A_f. Noting that

$$\min_i |\lambda_i(A_s)| \geqslant \|A_s^{-1}\|^{-1} \quad \text{(lower bound)}$$

$$\max_j |\lambda_j(A_f)| \leqslant \|A_f\| \quad \text{(upper bound)}$$

The two-time-scale property can be expressed as

$$\|A_s^{-1}\|^{-1} \gg \|A_f\| \tag{3}$$

3. System forms and block diagonalization

A class of discrete systems possessing a two-time-scale property has the form

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} A_{11} & \mu^{1-j}\hat{A}_{12} \\ \mu^j \hat{A}_{21} & \mu \hat{A}_{22} \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u(k) \tag{4}$$

where A_{11}^{-1} exists, $0 \leqslant j \leqslant 1$, and μ is a small positive parameter and can be defined as $\|A_{22}\|/\|A_{11}\|$.

The permutation and/or scaling of states necessary to put two-time-scale continuous systems into specific forms is discussed in Avramovic (1979) and Chow (1975). Our purpose now is to transform (4) into form (2) and show that the resulting A_s and A_f submatrices satisfy the two-time-scale property of (3).

It will be convenient throughout the remainder of the paper to use the following notation.

$$A_{12} = \mu^{1-j}\hat{A}_{12}, \quad A_{21} = \mu^j \hat{A}_{21}, \quad A_{22} = \mu \hat{A}_{22} \tag{5}$$

These substitutions will be made interchangeably throughout the remainder of the text, depending on whether or not the relative magnitude of the submatrix norms is needed explicitly.

For (4) to possess the two-time-scale property as has been defined, it will be necessary that the spectrum of (4) consist of N_s 'slow' eigenvalues of magnitude $0(\|A_{11}\|)$ disjoint from N_f 'fast' eigenvalues of $0(\mu)$ magnitude. Various norm bounds can be derived to guarantee that (4) has this property (Feingold

and Varga 1962, Stewart 1971). Here, we apply a lemma from Kokotovic (1975) and extended results in Arvamovic (1979) and Phillips (1979) to achieve a bound on μ in terms of the submatrix norms such that (4) possesses the two-time-scale property defined by (2) and (3).

Lemma

Given system model (4) let

$$A_0 = \hat{A}_{22} - \hat{A}_{21} A_{11}^{-1} \hat{A}_{12} \tag{6}$$

and define the scalars

$$s = \|A_{11}^{-1} \hat{A}_{12}\|, \quad f = \|\hat{A}_{21} A_{11}^{-1}\|, \quad a = \|A_0\|, \quad c = \|A_{11}^{-1}\|$$
$$b_t = f \cdot \|\hat{A}_{12}\|, \quad b_s = s \cdot \|\hat{A}_{21}\|, \quad d_t = a + b_t, \quad d_s = a + b_s$$

If

$$0 \leqslant \mu < \frac{d_t}{c(d_t^2 + 8ab_t)} \tag{7}$$

Then there exists a unique $P^t \in R^{N_t \times N_s}$ satisfying

$$\|P^t\| \leqslant \mu^j f \left(1 + \frac{2a}{a + b_t}\right) \tag{8}$$

such that

$$\mathcal{R} \begin{pmatrix} I_{N_s} \\ P^t \end{pmatrix}, \quad \mathcal{R} \triangleq \text{range space} \tag{9}$$

is the invariant subspace of A corresponding to $\sigma(A_{11} + A_{12}P^t)$. Moreover, $\sigma(A)$ is the disjoint union of $\sigma(A_{11} - A_{12}P^t) \cup \sigma(A_{22} + P^t A_{12})$. Also, if

$$0 \leqslant \mu < \frac{d_s}{c(d_s^2 + 8ab_s)} \tag{10}$$

there exists a unique $P^s \in R^{N_s \times N_t}$ satisfying

$$\|P^s\| \leqslant \mu^{(1-j)} s \left(1 + \frac{2a}{a + b_s}\right) \tag{11}$$

such that

$$\mathcal{R} \begin{pmatrix} I_{N_s} \\ (P^s)^{\mathrm{T}} \end{pmatrix} \tag{12}$$

is the invariant subspace of A^{T} corresponding to $\sigma(A_{11} + P^s A_{21})$. Moreover, $\sigma(A)$ is the disjoint union of $\sigma(A_{11} + P^s A_{21}) \cup \sigma(A_{22} - A_{21} P^s)$.

Proof

An application of results obtained in Avramovic (1979), Kokotovic (1975) and Phillips (1979). An outline is given in Appendix B.

Consider now the transformation on (4)

$$y_2(k) = x_2(k) + P^t x_1(k) \tag{13}$$

where P^t is selected such that

$$R_t(P^t) \triangleq A_{21} + P^t A_{11} - A_{22} P^t - P^t A_{12} P^t = 0 \tag{14}$$

P^t transforms (4) into

$$\begin{bmatrix} x_1(k+1) \\ y_2(k+1) \end{bmatrix} = \begin{bmatrix} A_{11} - A_{12}P^t & A_{12} \\ 0 & A_{22} + P^t A_{12} \end{bmatrix} \begin{bmatrix} x_1(k) \\ y_2(k) \end{bmatrix} + \begin{bmatrix} B_1 \\ B_1 + P^t B_2 \end{bmatrix} u(k) \tag{15}$$

To complete the block diagonalization, let

$$y_1(k) = x_1(k) - Q^t y_2(k) \tag{16}$$

where Q^t is the solution to the Lyapunov equation

$$(A_{11} - A_{12}P^t)Q^t - Q^t(A_{22} + P^t A_{12}) + A_{12} = 0 \tag{17}$$

Iterative techniques (Avramovic 1979, Kokotovic 1975) for obtaining solution to (14) and (17) are briefly reviewed in Appendix A. Convergence of the iterative algorithms is assured for every μ satisfying (7). (13) and (16) give a net transformation

$$y(k) = \begin{bmatrix} I_1 - Q^t P^t & -Q^t \\ P^t & I_2 \end{bmatrix} x(k) \tag{18}$$

which has the explicit inverse

$$x(k) = \begin{bmatrix} I_1 & Q^t \\ -P^t & I_2 - P^t Q^t \end{bmatrix} y(k) \tag{19}$$

This will be called the ' F ' transformation. When applied to (4), this transformation gives

$$y(k+1) = \begin{bmatrix} A_{11} - A_{12}P^t & 0 \\ 0 & A_{22} + P^t A_{12} \end{bmatrix} y(k) + \begin{bmatrix} (I - Q^t P^t)B_1 - Q^t B_2 \\ P^t B_1 + B_2 \end{bmatrix} u(k) \tag{20}$$

If the A_{12} block has been removed from (4) first, a dual transformation to (16) results which will be called the ' S ' transformation. Let

$$y_1(k) = x_1(k) + P^s x_2(k) \tag{21}$$

where P^s is the solution to

$$R_s(P^s) \triangleq A_{12} - A_{11}P^s - P^s A_{21} P^s + P^s A_{22} = 0 \tag{22}$$

P^s transforms (4) into

$$\begin{bmatrix} y_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} A_{11} + P^s A_{21} & 0 \\ A_{21} & A_{22} - A_{21}P^s \end{bmatrix} \begin{bmatrix} y_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} B_1 + P^s B_2 \\ B_2 \end{bmatrix} u(k) \tag{23}$$

To complete the block diagonalization

$$y_2(k) = x_2(k) - Q^s y_1(k) \tag{24}$$

where Q^s is the solution of the Lyapunov equation

$$(A_{22}-A_{21}P^s)Q^s-Q^s(A_{11}+P^sA_{21})+A_{21}=0 \tag{25}$$

Again, convergence of the iterative algorithms for obtaining solutions to (22) and (25) is assured for every μ satisfying (10). This transformation takes the form

$$y(k)=\begin{bmatrix} I_1 & P^s \\ -Q^s & I_2-Q^sP^s \end{bmatrix} x(k) \tag{26}$$

which possesses the explicit inverse

$$x(k)=\begin{bmatrix} I_1-P^sQ^s & -P^s \\ Q^s & I_2 \end{bmatrix} y(k) \tag{27}$$

and when applied to (4), the transformed system takes the form

$$y(k+1)=\begin{bmatrix} A_{11}+P^sA_{21} & 0 \\ 0 & A_{22}-A_{21}P^s \end{bmatrix} y(k) + \begin{bmatrix} B_1+P^sB_2 \\ -Q^sB_1+(I-Q^sP^s)B_2 \end{bmatrix} u(k) \tag{28}$$

It is easy to see from the lemma that

$$\|P^s\| \triangleq 0(\mu^{1-j}), \quad \|P^t\| \triangleq 0(\mu^j)$$

If we let

$$\mu^{(1-j)}\hat{P}^s=P^s, \quad \mu^j\hat{P}^t=P^t \tag{29}$$

then (20) and (28) satisfy the two-time-scale property as defined by (3), since for μ sufficiently small

$$\|(A_{11}+\mu\hat{P}^s\hat{A}_{21})^{-1}\|^{-1}>\mu\|(\hat{A}_{22}-\hat{A}_{21}\hat{P}^s)\| \tag{30}$$

$$\|(A_{11}-\mu\hat{A}_{12}\hat{P}^t)^{-1}\|^{-1}>\mu\|(\hat{A}_{22}+\hat{P}^t\hat{A}_{12})\| \tag{31}$$

Inequalities (30) and (31) are given here to show our transformations lead to block diagonalizations that are consistent with our norm definition of a two-time-scale system (3). The set of values of μ that satisfy (30) and (31) will, in general, be a subset of the value defined by (7) and (10) respectively. In the remainder of this paper (7) and (10) will be used to determine if the system can be put into two-time-scale block diagonal form. It should be noted here that bounds obtained from norms tend to be conservative. That is, the methodology presented here is applicable to some systems not satisfying (7) and (10).

4. Reduced order modelling

One of the applications of singular perturbations is the ability to obtain low order well-conditioned models from high order ill-conditioned models of continuous linear systems. The approximation made in obtaining these reduced

order models is to assume that the real part of the stable fast eigenvalues go to minus infinity. Thus, all fast modes are assumed to decay instantaneously. In the discrete case, this is analogous to approximating all fast eigenvalues as 'deadbeat', that is $|\lambda| = 0$. For model (4), this means approximating the group of N_f eigenvalues clustered within an $0(\mu)$ radius of the origin of the complex plane as zero eigenvalues. Thus,

$$y_2(k) = 0, \quad k > 0 \tag{32}$$

and from the 'F' transformation

$$x_1(k) = y_1(k), \quad k > 0 \tag{33}$$

$$x_2(k) = -P^t y_1(k), \quad k > 0 \tag{34}$$

Applying (33) and (34) for all k, our reduced order model becomes

$$\hat{x}(k+1) = \hat{A}_s \hat{x}_1(k), \quad \hat{x}_1(0) = x_1(0) \tag{35}$$

and the 'fast' states appear only as quasi steady-state functions of $\hat{x}_1(k)$

$$\hat{x}_2(k) = -P^t \hat{x}_1(k) \tag{36}$$

In (36), we eliminate any dependence of $x_2(k)$ on $x_2(0)$. Thus, from $k = 0$ to some $k = k^*$, (36) may differ considerably from the actual $x_2(k)$ states. Since all fast modes are stable in this analysis, the question is not whether $\hat{x}_2(k)$ will converge to $x_2(k)$, but how soon

$$|\hat{x}_2(k) - x_2(k)| < \gamma \tag{37}$$

for some $\gamma > 0$ and $k > 0$.

The interval $[0, k^*]$ is referred to as the 'boundary layer' in the analysis of continuous singularly perturbed systems (Kokotovic *et al.* 1976). For μ small, k^* can be as small as 1.

Example

From Calovic (1971), the discrete model of a steam power system is given as

$$x(k+1) = \begin{bmatrix} 0{\cdot}9014 & 0{\cdot}1179 & 0{\cdot}0525 & 0{\cdot}0167 & 0{\cdot}02104 \\ -0{\cdot}0196 & 0{\cdot}8743 & 0 & 0{\cdot}025 & 0{\cdot}02934 \\ -0{\cdot}0071 & 0{\cdot}7342 & 0{\cdot}20175 & 0{\cdot}013 & 0{\cdot}21067 \\ -0{\cdot}75 & -0{\cdot}0557 & -0{\cdot}032 & 0{\cdot}19357 & -0{\cdot}014076 \\ -0{\cdot}306 & -0{\cdot}01894 & -0{\cdot}011 & 0{\cdot}14278 & 0{\cdot}013217 \end{bmatrix} x(k) \tag{38}$$

This model was found to fit the model (4) with $N_s = 2$, $N_f = 3$. The corresponding submatrix norms were

$$\|A_{11}\| = 0{\cdot}9415, \quad \|A_{12}\| = 0{\cdot}0625, \quad \|A_{21}\| = 0{\cdot}8184, \quad \|A_{22}\| = 0{\cdot}2441$$

Condition (7) is satisfied with $0 \leqslant 0{\cdot}259\,25 < 0{\cdot}338\,38$. Using one iteration of the P^t matrix recursion

$$P_1{}^t = \begin{bmatrix} -0{\cdot}02583 & -1{\cdot}029 \\ 1{\cdot}0207 & -0{\cdot}09667 \\ 0{\cdot}4817 & -0{\cdot}06101 \end{bmatrix} \tag{39}$$

we obtain the slow subsystem

$$x_s(k+1) = \begin{bmatrix} 0.875521 & 0.17486 \\ -0.05932 & 0.87798 \end{bmatrix} x_s(k) \tag{40}$$

which has eigenvalues

$$\lambda_{1,2} = 0.876\,75 \pm 0.101\,84j$$

which compares favourably to the actual eigenvalues of (38) given here for comparison purposes

$$\lambda_{1,2} = 0.8777 \pm 0.1054j$$
$$\lambda_3 = 0.0179$$
$$\lambda_{4,5} = 0.2055 \pm 0.0236j$$

The entire system dynamics of (38) will now be simulated using

$$\hat{x}_1(k+1) = A_s \hat{x}_1(k), \quad \hat{x}_1(0) = x_1(0)$$

where the remaining three states appear as output functions or

$$\hat{x}_2(k) = -P_1{}^t\, \hat{x}_1(k)$$

To compare this second order model with our fifth order model, the response of the system to an initial perturbation of

$$x_0 = (1,\ -0.8,\ 0.5,\ 0.2,\ 0.6)$$

is plotted versus its lower order approximation (Figs. 1 to 5). The actual (high order) states will be designated $x_i(k)$, while the approximated state will be identified by $\hat{x}_i(k)$. Using just *one* iteration of $P_k{}^t$ results in good accuracy.

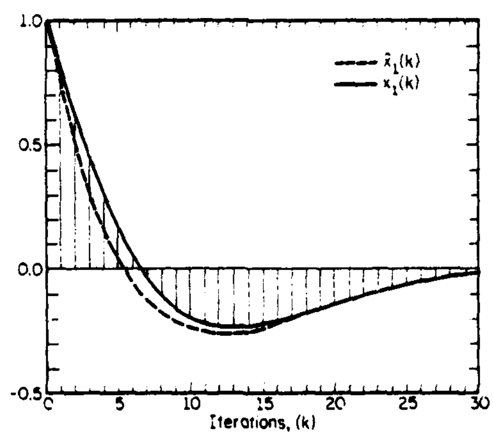

Figure 1. Reduced order approximation of state $x_1(k)$.

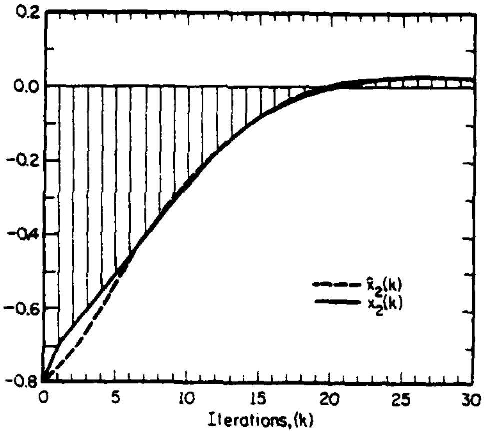

Figure 2. Reduced order approximation of state $x_2(k)$.

5. Control law design

The explicit inverses of the block diagonalizing transformations and the two-time-scale nature of the proposed model, enable partial or total pole placement to be carried out by solving reduced order pole placement problems. First, two cases for partial pole placement will be covered. Then, a two stage design for total pole placement will be outlined.

Case 1

Only slow eigenvalues need to be altered.

Using transformation (21), our resulting system is of the form (23). Let $A_s = A_{11} + P^s A_{21}$ and $B_s = B_1 + P^s B_2$ and observe that the pair (A_s, B_s) spans only the 'slow' controllable subspace.

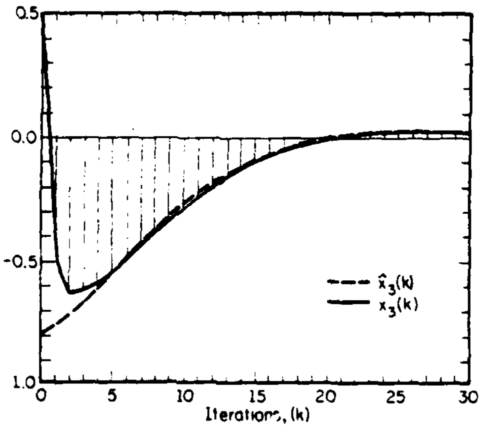

Figure 3. Reduced order approximation of state $x_3(k)$.

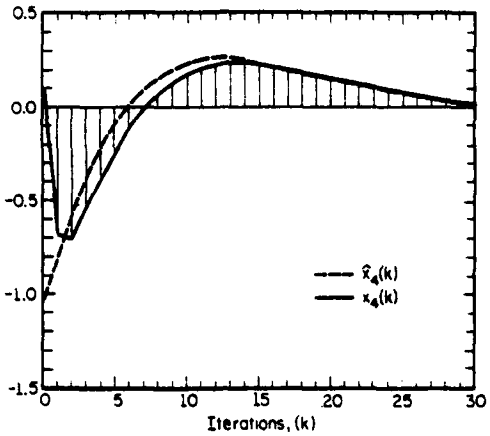

Figure 4. Reduced order approximation of state $x_4(k)$.

Design G_s such that the eigenvalues of $(A_s + B_s G_s)$ are at N_s desired locations. This gives a closed loop system

$$\begin{bmatrix} y_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} (A_s + B_s G_s) & 0 \\ (A_{21} + B_2 G_s) & A_f \end{bmatrix} \begin{bmatrix} y_1(k) \\ x_2(k) \end{bmatrix} \tag{41}$$

where $A_f = A_{22} - A_{21} P^s$.

This system has N_s eigenvalues of $A_s + B_s G_s$ and N_f eigenvalues of A_f. The feedback control takes the form

$$\begin{aligned} u(k) &= G_s y_1(k) \\ &= [G_s \vdots G_s P^s] x(k) \end{aligned} \tag{42}$$

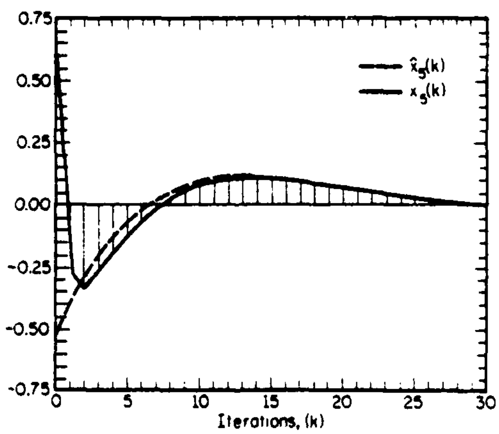

Figure 5. Reduced order approximation of state $x_5(k)$.

Case 2

Only fast eigenvalues need to be altered.

Using transformation (13), we obtain the upper block triangular system (15). Let $A_f = A_{22} + P^t A_{12}$ and $B_f = B_2 + P^t B_1$ and observe that the pair (A_f, B_f) spans only the fast controllable subspace.

Design G_f such that the eigenvalues of $(A_f + B_f G_f)$ are at N_f desired locations. This gives a closed loop system

$$\begin{bmatrix} x_1(k+1) \\ y_2(k+1) \end{bmatrix} = \begin{bmatrix} A_s & A_{12} + B_1 G_f \\ 0 & A_f + B_f G_f \end{bmatrix} \begin{bmatrix} x_1(k) \\ y_2(k) \end{bmatrix} \tag{43}$$

where $A_s = A_{11} - A_{12} P^t$.

This system has N_s eigenvalues of A_s and N_f eigenvalues of $A_f + B_f G_f$. The feedback control takes the form

$$\begin{aligned} u(k) &= G_f y_2(k) \\ &= [G_f P^t \vdots G_f] x(k) \end{aligned} \tag{44}$$

If the design requirement entails moving both slow and fast eigenvalues, then a two stage procedure can be implemented.

Given a system of form (4) that has been put into block diagonal form via either the ' S ' or ' F ' transformation,

$$\begin{bmatrix} y_1(k+1) \\ y_2(k+1) \end{bmatrix} = \begin{bmatrix} A_s & 0 \\ 0 & A_f \end{bmatrix} \begin{bmatrix} y_1(k) \\ y_2(k) \end{bmatrix} + \begin{bmatrix} B_s \\ B_f \end{bmatrix} u(k) \tag{45}$$

We can design for either slow or fast subsystems pairs (A_s, B_s) or (A_f, B_f). Arbitrarily, the pair (A_s, B_s) is selected first.

Find a feedback gain G_s such that $A_s + B_s G_s$ has N_s desired ' slow ' eigenvalues. The resulting partially closed loop system is of the form

$$\begin{bmatrix} y_1(k+1) \\ y_2(k+1) \end{bmatrix} = \begin{bmatrix} A_s + B_s G_s & 0 \\ B_f G_s & A_f \end{bmatrix} \begin{bmatrix} y_1(k) \\ y_2(k) \end{bmatrix} + \begin{bmatrix} B_s \\ B_f \end{bmatrix} u_f(k) \tag{46}$$

where

$$u(k) = G_s y_1(k) + u_f(k)$$

Now, let

$$z_2(k) = y_2(k) - S y_1(k) \tag{47}$$

which results in

$$z_2(k+1) = [B_f G_s + A_f S - S(A_s + B_s G_s)] y_1(k) + A_f z_2(k) + (B_s - S B_f) u_f(k)$$

Choose S such that

$$A_f S - S(A_s + B_s G_s) + B_f G_s = 0 \tag{48}$$

This Lyapunov type equation has a unique solution if

$$\sigma(A_f) \cap \sigma(A_s + B_s G_s) = \phi \tag{49}$$

Thus, one of the design requirements of the first stage is that the desired slow spectrum be disjoint from the open loop fast spectrum. The solution to (48) can be found iteratively like Q^f and Q^s or algebraically (Bartels and Stewart 1977). (46) now becomes

$$\begin{bmatrix} y_1(k+1) \\ z_2(k+1) \end{bmatrix} = \begin{bmatrix} A_s + B_s G_s & 0 \\ 0 & A_f \end{bmatrix} \begin{bmatrix} y_1(k) \\ z_2(k) \end{bmatrix} + \begin{bmatrix} B_s \\ B_f - SB_s \end{bmatrix} u_2(k) \tag{50}$$

Now, design a feedback gain G_f such that $(A_f + (B_f - SB_s)G_f)$ has N_f desired eigenvalue locations.

The composite feedback is of the form

$$\begin{aligned} u(k) &= u_s(k) + u_f(k) \\ &= G_s y_1(k) + G_f z_2(k) \\ &= (G_s - G_f S) y_1(k) + G_f y_2(k) \end{aligned} \tag{51}$$

Depending on whether the ' S ' or ' F ' transformation was used, $y_1(k)$ and $y_2(k)$ can be expressed as functions of our original states. For example, using the ' S ' transformation

$$\begin{aligned} y_1(k) &= [I \vdots P^s] x(k) \\ y_2(k) &= [-Q^s \vdots I - Q^s P^s] x(k) \end{aligned}$$

and (51) becomes

$$u(k) = [G_s - G_f(S + Q^s) \vdots G_s P^s + G_f - G_f(S + Q^s) P^s] x(k) \tag{52}$$

which places N_s eigenvalues of (4) according to $\lambda(A_s + B_s G_s)$ and N_f eigenvalues according to $\lambda(A_f + (B_f - SB_s)G_f)$.

Example

The discrete model of an eighth order power system (Calovic 1971) is given as

$$A = \begin{bmatrix} 0{\cdot}835 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0{\cdot}096 & 0{\cdot}861 & 0 & 0 & 0 & 0 & 0 & 0{\cdot}029 \\ -0{\cdot}002 & -0{\cdot}005 & 0{\cdot}882 & -0{\cdot}253 & 0{\cdot}041 & -0{\cdot}003 & -0{\cdot}025 & -0{\cdot}001 \\ 0{\cdot}007 & 0{\cdot}014 & -0{\cdot}029 & 0{\cdot}928 & 0 & 0{\cdot}006 & 0{\cdot}059 & 0{\cdot}002 \\ -0{\cdot}03 & -0{\cdot}061 & 2{\cdot}028 & -2{\cdot}303 & 0{\cdot}088 & -0{\cdot}021 & -0{\cdot}224 & -0{\cdot}008 \\ 0{\cdot}048 & 0{\cdot}758 & 0 & 0 & 0 & 0{\cdot}165 & 0 & 0{\cdot}023 \\ -0{\cdot}012 & -0{\cdot}027 & 1{\cdot}209 & -1{\cdot}4 & 0{\cdot}161 & -0{\cdot}013 & 0{\cdot}156 & 0{\cdot}006 \\ 0{\cdot}815 & 0 & 0 & 0 & 0 & 0 & 0 & 0{\cdot}011 \end{bmatrix} \tag{53}$$

$$B^T = \begin{bmatrix} 0{\cdot} & 0 & 0{\cdot}294 & -0{\cdot}038 & 2{\cdot}762 & 0 & 1{\cdot}473 & 0 \\ 3{\cdot}295 & 0{\cdot}152 & -0{\cdot}003 & 0{\cdot}01 & -0{\cdot}051 & 0{\cdot}056 & -0{\cdot}015 & 2{\cdot}477 \end{bmatrix} \tag{54}$$

which is of the form (4), with $N_s = N_f = 4$, and $\mu = 0{\cdot}259\,04$.

After three iterations,

$$P_3{}^t = \begin{bmatrix} 0.012 & 0.035 & -2.102 & 1.809 \\ 0.095 & -1.083 & 0 & 0 \\ -0.020 & -0.006 & -2.168 & 1.546 \\ -0.909 & 0 & 0 & 0 \end{bmatrix} \tag{55}$$

$$\zeta_3{}^t = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.034 \\ -0.043 & 0 & -0.005 & 0 \\ -0.004 & -0.009 & -0.109 & -0.003 \end{bmatrix} \tag{56}$$

The slow subsystem is given as

$$A_s = \begin{bmatrix} 0.835 & 0 & 0 & 0 \\ 0.126 & 0.861 & 0 & 0 \\ 0.004 & -0.009 & 0.913 & -0.288 \\ 0.010 & 0.021 & 0.098 & 0.836 \end{bmatrix} \tag{57}$$

$$B_s{}^{\mathrm{T}} = \begin{bmatrix} 0 & 0 & 0.394 & 0.054 \\ 3.295 & 0.125 & -0.002 & 0.003 \end{bmatrix} \tag{58}$$

The desired slow eigenvalues are

$$\lambda_{1,2}(\mathrm{Des}) = 0.9 \pm 0.05j$$
$$\lambda_3(\mathrm{Des}) = 0.85$$
$$\lambda_4(\mathrm{Des}) = 0.8$$

and the feedback

$$G_s = \begin{bmatrix} 0.032 & -0.01 & 0 & 0 \\ -0.069 & -0.027 & -0.347 & 0.676 \end{bmatrix} \tag{59}$$

places the eigenvalues of $A_s + B_s G_s$ at these desired locations.

This gives S as

$$S = \begin{bmatrix} -0.170 & -0.121 & -1.013 & 1.508 \\ 0.01 & -0.003 & 0 & 0 \\ -0.134 & -0.131 & -0.904 & 1.076 \\ -0.028 & 0.008 & 0 & 0 \end{bmatrix} \tag{60}$$

and the resulting fast subsystem

$$A_f = \begin{bmatrix} 0\cdot003 & -0\cdot005 & -0\cdot064 & 0 \\ 0 & 0\cdot165 & 0 & -0\cdot009 \\ 0\cdot07 & 0\cdot001 & 0\cdot302 & 0\cdot011 \\ 0 & 0 & 0 & 0\cdot011 \end{bmatrix} \tag{61}$$

$$(B_f - SB_s)^T = \begin{bmatrix} 2\cdot392 & 0 & 1\cdot074 & 0 \\ 0\cdot587 & 0\cdot176 & 0\cdot394 & -0\cdot692 \end{bmatrix} \tag{62}$$

The desired fast eigenvalues are

$$\lambda_{5,6}(\text{Des}) = 0\cdot2 \pm 0\cdot01j$$
$$\lambda_7(\text{Des}) = 0\cdot15$$
$$\lambda_8(\text{Des}) = 0\cdot1$$

and the feedback

$$G_f = \begin{bmatrix} 0 & -0\cdot039 & 0 & -0\cdot333 \\ 0 & 0\cdot010 & -0\cdot052 & 0\cdot029 \end{bmatrix} \tag{63}$$

places the eigenvalues of $A_f + (B_f - SB_s)^T G_f$ at these desired locations.

The composite feedback analogous to (52), places the closed loop eigenvalues at

$$\lambda_{1,2}(\text{CL}) = 0\cdot900 \pm 0\cdot051j$$
$$\lambda_3(\text{CL}) = 0\cdot846$$
$$\lambda_4(\text{CL}) = 0\cdot799$$
$$\lambda_{5,6}(\text{CL}) = 0\cdot199 \pm 0\cdot014j$$
$$\lambda_7(\text{CL}) = 0\cdot161$$
$$\lambda_8(\text{CL}) = 0\cdot093$$

Thus, a maximum error of 7·3% after 3 iterations, and an absolute error well within the convergence rate bound $0(\mu^3)$, where

$$\mu^3 = 0\cdot017$$

6. Conclusions

A model for a class of discrete systems having a two-time-scale property has been defined. By satisfying certain subsystem norm conditions, reduced order models may be derived without *a priori* knowledge of the eigenvalues of the system. This is appealing to exceptionally large discrete systems (i.e. economic, sociological, etc.) where the computation of eigenvalues may be impossible. Dual block diagonalizing transformations are used to obtain reduced order fast and slow models. Order reduction is achieved by approximating the fast modes at deadbeat. Thus, the original fast states are approximated as quasi steady state output functions of the slow states. Partial or total pole placement for the higher order system is implemented using designs based on the reduced order fast and slow subsystems.

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Appendix A

Iterative techniques to obtain solutions to (14), (17), (22), and (25) appeared in Avramovic (1979) and Kokotovic (1975). The subspace method (Avramovic 1979) leads to the following matrix recursion equation for P^t

$$P_{k+1}{}^t = P_k{}^t - R_t(P_k{}^t)(A_{11} - A_{12}P_k{}^t)^{-1} \tag{A 1}$$

$$P_0{}^t = -A_{21}A_{11}{}^{-1} \tag{A 2}$$

In the case of (4), the convergence of (A 1) is guaranteed for all μ satisfying (7) and the convergence rate will be $0(\mu^k)$ since

$$\frac{\sup|\sigma(A_{22} + P^t A_{12})|}{\inf|\sigma(A_{11} - A_{12}P^t)|} \triangleq 0(\mu) \tag{A 3}$$

A computationally more efficient form of (A 1) is proposed in Kokotovic (1975), where the approximation

$$(A_{11} - \mu\hat{A}_{12}\hat{P}^t)^{-1} \approx A_{11}{}^{-1} \tag{A 4}$$

leaves (A 1) in the form

$$P_{k+1}{}^t = (A_{22}P_k{}^t + P_k{}^t A_{12}P_k{}^t - A_{21})A_{11}{}^{-1} \tag{A 5}$$

which possesses the same local convergence properties of (A 1). Bounds on μ for the convergence of (A 5) have been considered in Phillips (1979).

The subspace method of Avramovic (1979) can be extended to find an iterative solution for P^s. This is seen by letting $P^s = M_1{}^{-1} M_2$, where the rows of $[M_1M_2]$ are left eigenvectors of the system matrix in (4) spanning the dominant N_s dimensional eigenspace. The iterative solution can then be shown to be of the form

$$P_{k+1}{}^s = P_k{}^s + (A_{11} + P_k{}^s A_{21})^{-1} \,.\, R_s(P_k{}^s) \tag{A 6}$$

$$P_0{}^s = A_{11}{}^{-1} A_{12} \tag{A 7}$$

The conditions for convergence and the convergence rate of (A 6) are analogous to those of (A 1).

Again, from Kokotovic (1975), making the approximation

$$(A_{11} + \mu\hat{P}_k{}^s \hat{A}_{21})^{-1} \approx A_{11}{}^{-1} \tag{A 8}$$

(A 6) becomes

$$P_{k+1}{}^s = A_{11}{}^{-1}(A_{12} + P_k{}^s A_{22} - P_k{}^s A_{21}P_k{}^s) \tag{A 9}$$

which is also considered in Phillips (1979).

Solutions to (17) and (25) have been well documented and can be found algebraically (Bartels and Stewart 1977) or recursively (Kokotovic 1975). From Kokotovic (1975), the following successive approximations converge under mild bounds at an $0(\mu^k)$ rate

$$Q_{k+1}^t = A_{11}^{-1}(A_{12}P^tQ_k^t + Q_k^t(A_{22} + P^tA_{12}) - A_{12}) \tag{A 10}$$

$$Q_0^t = -A_{11}^{-1} A_{12} \tag{A 11}$$

$$Q_{k+1}^s = ((A_{22} - A_{21}P^s)Q_k^s - Q_k^s P^s A_{21} + A_{21})A_{11}^{-1} \tag{A 12}$$

$$Q_0^s = A_{21}A_{11}^{-1} \tag{A 13}$$

In the example (A 10) and (A 11) have been used.

Appendix B

The existence of an equilibrium solution to (A 5) is guaranteed if μ is bounded by (7) which also establishes (8). These results were first derived in Kokotovic (1975) and applied to discrete systems in Phillips (1979). However, Avramovic (1979) has shown that (A 5) is a simplified form to the matrix recursion (A 1) in that convergence of (A 5) assures convergence of (A 1) to an equivalent result. From Avramovic (1979), the sequences of (A 1) are shown to be equivalent to the sequences of the following simultaneous iteration for computing the basis of a dominant eigenspace of A

$$V^{k+1} = A\, V^k S^k \tag{A 14}$$

where

$$V_k = \begin{bmatrix} V_1^k \\ V_2^k \end{bmatrix} \in R^{N \times N_s}$$

and S^k is a scaling matrix. Thus,

$$P_k^t = -V_2^k(V_k^k)^{-1} \tag{A 15}$$

and the spectrum separation property and (8) are established.

The existence of an equilibrium solution to (A 9) is guaranteed if μ is bounded by (10) which also establishes (11). The left eigenvector approach to the derivation in Avramovic (1979) results in (A 6) and (A 7) of which (A 9) has been shown to be a simplified form. Thus, the dominant eigenspace iterations of Avramovic (1979) can again be applied in analogous fashion to establish (12) and the spectrum separation property.

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SECTION 3

TWO-TIME-SCALE MODELING OF POWER SYSTEMS

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Multi-Time-Scale Analysis of a Power System*

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Singular perturbations, applied to a model of a three machine power system, provided reduced models which yield good eigenvalue and time response approximations of the original system.

Key Words—Computational methods; time scale modeling; system order reduction; iterative methods; power system simulation; dynamic response; large scale systems.

Abstract—A time-scale separation procedure is outlined and applied to a three machine interconnected power system modeled with flux linkage and voltage regulator dynamics. Partial models such as the electromechanical model and single machine-infinite bus model are used to identify the slow and fast states of the systems. Linear simulation results in two- and four-time-scales demonstrate the potential applicability of the singular perturbation approach to long-term dynamic studies of power systems.

1. INTRODUCTION

THIS paper presents an application of the singular perturbation method for separation of time scales described in a companion paper (Kokotovic and co-workers, 1980). A linearized 20th order model of a three machine power system with realistic data is analyzed in two- and four-time-scales. The model includes the electromechanical, flux linkage and excitation system dynamics. Following an electrical disturbance the model exhibits a rich frequency spectrum of restoring motions. Due to the strong interactions between machines, the individual machine variables are found to be mixed and hence is not suitable for direct state separation into a slow and a fast subsystem. The identification and the reformulation of the slow and fast variables are therefore among the major problems.

A state-space model of a multi-time-scale system is said to be *state separable if the fast parts of some of its states are small compared with their slow parts and with the fast parts of the other states.* The fast parts may arise due to either modes that are well-damped, that is, eigenvalues with large negative real parts, or high frequency oscillatory modes, that is, eigenvalues with large imaginary parts. Such models can be put in the standard singular perturbation form (Kokotovic, O'Malley and Sannuti, 1976) and the time scale separation method discussed in the companion paper is directly applicable. However, in a large scale system the situation is more complex. Even if the subsystem models are state separable, their interconnections may introduce new phenomena and change the speeds of some of the states. Then a new choice of state variables may be needed to make the interconnected model state separable. An example is the angle transformation used here to deal with electromechanical interactions. To make the determination of fast and slow states more systematic, we propose the following separation procedure. First, we study readily identifiable phenomena, in this case the electromechanical interactions and the single machine characteristics to identify the fast and slow states. The next step is to apply the iterative scheme to validate the choice of the slow and fast variables and to improve the accuracies of the slow and the fast subsystem.

The paper is organized as follows. The model of the three machine system is given in Section 2. The separation procedure is proposed in Section 3. Time scales of the electromechanical model are discussed in Section 4. In Sections 5 and 6, the full model is analyzed as a two-time-scale system and the simulation results are discussed in Section 7. Section 8 is an extension to a four-time-scale analysis.

2. THE TEST SYSTEM

For the three machine test system in Fig. 1 the disturbance of interest is a five cycle three phase fault on bus 8 followed by the loss of line 8–9. Constant impedance loads are assumed. The six electromechanical equations are

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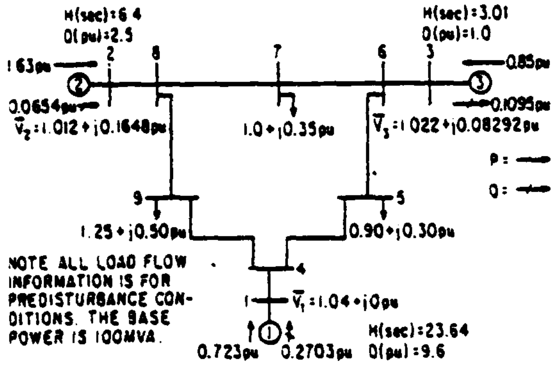
$$\delta_i = 377(\omega_i - 1) \qquad i = 1, 2, 3 \tag{1}$$

$$\dot{\omega}_i = \frac{1}{2H_i}\left[\frac{P_i}{\omega_i} - D_i(\omega_i - 1) - \sum_{j=1}^{3} V_i V_j Y_{ij} \cos(\theta_{ij} + \delta_j - \delta_i)\right] \qquad i = 1, 2, 3 \tag{2}$$

where V_i are generator voltages, $Y_{ij}\angle\theta_{ij}$ interconnecting admittances, D_i damping terms, P_i mechanical input powers and H_i inertias.



Line #	From	To	R (pu)	X (pu)	B/2 (pu)
1	1	4	0	0.0567	0
2	4	5	0.017	0.092	0.079
3	5	6	0.039	0.170	0.179
4	3	6	0	1.0586	0
5	6	7	0.0119	0.1008	0.1045
6	7	8	0.0085	0.072	0.0745
7	8	2	0	0.0625	0
8	8	9	0.032	0.161	0.153
9	9	4	0.01	0.085	0.088

FIG. 1. 3 machine, 9 bus test system.

TABLE 1. SYNCHRONOUS MACHINE DATA (100 MVA BASE)

Parameter	Machine # 1	2	3
H (sec)	23.64	6.4	3.01
D (pu)	9.6	2.5	1.0
X_d(pu)	0.6	0.8958	0.9
X_q(pu)	0.58	0.8645	0.85
X_ℓ(pu)	0.056	0.110	0.18
X'_d(pu)	0.0608	0.1198	0.1813
X'_q(pu)	0.0608	0.1198	0.1813
T'_{d0}(sec)	4.0	6.0	5.0
T'_{q0}(sec)	0.25	0.54	0.65

The voltages induced in the d-axis and the q-axis by the flux linkages (Schulz, 1972) for $i = 1, 2, 3$

$$\dot{e}'_{di} = \frac{1}{T'_{q0i}}\left[-e'_{di} + (X_{qi} - X'_{qi})i_{qi}\right] \tag{3}$$

$$\dot{e}'_{qi} = \frac{1}{T'_{d0i}}\left[-e'_{qi} - (X_{di} - X'_{di})i_{di} + E_{fdi}\right] \tag{4}$$

where E_{fdi} is field voltage and the currents are

$$i_{di} = \sum_{j=1}^{3} Y_{ij}\left[e'_{dj}\cos(\theta_{ij} + \delta_j - \delta_i) - e'_{qj}\sin(\theta_{ij} + \delta_j - \delta_i)\right] \tag{5}$$

$$i_{qi} = \sum_{j=1}^{3} Y_{ij}\left[e'_{qj}\cos(\theta_{ij} + \delta_j - \delta_i) + e'_{dj}\sin(\theta_{ij} + \delta_j - \delta_i)\right]. \tag{6}$$

The machine data are given in Table 1.

The same IEEE Type 1 voltage regulator (IEEE Committee Report, 1968) is used for all machines with the exponential saturation function

$$S_E(E_{fd}) = A_{sat}\exp\left[B_{sat}E_{fd}\right] \tag{7}$$

retained but limit type nonlinearities neglected. The amplifier, exciter and feedback compensator equations are, for $i = 1, 2, 3$

$$\dot{V}_{Ri} = \frac{1}{T_{Ai}}\left[K_{Ai}(K_{Fi}(R_{fi} - E_{fdi})/T_{Fi} - V_i + V_{Refi}) - V_{Ri}\right] \tag{8}$$

$$\dot{E}_{fdi} = \frac{1}{T_{Ei}}\left\{-\left[K_{Ei}E_{fdi} + S_E(E_{fdi})\right] + V_{Ri}\right\} \tag{9}$$

$$\dot{R}_{fi} = \frac{1}{T_{Fi}}(-R_{fi} + E_{fdi}) \tag{10}$$

$$V_i^2 = (e'_{di} + X'_{di}i_{qi})^2 + (e'_{qi} - X'_{qi}i_{di})^2 \tag{11}$$

and their parameters are listed in Table 2.

TABLE 2. VOLTAGE REGULATOR CONSTANTS

$T_A = 0.06$ s	$K_E = -0.0445$
$T_E = 0.5$ s	$K_F = 0.16$
$T_F = 1.0$ s	$A_{sat} = 0.001123$
$K_A = 25$	$B_{sat} = 0.3043$

This model is described by twenty-one differential equations whose linearized form will be analyzed in two- and in four-time-scales. It is well-known that for machines with non-uniform damping, the system order can be reduced by one as one of the angles can be used as a reference (Prabhakara and El-Abiad, 1975). However, for illustrative purposes, we use the individual machine angles and eliminate the extraneous angle variable only after we have introduced the angle transformation. This model will be referred to as the full 20th order model.

3. THE SEPARATION PROCEDURE

The iterative scheme developed in the companion paper starts with a model in the state separable form

$$\begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} \tag{12}$$

where x is predominantly slow and z contains fast transients or oscillations superimposed on slowly varying quasi-steady-state (qss). Note that the scale factor ε is incorporated in C and D (see *Remark* in Section 4 of the companion paper). This full system is then decomposed into a slow subsystem

$$\dot{\xi}_j = A_{kj}\xi_j \tag{13}$$

and a fast subsystem

$$\dot{\eta}_k = D_{kj}\eta_k, \tag{14}$$

and the original variables x and z are approximated by

$$\begin{aligned} x_{kj} &= \xi_j + H_{kj}\eta_k \\ z_{kj} &= \eta_k - L_k x_{kj} \end{aligned} \tag{15}$$

where k and j denote the number of iterations performed on the fast and slow variables, respectively.

For completeness, we summarize the iterative procedure (57), (58) in the companion paper as follows. We first compute L_k from

$$L_i = D^{-1}C + D^{-1}L_{i-1}(A - BL_{i-1}) \qquad L_1 = D^{-1}C \tag{16}$$

$i = 2, \ldots, k$, and obtain

$$\begin{aligned} A_k &= A - BL_k \\ C_k &= C - DL_k + L_k A_k \\ D_k &= D + L_k B. \end{aligned} \tag{17}$$

Then, we compute H_{kj} from

$$H_{ki} = BD_k^{-1} + (A_k - H_{ki-1}C_k)H_{ki-1}D_k^{-1}, \qquad H_{k1} = BD_k^{-1} \tag{18}$$

$i = 2, \ldots, j$, and obtain

$$\begin{aligned} A_{kj} &= A_k - H_{kj}C_k \\ B_{kj} &= B - H_{kj}D_k + A_{kj}H_{kj} \\ D_{kj} &= D_k + C_k H_{kj}. \end{aligned} \tag{19}$$

The iterations (16), (17) are used for correcting the fast variables η_k and (18), (19) for the slow variables ξ_j. Different j and k are possible depending on the accuracy requirements for the slow and the fast variables. If a better accuracy of the slow variables is required while some inaccuracy of the fast variables can be tolerated, then a few more H-iterations are used and vice versa. Note that the quasi-steady-state models as defined in the companion paper is obtained by substituting A_1, D, 0 and L_1 for A_{kj}, D_{kj}, H_{kj} and L_k in (13)–(15).

In contrast to modal analysis, A_{kj} and D_{kj} approximate the slow and the fast modes of (12), respectively, in groups rather than individually. There are two distinct advantages in this approach. First, (15) indicates that ξ_j retains the physical nature of the original variable x while η_k retains that of the variable z. Second, the expressions (16)–(19) for A_{kj} and D_{kj} are expressed in terms of the original matrices A, B, C and D, and the calculations are straightforward. The price to be paid for these advantages is that a model has to be in a state separable form (12). In this paper we demonstrate how this is done in practice using a 5th order electromechanical model and a 20th order model of the test system.

The separation procedure which we follow in the examples consists of two phases.

Modeling phase. As is the case in power systems, large scale systems often are composed of individual systems having the same types of components. If these individual systems are weakly interconnected, the slow and fast states of the overall system could be obtained by determining the slow and fast states of the individual systems. However, it is usually the case that interconnections result in an altering of the slow and fast variables as determined from the study of the

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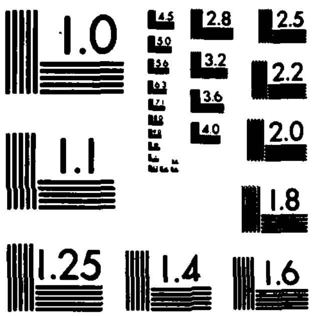

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individual systems. As a result transformations may be required to put the total system into state separable form. This may be achieved by a global study of the appropriate interacting components. Such a study will allow us to propose candidates for the slow states x and the fast states z.

Validation phase. Norm conditions such as those in Kokotovic (1975) usually yield conservative results and hence are not used here. Experience has shown eigenvalues to be a better indication of convergence. Given a proposed model in the form (12) the qss model and models obtained by using the iterative method with one or two levels of correction ($k=j=1$ or 2) are used in this validation phase. If the increments of the eigenvalues in successive A_{kj} or D_{kj} are small then the choice of slow and fast variables is adequate. Otherwise, a different selection of states is required, that is, a return to the modeling phase. Alternatively, eigenvectors of A_{kj} or D_{kj} may be used to indicate convergence.

machine against an infinite bus. Finally the fastest is the motion of the two smaller machines relative to each other. These motions are better exhibited in a new set of variables

$$\delta_r = \frac{H_1\delta_1 + H_2\delta_2 + H_3\delta_3}{H_1 + H_2 + H_3}$$

$$\delta_c = \frac{H_2\delta_2 + H_3\delta_3}{H_2 + H_3} - \delta_r$$

$$\delta_d = \delta_2 - \delta_3 \tag{20}$$

and a similar set of variables ω_r, ω_c, ω_d in terms of ω_1, ω_2, ω_3. Since the dynamics of the other five variables do not depend on δ_r, this angle is not included and the model reduces to fifth order. The ω_r coordinate (20) is commonly used in stability analysis (Stanton, 1972; Luini, Schulz and Turner, 1975). The linearized post-fault system is

$$\begin{bmatrix}\Delta\omega_r\\ \Delta\delta_c\\ \Delta\omega_c\\ \Delta\delta_d\\ \Delta\omega_d\end{bmatrix} = \begin{bmatrix} -0.198 & 0.00756 & 0.00486 & 0.00733 & -0.00181\\ 0 & 0 & 377 & 0 & 0\\ 0.0122 & -0.133 & -0.191 & 0.0304 & -0.00454\\ 0 & 0 & 0 & 0 & 377\\ -0.292 & 0.163 & -0.0292 & -0.426 & -0.175 \end{bmatrix} \begin{bmatrix}\Delta\omega_r\\ \Delta\delta_c\\ \Delta\omega_c\\ \Delta\delta_d\\ \Delta\omega_d\end{bmatrix}. \tag{21}$$

The models (13) and (14) which passed the validation phase may be iteratively improved using (16)–(19) to match the needs of applications such as simulation in two-time-scales and decomposed control design. In this paper, only the linear simulation results are presented. The above procedure can be extended to multi-time-scales by a repeated application of the two-time-scale procedure.

4. THE ELECTROMECHANICAL MODEL

It can be expected that the time scales introduced in the single-machine analysis in Section 5 of the companion paper will have to be modified here primarily due to electromechanical interactions between the machines. These interactions are considered first by assuming that the voltage V_1, V_2, V_3 in (1), (2) are constant. From the fault location and the physical parameters of the system, three different speeds of system dynamics are to be expected. The slowest is the motion of the whole system as a single unit. The second is the motion of the two smaller machines moving together against the center of inertia, which is analogous to the motion of the single

To test whether this model can be separated into three subsystems, we apply the iterative scheme to decompose (21) into the slow subsystem ($\Delta\omega_r$, $\Delta\delta_c$, $\Delta\omega_c$) and the fast subsystem ($\Delta\delta_d$, $\Delta\omega_d$), and then further decompose the slow subsystem into two subsystems ($\Delta\omega_r$) and ($\Delta\delta_c$, $\Delta\omega_c$). The qss models are

$$\Delta\dot\omega_r = -0.199\Delta\omega_r$$

$$\begin{bmatrix}\Delta\dot\delta_c\\ \Delta\dot\omega_c\end{bmatrix} = \begin{bmatrix}0 & 377\\ -0.102 & -0.193\end{bmatrix}\begin{bmatrix}\Delta\delta_c\\ \Delta\omega_c\end{bmatrix} \tag{22}$$

$$\begin{bmatrix}\Delta\dot\delta_d\\ \Delta\dot\omega_d\end{bmatrix} = \begin{bmatrix}0 & 377\\ -0.426 & -0.175\end{bmatrix}\begin{bmatrix}\Delta\delta_d\\ \Delta\omega_d\end{bmatrix}.$$

Since the order of the full system is small we compare the eigenvalues of the qss models directly to the eigenvalues of the full system. As Table 3 shows, their eigenvalues approximate the eigenvalues of the full model (21) within 2%.

This excellent separation of the time-scales motivates the use of the variables (20) in the 20th order system.

5. TWO-TIME-SCALE MODELING

Now we proceed to perform a two-time-scale

TABLE 3. EIGENVALUES OF THE FULL MODEL (21) AND THE SUBSYSTEMS (22)

Full model	qss models
-0.199	-0.199
$-0.0969 \pm j6.09$	$-0.0965 \pm j6.20$
$-0.0857 \pm j12.9$	$-0.0877 \pm j12.7$

decomposition on the full 20th order model. The choice of the slow and fast variables for this model will be based on the results obtained from two partial models, namely, the single machine model in the companion paper and the electromechanical model in the previous section.

In the 7th order single machine model using data close to those of machine 2, the variables ($\Delta e'_q$, ΔR_f) have been identified as slow and the other variables ($\Delta e'_d$, $\Delta\delta$, $\Delta\omega$, ΔV_R, ΔE_{fd}) as fast. Similar conclusions are obtained using data from each of the three machines. On the other hand, the electromechanical model from Section 4 exhibits one state ($\Delta\omega_r$) whose speed is comparable to $\Delta e'_q$ and ΔR_f, and four states ($\Delta\delta_c$, $\Delta\omega_c$, $\Delta\delta_d$, $\Delta\omega_d$) whose speeds are comparable with the fast variables in the single machine model.

Whether the choice of states suggested by the two partial models is applicable to the full model depends on the interactions. From this point of view, the voltage regulator variables ΔV_{Ri} and ΔE_{fdi} should be retained as fast, and ΔR_{fi} as slow in the full model. Furthermore, the $\Delta\omega_r$ variable is assigned to the slow time-scale and $\Delta\delta_c$, $\Delta\omega_c$, $\Delta\delta_d$, $\Delta\omega_d$ are assigned to the fast time-scale. Their interaction with the voltage regulator is known to significantly change the damping, but not the frequencies. The decision on the remaining $\Delta e'_{qi}$, $\Delta e'_{di}$ is considerably more complex and likely to depend on the parameters of the specific system. The single machine separation of $\Delta e'_{qi}$ as slow and $\Delta e'_{di}$ as fast seems a good starting point for systems with weak and moderate interactions.

In view of the above discussion, we propose the (7, 13) decomposition

$$x^{\mathrm{T}} = (\Delta\omega_r, \Delta e'_{q1}, \Delta R_{f1}, \Delta e'_{q2}, \Delta R_{f2}, \Delta e'_{q3}, \Delta R_{f3})^{\mathrm{T}}$$

$$z^{\mathrm{T}} = (\Delta e'_{d1}, \Delta e'_{d2}, \Delta e'_{d3}, \Delta\delta_c, \Delta\omega_c, \Delta V_{R1}, \Delta E_{fd1}, \Delta V_{R2}, \Delta E_{fd2}, \Delta V_{R3}, \Delta E_{fd3}, \Delta\delta_d, \Delta\omega_d)^{\mathrm{T}}. \quad (23)$$

The system matrix for this ordering of the states is given in Fig. 2.

A particular merit of this grouping of state variables is that x contains the system frequency, all the slow flux linkage variables and all the slow regulator modes, while z contains all the fast flux linkage variables, the fast regulator modes and the swing modes. This grouping of variables of similar physical nature simplifies the identification of the slow and fast variables. With the slow and fast states tentatively identified, we may proceed to the validation phase immediately. Note that the above decomposition does not depend on the knowledge of the exact eigenvalues of the full system.

An alternative approach is possible if the exact eigenvalues are given as they are in Table 4. Then we first note that there is a gap between the 7 small eigenvalues and the 13 large eigenvalues, and hence, the order of the slow subsystem is

$\Delta\omega_r$	-.246	-.0612	0	-.0142	0	-.0161	0	.00677	.0111	.00665	.00477	-.0277	0	0	0	0	0	0	.00146	-.000903
$\Delta e'_{q1}$	0	-.560	0	.0736	0	.138	0	.153	-.0629	-.0462	-.0154	0	0	.25	0	0	0	0	-.0160	0
ΔR_{f1}	0	0	-1.0	0	0	0	0	0	0	0	0	0	0	1.0	0	0	0	0	0	0
$\Delta e'_{q2}$	0	-.0467	0	-.427	0	.134	0	.0816	.0484	.0830	-.121	0	0	0	0	.167	0	0	-.152	0
ΔR_{f2}	0	0	0	0	-1.0	0	0	0	0	0	0	0	0	0	0	1.0	0	0	0	0
$\Delta e'_{q3}$	0	.122	0	.171	0	-.538	0	.0954	-.0613	.0387	-.161	0	0	0	0	0	0	.20	.103	0
ΔR_{f3}	0	0	0	0	0	0	-1.0	0	0	0	0	0	0	0	0	0	0	1.0	0	0
$\Delta e'_{d1}$	0	-2.36	0	.970	0	.711	0	-8.77	1.13	2.13	4.64	0	0	0	0	0	0	0	-1.08	0
$\Delta e'_{d2}$	0	-.868	0	-.516	0	-.885	0	.476	-6.63	1.63	-.406	0	0	0	0	0	0	0	-1.01	0
$\Delta e'_{d3}$	0	-.683	0	.296	0	-.277	0	.875	1.22	-3.96	-.989	0	0	0	0	0	0	0	1.31	0
$\Delta\delta_c$	0	0	0	0	0	0	0	0	0	0	0	377	0	0	0	0	0	0	0	0
$\Delta\omega_c$	-.0694	.00826	0	-.0652	0	-.0685	0	[illegible]	.00499	-.0320	-.101	-.288	0	0	0	0	0	0	.0313	-.00227
ΔV_{R1}	0	-.355	66.7	-10.5	0	-23.0	0	59.4	14.8	14.7	16.6	0	-16.7	-66.7	0	0	0	0	-.276	0
ΔE_{fd1}	0	0	0	0	0	0	0	0	0	0	0	0	2.0	.3806	0	0	0	0	0	0
ΔV_{R2}	0	-32.7	0	-270	66.7	-61.6	0	-15.6	171	4.36	29.8	0	0	0	-16.7	-66.7	0	0	[illegible]	0
ΔE_{fd2}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	2.0	.0781	0	0	0	0
ΔV_{R3}	0	-81.0	0	-48.2	0	-220	66.7	-12.6	63.0	99.2	38.9	0	0	0	0	0	-16.7	-66.7	[illegible]	0
ΔE_{fd3}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	2.0	.0803	0	0
$\Delta\delta_d$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	377
$\Delta\omega_d$	-.0146	.0140	0	-.290	0	.229	0	-.130	-.204	.326	.170	-.0146	0	0	0	0	0	0	-.333	-.110

FIG. 2. System matrix of full model.

TABLE 4. EIGENVALUE AND EIGENVECTOR APPROXIMATIONS OF THE 7,13 DECOMPOSITION

Eigenvalues of the 20th Order System	Subsystems	Eigenvalues of the Subsystems			Cosines of the principal angles between the eigenvectors of the full system and subsystems		
		qss Models	First Level Correction	Second Level Correction	qss Models	First Level Correction	Second Level Correction
-.245		-.245	-.245	-.245	.998	.998	1.
-.342±j.502	Slow	-.372±j.538	-.336±j.505	-.342±j.502	.990, .977	.991, .977	1., 1.
-.467∓j.777		-.511∓j.770	-.463∓j.774	-.466∓j.776	.987, .930	.987, .932	1., .998
-1.14∓j.866		-1.06∓j.673	-1.15∓j.784	-1.17∓j.866	.982, .806	.989, .815	1., .984
-2.11		-2.33	-2.08	-2.04	.831	.956	.993
-4.17		-4.05	-4.17	-4.17	.979	.999	1.
-.759±j4.63		-.667±j4.71	-.763±j4.62	-.759±j4.63	.988, .939	1., .999	1., 1.
-7.36	Fast	-7.59	-7.37	-7.36	.987	1.	1.
-8.17±j7.70		-8.29±j7.95	-8.16±j7.71	-8.16±j7.70	.927, .771	1., .998	1., 1.
-8.46∓j8.06		-8.29∓j7.95	-8.46∓j8.06	-8.46∓j8.06	.853, .547	1., 1.	1., 1.
-8.55∓j8.24		-8.29∓j7.95	-8.55∓j8.24	-8.55∓j8.23	.975, .313	1., 1.	1., 1.
-1.42∓j11.1		-1.33∓j11.0	-1.42∓j11.1	-1.42∓j11.1	.999, .997	1., 1.	1., 1.

tentatively set at 7, while that of the fast subsystem is set at 13.

To identify the slow and fast variables, we attempt to correlate groups of eigenvalues of the partial models with the exact eigenvalues. For example, the swing modes are recognizable in Table 4 by being close to the swing modes in Table 3. To be specific, the pair $-1.42 \pm j11.1$ corresponds to the intermachine swing modes $\Delta\delta_4$, $\Delta\omega_4$, the pair $-0.759 \pm j4.63$ corresponds to the center of swing $\Delta\delta_c$, $\Delta\omega_c$ of machines 2 and 3, and the mode -0.245 corresponds to the system frequency $\Delta\omega_s$. Note that the inclusion of voltage regulators has improved the damping, but does not significantly alter the frequencies of these modes.

Another group of eigenvalues which can also be easily correlated are the pairs $-8.17 \pm j7.70$, $-8.46 \pm j8.06$ and $-8.55 \pm j8.24$, which correspond to the fast voltage regulator modes ΔV_{Ri}, ΔE_{fdi} (see the companion paper). Furthermore, the modes -7.36, -4.17 and -2.11 are close to the eigenvalues -9.16, -5.64 and -2.56 of the 3×3 $\Delta e'_{di}$ submatrix obtained from the A matrix, implying that they are associated with $\Delta e'_{di}$. The remaining eigenvalues therefore correspond to the $\Delta e'_{qi}$, ΔR_i modes. This analysis deals with groups of modes rather than the individual modes. The same 7 slow and 13 fast state separation (23) is obtained.

6. TWO-TIME-SCALE VALIDATION

We now apply the iterative scheme to validate the choice of the slow and the fast variables by comparing the eigenvalues of the qss models with the eigenvalues of the first and second level corrected models $(k = j = 1, 2)$ given in Table 4. The eigenvalue approximation between the qss models and the first level $(k = j = 1)$ corrected models is within 11%. Between subsystems with first level corrected models and second level corrected models $(k = j = 2)$, the increment is within 5%. This indicates that the choice of state variables (23) for the fast and slow subsystems is appropriate. Note that even though the fast-slow ratio defined as $|-2.11|/|-1.14 \pm j0.866| = 1.5$ is not much larger than one, the iterative scheme is still applicable.

We now illustrate how eigenvectors may be used as an indication of convergence. For real eigenvectors, we compute the angle $\theta = \cos^{-1}\{(u, v)/|u||v|\}$ between the eigenvector u corresponding to the accurate eigenvalue λ and the eigenvector v corresponding to the eigenvalue of (13), (14) and (15) approximating λ, where $(.)$ denotes the dot product between two vectors. For the complex eigenvector $u = u_1 \pm ju_2$ and its approximation $v = v_1 \pm jv_2$, we compute the inclination of the subspace $S' = \{v_1, v_2\}$ spanned by the vectors v_1, v_2 with respect to the subspace $S = \{u_1, u_2\}$ spanned by the vectors u_1, u_2. This inclination can be measured by the principal angles θ_1, θ_2 (Bjorck and Golub, 1973) which are defined as follows:

1. θ_1 is the smallest angle between any pair of vectors $a_1 \in S$ and $b_1 \in S'$, that is, $a_1 = c_1u_1 + c_2u_2$, $b_1 = c_3v_1 + c_4v_2$, where c_i are scalar quantities.
2. θ_2 is the smallest angle between any pair of vectors $a_2 \in S$ and $b_2 \in S'$ subject to the constraints $a_2 \perp a_1$, $b_2 \perp b_1$.

The smaller the angles θ_1, θ_2 are, the better the approximation of the eigenvectors computed from the subsystems will be. This is because the orientation of the subspaces S and S' is almost parallel. From the definition of θ_1 and θ_2, it seems that a search scheme is required to find θ_1

and θ_2. However, it is shown in Laub (1977) that $\cos\theta_1$, $\cos\theta_2$ are directly given by the square roots of the eigenvalues of the 2×2 matrix $U^{+}VV^{+}U$ where $U^{+}=(U^TU)^{-1}U^T$, $V^{+}=(V^TV)^{-1}V^T$, $U=[u_1u_2]$, $V=[v_1v_2]$.

For large systems, comparisons between the eigenvectors of the subsystem models with different levels of correction may be used. Alternatively if the order of the full system is not large, as in this case, comparisons between the full order system and the subsystem models may be performed. The cosines of θ_1 and θ_2 between the accurate eigenvectors and the eigenvectors constructed from the subsystems are shown in Table 4. With first level correction, the cosine of the worst principal angle is 0.815, which improves to 0.984 with one more level of correction.

7. TWO-TIME-SCALE SIMULATION

From the validation phase it can be expected that the 7th order slow subsystem model (13) and the 13th order fast subsystem model (14) with first level correction will provide a satisfactory approximation of the 20th order model. The subsystem models are used to simulate a five cycle three phase fault on bus 8 followed by the loss of line 8–9. The original state variables x, z may be obtained from the subsystem variables ξ, η by using (15).

Figures 3(a–d) show the responses of two slow variables $\Delta\omega_r$, $\Delta e'_{q1}$, and two fast variables $\Delta\omega_c$, ΔV_{R1}. Other responses are similar and with even smaller approximation error. For the fast variables, the accuracy is excellent as the difference between the exact curves and the first level approximate curves is virtually indistinguishable. In the slow variables, the error is noticeable but small.

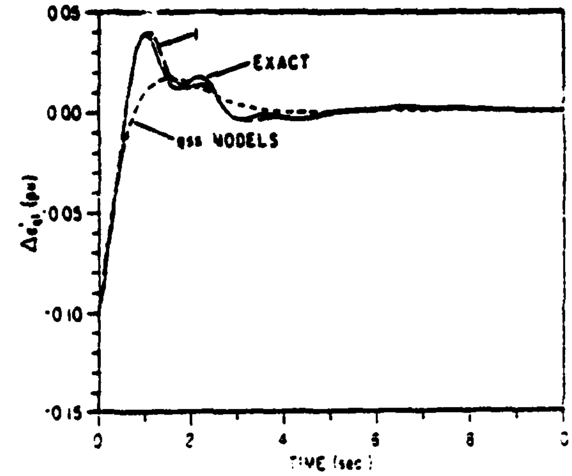

FIG. 3(b). 7,13 decomposition of $\Delta e'_{q1}$ with no and first level corrections.

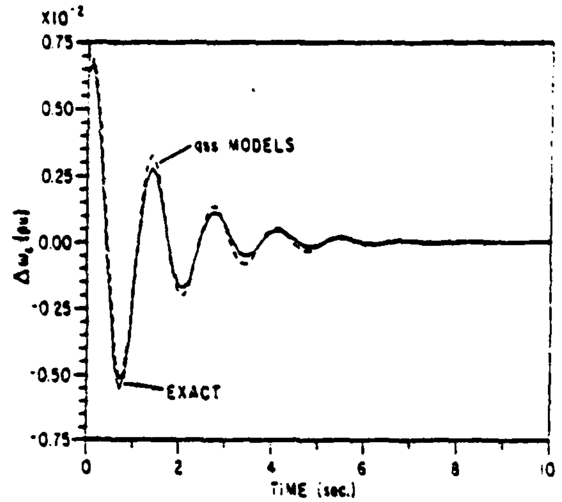

FIG. 3(c). 7,13 decomposition of $\Delta\omega_c$ with no and first level corrections.

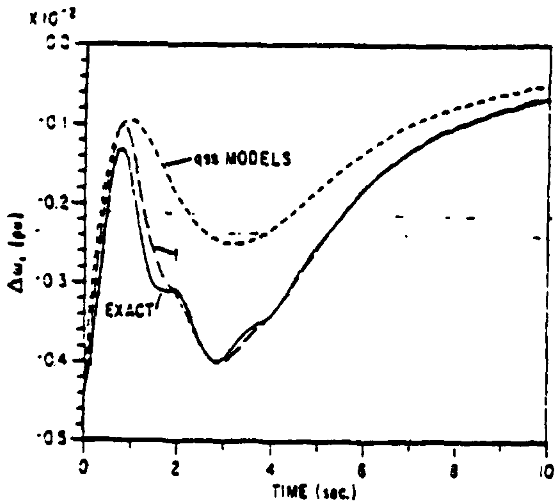

FIG. 3(a). 7,13 decomposition of $\Delta\omega_r$ with no and first level corrections.

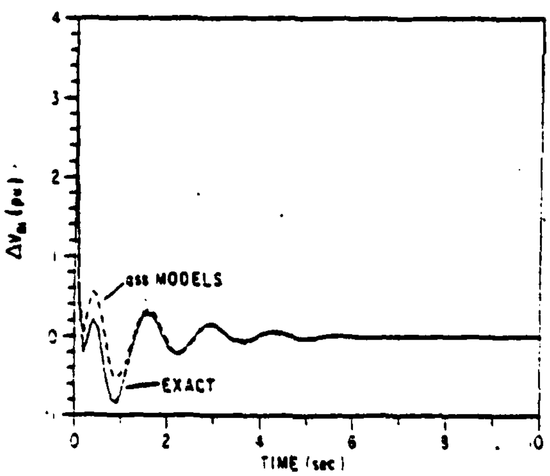

FIG. 3(d). 7,13 decomposition of ΔV_{R1} with no and first level corrections.

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AREA DECOMPOSITION FOR ELECTROMECHANICAL MODELS OF POWER SYSTEMS

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Abstract. The notion of slow coherency is introduced as a less demanding definition of coherency, which allows for a lack of coherency in the fast part of machine transients. The relationship between the time scale properties and the slow coherency is shown to be the dichotomic solution of a matrix Riccati equation. A grouping algorithm is presented which reduces the area decomposition problem to one of obtaining a basis for the slow subsystem and performing a Gaussian elimination. A geometric interpretation of this area grouping algorithm is also presented. The procedure is illustrated with a 3-machine and a 16-machine example.

Keywords. Power system modeling; large scale systems; system order reduction; singular perturbations; identification.

INTRODUCTION

The size of any present day power system is such that full scale simulation of even a basic multimachine electromechanical model may be too costly. It has been observed that in post-fault transients represented by this model only some machines closer to the fault respond as individual units, while other machines more distant from the fault swing together with "in-phase" slow motion. In a typical study, each of these groups is considered to be a "coherent area". Then, only the faulted area is modeled in detail, while other areas are represented by equivalent machines.

A critical step in such studies is the grouping of the machines into areas. Coherent machines are identified either from actual or simulated machine responses, (Marconato, Mariani and Saccomano, 1973; Podmore, 1978), or by an algebraic evaluation of the modes present in the linearized response of each machine (Pai and Adgaonkar, 1979; Lawler and others, 1979; Saccomano, 1974a, 1974b; DiCaprio and Marconato, 1978; Price and others, 1978; Bhatt, Kwatny and Mablekos, 1976). Most analytical techniques require that machines be coherent throughout the duration of their transients. In this paper we introduce a less demanding definition of "slow coherency," which allows a lack of coherency in the fast part of the transients. It may be interpreted as a requirement that the equivalent machines of the areas represent as closely as possible a preselected group of the slowest modes. The resulting area decomposition is independent of various fault locations.

With our approach the slow and the fast modes are dichotomically separated as groups using a transformation well known from the singular perturbation technique (Chow, Allemong and Kokotovic, 1978; Kokotovic and others, 1980). Grouping machines according to the slow coherency criterion means in singular perturbation terms that the equivalent machines constitute the slow subsystem. The fast subsystem is then formulated to represent the fast oscillations within each area. The slow and fast subsystem models are obtained from the dichotomic transformation matrices L and M, which define a set of physically meaningful state variables. In the ideal slow coherency case the dichotomic L is a "grouping" matrix, whose elements are zeros and ones, and the state variables of the fast subsystem are machine angle differences within areas. On the other hand, the matrix M, which separates the slow subsystem, actually defines the slow variables as the area centers of inertias (Stanton, 1971; Marconato, Mariani and Saccomano 1973; Saccomano, 1972). In a nonideal case our approach is to search for a dichotomic L whose elements are in some sense close to zeros and ones. This dichotomic L is then approximated by a grouping matrix and the corresponding M is computed as a function of this matrix. This results in areas which contain machines that are near-coherent in their slow modes, and in weakly coupled rather than decoupled slow and fast subsystems.

In the literature on power system dynamic equivalents, there has been a continuing interest in the development of a systematic area decomposition procedure. Our grouping algorithm reduces the decomposition procedure

to the calculation of a basis for the slow subsystem and a Gaussian elimination. In the formulation of our algorithm we have benefited from insights and results of the above referenced authors. In particular, a motivation to relate coherency and singular perturbations is found in DiCaprio and Marconato (1978) and some important properties of what we call r-decomposable systems appear or are alluded to in Saccomano (1974a). Our analysis incorporates these properties in a unified framework of dichotomic solutions of Riccati equations and establishes new properties. These properties are the tools for the development of the algorithm and the separation of time scales.

In the next section we first review some properties of the model used in this paper. The third section defines the notion of slow coherency and reveals the structure of ideally decomposable systems. A grouping algorithm is developed for near-decomposable systems in the fourth section. The fifth section introduces the slow variables. While the original states contain a mix of fast and slow phenomena, the new states make it possible to apply singular perturbation techniques to nonlinear electromechanical and potentially more extensive power system models. The presentation in the fourth and fifth sections is illustrated by a 16-machine example.

ELECTROMECHANICAL MODEL

The well-known electromechanical model (Anderson and Fouad, 1977) of an n-machine power system is

$$\dot{\delta}_i = \Omega(\omega_i - 1), \qquad (2.1)$$

$$2H_i\dot{\omega}_i = -D_i(\omega_i - 1) + (P_{mi} - P_{ei}), \qquad (2.2)$$

where

$i = 1, 2, \ldots, n,$

δ_i = rotor angle of machine i (radians),

ω_i = speed of machine i (per unit),

P_{mi} = mechanical input power of machine i (per unit),

P_{ei} = electrical output power of machine i (per unit),

H_i = inertia constant of machine i (seconds),

D_i = damping constant of machine i (per unit),

Ω = base frequency (radians per second).

In this model disturbances are represented by appropriate selection of initial conditions, and the following assumptions are usually made.

(A1) Mechanical input power P_{mi} is constant.
(A2) The electrical output power is

$$P_{ei} = \sum_{\substack{j=1 \\ j\neq i}}^{n} V_iV_jB_{ij}\sin(\delta_i - \delta_j) + V_i^2G_{ii},$$

$$i = 1, 2, \ldots, n. \qquad (2.3)$$

where the per unit voltage V_i behind transient reactance is assumed to be constant and saliency is neglected. Loads are represented by passive impedances, and G and B are the real and imaginary parts of the reduced admittance matrix Y at the internal machine nodes. The off-diagonal resistive terms of Y are neglected.

The intermachine motions are largely determined by the natural frequencies and the mode shapes of the linearized electromechanical model around the stable equilibrium δ_i^* and $\omega_i^* = 1.0$. The linearized model is

$$\Delta\dot{\delta}_i = \Omega\Delta\omega_i, \qquad (2.4)$$

$$2H_i\Delta\dot{\omega}_i = -D_i\Delta\omega_i - \sum_{j=1}^{n} k_{ij}\Delta\delta_j, \qquad (2.5)$$

where

$$\Delta\delta_i = \delta_i - \delta_i^*, \qquad (2.6)$$

$$\Delta\omega_i = \omega_i - 1, \qquad (2.7)$$

$$k_{ii} = \sum_{\substack{j=1 \\ j\neq i}}^{n} V_iV_jB_{ij}\cos(\delta_i - \delta_j)\Big|_{\delta^*}, \qquad (2.8)$$

$$k_{ij} = V_iV_jB_{ij}\cos(\delta_i - \delta_j)\Big|_{\delta^*}, \ j\neq i. \qquad (2.9)$$

At δ^* and ω^*, the eigenvalues of (2.4) and (2.5) are of the following three types:

1. a zero eigenvalue corresponding to the motion of all the machine angles,
2. a small negative real eigenvalue corresponding to the aggregate speed of all the machines, and
3. (n-1) pairs of lightly damped oscillatory modes which typically range in frequency from 1/2 to 2 Hz.

Models involving more details such as excitation systems and governors would still contain the above set of eigenvalues modified mostly in the damping and not in the frequencies (Podmore, 1978). Since the small damping constants D_i do not significantly affect the frequencies of the oscillatory modes (DiCaprio and Sccomano, 1970) they may be neglected. Thus, the model used in this paper is

$$\ddot{x} = -(1/2)\Omega H^{-1}K\,x \overset{\Delta}{=} Ax, \qquad (2.10)$$

where

$x_i = \Delta\delta_i$

$H = \text{diag}\,(H_1, H_2, \ldots, H_n)$

$$K = (k_{ij}). \qquad (2.11)$$

Therefore instead of dealing with a system of order 2n, we only need to deal with the nxn matrix A. *Due to Assumption (A2)*, K is symmetric if Y is symmetric which is true for networks without phase shifters.

The properties of the eigenvalues of the A matrix are as follows:

(P1) A has a zero eigenvalue whose eigenvector is

$$v_o = (\ 111\ \ldots\ 1\ )' \qquad (2.12)$$

Property (P1) follows from $Av_o=0$, which is due to (2.8) and (2.9) as the sum of each row in $A=(a_{ij})$ is

$$\sum_{j=1}^{n} a_{ij} = 0 \qquad i = 1, 2, \ldots, n \qquad (2.13)$$

(P2) When K is symmetric A is diagonalizable because it is similar to the symmetric matrix

$$-(1/2)\Omega H^{-1/2} K H^{-1/2}$$

where $H^{1/2}$ is the square root of H.

Thus, all the eigenvalues λ_i of A are real. It follows that the eigenvalues of the second order system (2.10) are $\pm\sigma_i$, where

$$\sigma_i = \sqrt{\lambda_i} \qquad (2.14)$$

For λ_i negative, they are on the imaginary axis close to the slightly damped eigenvalues of (2.4) and (2.5). The double eigenvalue $\sigma_i=0$ corresponds to the aggregate motion of the machine angles and speeds. In the following analysis, it is important to note that the low frequency modes of (2.4) and (2.5) are the slow modes of A.

SLOW COHERENCY

In most actual and simulated responses the groups of machines "swinging together" are discernible only in slow motion. This motivates the following definition of coherency which allows responses of coherent machines to have different fast dynamics.

Definition 3.1

Given r smallest in magnitude eigenvalues (slowest modes) of A in (2.10). Then machines "i" and "j" are slowly coherent if for all t of interest, possibly $t\epsilon[0,\infty)$, their angles $x_i(t)$ and $x_j(t)$ satisfy

$$x_i(t) - x_j(t) = z_{ij}(t) \qquad (3.1)$$

where $z_{ij}(t)$ contains none of the r slow modes. A coherent area consists of all the machines coherent to each other.

This definition of coherency may be interpreted as a row property of the nxr matrix V of slow eigenvectors. Machines i and j are coherent if rows i and j of V are identical. It is not hard to see that this remains true for the columns of V that form any basis of the slow eigensubspace.

We note that in this definition no machines from different areas can be coherent, that is *no coherent area can be divided into more* areas. An individual machine can constitute an area if it is not coherent with any other machine.

Although Definition 3.1 does not require that the number of coherent areas be equal to the number of slow modes, systems with this property, which will be called <u>r-decomposable systems</u>, are of particular interest for separation of time scales. The study of r-decomposable systems is an essential step toward the analysis of more common "near-decomposable" systems, that is systems with near-coherent rather than coherent areas.

Definition 3.2

The machines "i" and "j" are near-coherent if in Definition 3.1 the contribution of the slow modes in $z_{ij}(t)$ is small. A near-coherent area consists of all machines which are near-coherent to each other. An r near-decomposable system consists of r near-coherent areas.

Our approach to area determination is to first consider r-decomposable systems. We show that in this idealized case the dichotomic solution of a matrix Riccati equation automatically groups the machines into areas. We then use this result to develop a grouping algorithm for near-decomposable systems.

To define a compact notation for areas we introduce a <u>reference set</u> of machines and a <u>grouping matrix</u>. In each area we pick an arbitrary machine as the reference machine. The reference machine angles are considered as components of an r-vector x^1, while all other angles form the (n-r)-vector x^2. Equation (3.1) motivates the use of a grouping matrix L_g to assign machines to areas. The (i,j) entry of L_g is 1 if machines x_j^1 and x_i^2 are in the same area, and is zero otherwise. Thus, given x^1, x^2 and L_g the areas are uniquely determined. However, given the areas there is no unique choice of x^1 and x^2, and hence many possible choices for L_g exist.

As an illustration consider a three area five-machine system. Given

$$x^1 = (x_1, x_2, x_4)'$$

$$x^2 = (x_3, x_5)' \qquad (3.1)$$

$$L_g = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad (3.2)$$

the three areas, which are composed of machines 1 and 3, machines 2 and 5 and machine 4, are uniquely defined. For the same areas a different choice of x^1 and x^2, such as

$$x^1 = (x_4, x_3, x_2)'$$

$$x^2 = (x_1, x_5)' \qquad (3.3)$$

will result in a different L_g, that is

$$L_g = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad (3.4)$$

Note that the zero column in L_g of (3.2) or (3.4) indicates the presence of a single machine area.

Using L_g, (3.1) is rewritten more compactly as

$$x^2(t) - L_g x^1(t) = z^2(t) \qquad (3.5)$$

where the components of $z^2(t)$ are the corresponding functions $z_{ij}(t)$. In the case where an area contains k machines there will be exactly (k-1) elements in $z^2(t)$ for this area.

We interpret (3.5) as a special case of a more general coordinate transformation (Kokotovic and others, 1980)

$$\begin{bmatrix} x^1 \\ z^2 \end{bmatrix} = \begin{bmatrix} I & 0 \\ -L & I \end{bmatrix} \begin{bmatrix} x^1 \\ x^2 \end{bmatrix} = T_L \begin{bmatrix} x^1 \\ x^2 \end{bmatrix} \qquad (3.6)$$

where the (n-r) x r matrix L is not necessarily a grouping matrix. The substitution of (3.5) into (2.10), that is into $\ddot{x}=Ax$, results in

$$\begin{bmatrix} \ddot{x}^1 \\ \ddot{z}^2 \end{bmatrix} = \begin{bmatrix} B_1 & A_{12} \\ R(L) & B_2 \end{bmatrix} \begin{bmatrix} x^1 \\ z^2 \end{bmatrix} \qquad (3.7)$$

where

$$B_1 = A_{11} + A_{12}L \ , \ B_2 = A_{22} - LA_{12} \qquad (3.8)$$

$$R(L) = A_{22}L - LA_{11} - LA_{12}L + A_{21} \qquad (3.9)$$

and A_{11}, A_{12}, A_{21}, A_{22} are the submatrices of A conformal with x^1 and x^2. We are particularly interested in L which satisfies

$$R(L) = A_{22}L-LA_{11}-LA_{12}L+A_{21} = 0 \qquad (3.10)$$

and

$$|\lambda_j(B_1)| < |\lambda_i(B_2)| \qquad (3.11)$$

for all i=1,2,...,n-r and j=1,2,...,r. Such an L is called <u>dichotomic</u> and denoted by L_d. It is known from Wu (1977), Martensson (1971), Avramovic (1979), and Medanic (1979), that for diagonalizable A the generalized Riccati equation (3.10) can have at most one dichotomic solution $L=L_d$. The following property of an r-decomposable system relates L_d to a grouping matrix L_g.

Theorem 3.1

In an n-machine system let x^1 be the angles of r non-coherent machines and x^2 be the angles of the other n-r machines. This system is r-decomposable if and only if the dichotomic solution $L=L_d$ of the corresponding equation R(L)=0 is a grouping matrix L_g, that is if and only if

$$L_d = L_g \qquad (3.12)$$

Proof: If the system is r-decomposable, then for the given x^1 and x^2 a grouping matrix L_g exists such that (3.5) is satisfied and z^2 contains no slow modes. Substituting $L=L_g$ into (3.7) through (3.11) we can show that (3.10) and (3.11) must hold or else z^2 would contain slow modes. This is clear from

$$\begin{bmatrix} B_1 & A_{12} \\ R(L) & B_2 \end{bmatrix} \begin{bmatrix} W \\ 0 \end{bmatrix} = \begin{bmatrix} W \\ 0 \end{bmatrix} \Lambda_s \qquad (3.13)$$

where Λ_s is the diagonal matrix of the r slow eigenvalues and the rows of $[W'0']$ are the corresponding eigenvectors in the (x^1, z^2)-coordinates. Hence $R(L)W=0$, which implies $R(L)=0$. Then $\ddot{z}^2=B_2z^2$ and (3.11) must also hold. This proves the if part of the theorem, because L_d is unique. Conversely, if an L_d exists and is a grouping matrix, then the system is r-decomposable because L_g satisfies (3.10) and (3.11).

An interesting interpretation of $R(L_g)=0$ is *a set of slow coherency conditions satisfied* by the voltages, admittances, machine angles and inertias of an r-decomposable system. Let $\alpha(i)$ be the set of machines belonging to area i. Then using the structure of L_g and $R(L_g)=0$ we may conclude that

$$\frac{V_r}{H_r} \sum_{m\in\alpha(i)} V_m B_{rm} \cos(\delta_r - \delta_m) =$$

$$\frac{V_j}{H_j} \sum_{m\in\alpha(i)} V_m B_{jm} \cos(\delta_j - \delta_m) \qquad (3.14)$$

for all i, k=1, 2, ..., r, i≠k

where r is the reference machine in area k and j is any machine in area k. In other words, the sum of the interconnections between the reference machine r in area k and all the machines in area i is the same as the sum of the interconnections between any machine j in area k and all the machines in area i. These conditions hold for all areas. Such "tuned" conditions will generally not hold in practical situations. However, in practice, for relatively normal conditions, voltages are close to 1.0 p.u. and the cosines in (3.14) are close to 1.0. Thus, the coherency condition (3.14) is primarily determined by machine inertias and line admittances, that is by network configuration and much less by the operating conditions. A quantitative criterion for interpreting deviations from the conditions in (3.14) requires further investigation and is beyond the scope of this paper.

Suppose now that we know that a system is r-decomposable, but we do not know its areas. How can Theorem 3.1 help us to find them? First, we make a choice of x^1 and x^2, which in turn defines the corresponding equation R(L)=0. If our choice of x^1 does not contain coherent machines this equation

will have the dichotomic solution L_d which is the grouping matrix needed to find the areas. If our x^1 contains coherent machines, L_d will not exist. The negative outcome would mean that a new choice of x^1 would have to be made and a new equation $R(L)=0$ solved.

If the system is not r-decomposable, then no grouping matrix L_g will satisfy $R(L_g)=0$. Therefore z^2 in (3.5) will contain both fast and slow modes, that is

$$z^2 = z_f^2 + z_s^2 \tag{3.15}$$

For near decomposable systems there exist x^1 and L_g such that z_s^2 is small. Rewriting (3.5) for the slow parts

$$x_s^2 - L_g\, x_s^1 = z_s^2 \tag{3.16}$$

We now have the problem of finding x^1 and L_g such that some measure of the slow coherency error z_s^2 is minimized. From (3.6) with $L=L_d$ it follows that

$$x_s^2 - L_d\, x_s^1 = 0. \tag{3.17}$$

Substituting (3.17) into (3.15) we obtain

$$z_s^2 = (L_d - L_g)\, x_s^1 \tag{3.18}$$

An important conclusion is that the slow coherency error z_s^2 relative to the magnitude x_s^1 of the slow response of the reference machines is bounded by $\| L_d - L_g \|$, that is

$$\frac{|z_s^2|}{|x_s^1|} \le \| L_d - L_g \| \tag{3.19}$$

where the norm is

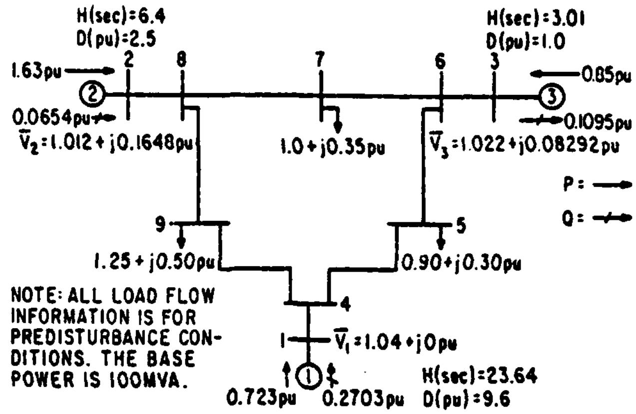
$$\| L \| = \max_i \sum_{j=1}^{r} |L_{ij}|, \quad i=1,\dots,n \tag{3.20}$$

Note that (3.19) is zero for r-decomposable systems. For near-decomposable systems (3.19) motivates a search for x^1 and L_g yielding the smallest $\| L_d - L_g \|$. In principle this search involves the comparison of all possible grouping matrices L_g with the dichotomic solutions L_d corresponding to all possible choices of reference machines x^1. However a much simpler grouping algorithm based on properties of L_d is given in the next section.

To motivate the grouping algorithm and to illustrate the use of (3.19) we consider in detail the three machine system given in (Figure 3.1). For the given numerical values the undamping linearized model is

$$\ddot{x} = \begin{bmatrix} -14.3 & 5.5 & 8.8 \\ 14.3 & -49.4 & 35.1 \\ 58.5 & 81.5 & -140.0 \end{bmatrix} x. \tag{3.21}$$

To decompose the system into two areas, there are only three possible choices of reference machines for this example, $x^1 = (x_1, x_3)'$, $x^1 = (x_2, x_3)'$ and $x^1 = (x_1, x_2)'$ and two possible choices of $L_g = [0,1]$ and $[1,0]$. As our first choice of reference machines consider $x^1 = (x_1, x_3)'$. Then the dichotomic solution of the corresponding equation $R(L)=0$ is



Line #	From	To	R (pu)	X (pu)	B/2 (pu)
1	1	4	0	0.0567	0
2	4	5	0.017	0.092	0.079
3	5	6	0.039	0.170	0.179
4	3	6	0	1.0586	0
5	6	7	0.0119	0.1008	0.1045
6	7	8	0.0085	0.072	0.0745
7	8	2	0	0.0625	0
8	8	9	0.032	0.161	0.153
9	9	4	0.01	0.085	0.088

Fig. 3.1. Three Machine Test System

$$L_d = [-0.470 \;\; 1.47]. \tag{3.22}$$

Of the two possible grouping matrics $L_g=[0\ 1]$ yields the minimum $\| L_d - L_g \|$

$$\| L_d - L_g \| = 0.94. \tag{3.23}$$

The second choice of $x^1=(x_2,x_3)'$ will result in a dichotomic solution of the corresponding equation $R(L)=0$

$$L_d = [-2.13 \;\; 3.13]. \tag{3.24}$$

Of the two possible grouping matrics $L_g=[0\ 1]$ yields the minimum $\| L_d - L_g \|$

$$\| L_d - L_g \| = 4.26. \tag{3.25}$$

The third possible choice of $x^1=(x_1,x_2)'$ will result in a dichotomic solution of the corresponding equation $R(L)=0$

$$L_d = [0.320 \;\; 0.680]. \tag{3.26}$$

Of the two possible grouping matrics $L_g=[0\ 1]$ yields the minimum $\| L_d - L_g \|$

$$\| L_d - L_g \| = 0.64. \tag{3.27}$$

Of these three cases, L_d of (3.26) yields the smallest $\| L_d - L_g \|$. For this case, L_g, x^1, and x^2 indicate that machine 1 is in one area and machines 2 and 3 form the other area.

This example shows that it is possible to have more than one element in D which results in the same area grouping as is the case with the first and third choices of reference machines. However, the grouping is most clearly shown by the third choice. It is also interesting that for the third choice $\|L_d\|$ is the smallest of all three. The second choice has the largest $\|L_d-L_g\|$ and the reference machines are from the same area. Another interesting observation about L_d is that the sum of row elements in all three cases is 1.

Due to $R(L_g)$ not being equal to zero, the eigenvalues of B_1 and B_2 will not be equal to those of A. However, the eigenvalues of B_1 and B_2 are close to those of A

$$\lambda_1(B_1)=0.0,\ \lambda_2(B_1)=-28.6,\ \lambda_1(B_2)=-175.0 \qquad (3.28)$$
$$\lambda_1(A)=0.0,\ \lambda_2(A)=-37.0,\ \lambda_3(A)=-166.0$$

which can be used as an indication that the areas are near-coherent.

The above direct search is presented only as a motivation for the systematic grouping algorithm proposed in the next section.

GROUPING ALGORITHM

From the three machine example it is apparent that finding the areas consists of two interdependent tasks: first, choosing the reference machines and, second, associating the other machines to the reference machines. The approach used in the three machine example is to exhaust all possible choices of x^1 and L_d, that is for each L_d, a particular L_g was found to minimize $\|L_d-L_g\|$. The best choice of x^1 is the one corresponding to the smallest of these minima. When the order of the system is large, this exhaustive search would be computationally prohibitive. Due to the properties of the set D established in Lemmas 4.1 and 4.2, the exhaustive search can be avoided. The algorithm presented in this section computes only one element of the set D, which does not necessarily minimize $\|L_d-L_g\|$, but still unambigously determines the areas. We also provide a geometric interpretation for this algorithm.

Lemma 4.1

Every element L_d of D has the property that

$$\sum_{j=1}^{r} L_{dij} = 1\ ,\ i=1,2,\ldots,n-r \qquad (4.1)$$

that is, the row sum of L_d is 1.

The proof of Lemma 4.1 is given in the Appendix. From (4.1), if all the entries of L_d are greater than or equal to zero, then $\|L_d\|$ is 1, which is the smallest norm achievable by any L_d in D. Furthermore, since the only nonzero entry in any row of L_g is 1, any grouping matrix L_g has row norm equal to 1. Thus, to minimize $\|L_d-L_g\|$, a necessary condition is that $\|L_d\|$ be close to 1. This motivates finding an L_d with a small norm instead of an L_d which minimizes $\|L_d-L_g\|$.

The following result indicates how $\|L_d\|$ depends on choice of reference machines.

Lemma 4.2

Let the angles of a given set of reference machines be ordered as components of x^1 and all the other machine angles as x^2, and let the columns of the nxr matrix

$$V = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \qquad (4.2)$$

where the rxr matrix V_1 is nonsingular, be a basis of the eigensubspace of the slow modes. Then

$$L_d = V_2V_1^{-1} \qquad (4.3)$$

is the unique dichotomic solution of the Riccati equation (3.10). Furthermore, let $\tilde{V}_1$ and $\tilde{V}_2$ be obtained by exchanging rows of V_1 for rows of V_2, that is by a permutation $\tilde{V}=PV$ of the rows of V. Provided that $\tilde{V}_1$ is nonsingular, the element of D corresponding to $\tilde{x}=Px$ is $\tilde{L}_d=\tilde{V}_2\tilde{V}_1^{-1}$.

The proof of Lemma 4.2 is given in the Appendix. This lemma establishes the connection between the Riccati approach presented here with the modal approach given in Saccomano (1974a). The lemma shows that to compute all the L_d elements in D, we only need to compute one V. Furthermore, multiplying both sides of (4.2) by V_1^{-1} we see that

$$V\,V_1^{-1} = \begin{bmatrix} I \\ L_d \end{bmatrix} \qquad (4.4)$$

is also a basis for the eigenspace of the slow modes. In the r-decomposable case, V_1 will be singular if two machines from the same area are in x^1. In the near-decomposable systems V_1 will be close to singular if two near-coherent machines are in x^1. This is due to the fact that the two rows involved are almost identical. When V_1 is close to singular $\|V_1^{-1}\|$ is large, resulting in a large L_d. Thus we aim at finding r large and most linearly independent rows of V. This would result in a V_1 with a large norm such that $\|L_d\|$ of (4.3) would be small.

To find this set of r rows, we use Gaussian elimination with complete pivoting. During the elimination, the rows and columns of V are permuted such that the (1,1) entry of the resulting V is the largest entry in magnitude. Note that permuting the rows of V is

equivalent to changing the ordering of the machines. This (1,1) entry of V is used as the pivot for performing the first step of the Gaussian elimination. Then the largest entry is chosen *from the remaining* $(n-1)x(r-1)$ submatrix of the reduced V and is used as the pivot for the next elimination step. The elimination terminates in r steps and the machines corresponding to the first r rows of the final reduced V matrix are designated as the reference machines. In this Gaussian elimination process, rows having small entries will not be used as the pivoting row because these small entries are the result of elimination with almost identical rows already used as pivoting rows. Thus, this algorithm does not put two near-coherent machines together as reference machines.

For the set of reference machines found by the algorithm the corresponding L_d is readily computed from

$$V_1'L_d' = V_2' \tag{4.5}$$

using the LU decomposition of V_1 obtained from the Gaussian elimination. The next step is to find an L_g approximating L_d, that is to find the machines belonging to each area. We examine each row of L_d and if the largest positive entry is the j-th entry in the row i, then in the matrix L_g entry (i,j) is 1. The resulting L_g will yield the minimum $\| L_d - L_g \|$.

We now summarize the grouping algorithm as follows:

Step 1: Decide on the number of areas.

Step 2: Compute a basis matrix V for a given ordering of the x variables.

Step 3: Apply Gaussian elimination with complete pivoting to V and obtain the set of reference machines.

Step 4: Compute L_d for the set of reference machines chosen in step 3. Construct the matrix L_g and find the machines in each area.

The main computational load is in the step 2. However, only a partial eigensubspace V of A is required and since A is similar to a symmetric matrix, eigenvalue-eigenvector computation is well conditioned (Wilkenson and Reinsch, 1971). Alternatively we can make an initial guess of the set of reference machines and apply the Riccati iterative algorithm in Kokotovic (1975) to calculate L_d. If the solution converges, then the resulting L_d can be used to construct the basis (4.4).

This area grouping algorithm which finds L_d of small norm is supported by the geometric interpretation of near-decomposable systems. For such systems, the row vectors of V corresponding to machines in the same area are almost identical. In other words, the row vectors of machines belonging to the same area are clustered in a cone. These cones are narrow for near-decomposable systems and degenerate to lines for r-decomposable systems. The role of Gaussian elimination is to select the most linearly independent vectors, one from each cone, which are then considered as the reference vectors for the areas. The entries in L_d are the projections of other vectors on the reference vectors. Therefore, it is easy to see that in each row of L_d the entry close to 1 corresponds to the projection of the vector on the corresponding reference vector, and the entries close to zero are projections of the vector to the other reference vectors.

We illustrate this area selection procedure on a 16 machine model (Figure 4.1) in Schulz and others (1974). The data are given in the reference and hence will not be repeated here. The model is linearized and the damping is neglected to obtain the A matrix in (2.10). *In the first step of the algorithm* we specify that we want 5 areas, that is r=5. From this point on the algorithm proceeds automatically giving the following results.

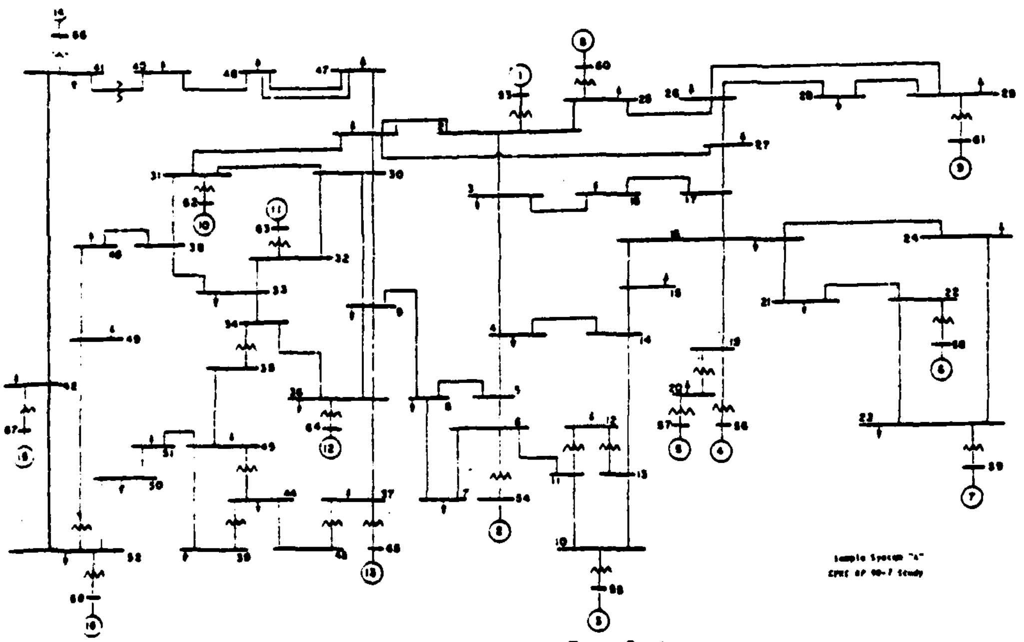


Fig. 4.1 Sixteen Machine Test System

AN ANALYSIS OF INTERAREA DYNAMICS OF MULTI-MACHINE SYSTEMS

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ABSTRACT

The slow coherency concept is introduced and an algorithm is developed for grouping machines having identical slow motions into areas. The singular perturbation method is used to separate the slow variables which are the area center of inertia variables and the fast variables which describe the intermachine oscillations within the areas. The areas obtained by this method are independent of fault locations. Three types of simulation approximations illustrated on a nonlinear 48 machine system indicate the validity of this algorithm.

1. INTRODUCTION

This paper presents a systematic procedure for grouping the machines of a power system into areas. The concept of an area is based upon the observation that in postfault transients only some machines close to the fault location respond with fast intermachine oscillations, while other machines more distant from the fault swing together in groups with "in phase" slow motion. Our approach is to define areas by grouping the machines which exhibit this slow coherency phenomenon. Allowing the machines in the same area to differ in their fast dynamics makes it possible to retain the same area grouping for different fault locations. The resulting conceptual simplifications and computational savings are significant in simulation and planning studies when many contingencies need to be examined.

The notion of slow coherency is expressed in the following way. If we consider the r slowest modes of the system's response to any fault, then machines "i" and "j" are slowly coherent if the difference of their angles $x_i(t)$ and $x_j(t)$

$$x_i(t) - x_j(t) = z_{ij}(t) \qquad (1.1)$$

contains none of the r slowest modes. This definition disregards differences of the fast dynamics of machines within the same area. In contrast to the more conventional definitions of coherency [1-6], which require that the total angular difference $z_{ij}(t)$ be within a specified tolerance, here the tolerance is specified only for the slow modes in $z_{ij}(t)$.

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Our approach to grouping machines starts with the linearized electromechanical model without damping, and separates its slow and fast modes using the so called dichotomic transformation from the singular perturbation technique [7]. The dichotomic transformation matrices L and M define a set of physically meaningful state variables. In the ideal slow coherency case the dichotomic L is a "grouping" matrix, whose elements are zeros and ones, and the state variables of the fast subsystem are machine angle differences within areas. On the other hand, the matrix M, which separates the slow subsystem, defines the slow variables as the area centers of inertias [1,2,4]. In a nonideal case we search for a dichotomic L whose elements are close to zeros and ones. This results in areas which contain machines that are near-coherent in their slow modes.

The slow interarea dynamics and the fast intra area dynamics are suitable for two time scale analysis of power systems by the singular perturbation method. This method is applicable to systems in the so called state separable form

$$\frac{d\xi}{dt} = f(\xi,\eta,t) \ , \ \xi(t_o) = \xi^o \qquad (1.2)$$

$$\varepsilon \frac{d\eta}{dt} = g(\xi,\eta,t) \ , \ \eta(t_o) = \eta^o \qquad (1.3)$$

where ξ and η represent the "slow" states and the "fast" states of the system respectively, and ε is a small positive parameter which accounts for small time constants, inverses of high gain coefficients, small inertias, etc. If the separation between time scales in (1.2) and (1.3) is large, ε will be small and may be approximated by $\varepsilon=0$. The model (1.2) and (1.3) with $\varepsilon=0$ then defines the quasi-steady-state $\xi_s(t)$, $\eta_s(t)$ as

$$\frac{d\xi_s(t)}{dt} = f(\xi_s, \eta_s, t) \ , \ \xi_s(t_o) = \xi^o \qquad (1.4)$$

$$0 = g(\xi_s, \eta_s, t) \qquad (1.5)$$

where the differential equations for η have been reduced to algebraic or transcendental equations.

In (1.2), (1.3) the variables ξ are predominantly slow, that is, $\xi(t) \cong \xi_s(t)$, while the variables $\eta(t)$ contain a significant fast component $\eta(t) - \eta_s(t)$ which becomes infinitely fast as $\varepsilon \to 0$. For application of the singular perturbation method it is necessary to express the system dynamics in the form (1.2) (1.3).

System models which describe fast and slow phenomena do not always appear in this form. For example, the electromechanical model using individual machine speeds and angles as the state variables does not exhibit this slow-fast separation. A new set of state variables which brings the model to the form (1.2), (1.3) are the interarea motions which represent the "slow" states ξ, and intra area motions of the machines within an area which represent the "fast" states η in (1.2), (1.3).

The procedure presented in the first part of this paper transforms the conventional nonlinear electromechanical model into the form (1.2), (1.3) by applying the area grouping obtained in the study of the linearized model. In the second part of the paper we demonstrate some properties of the transformed nonlinear models which are useful in understanding interarea motions and model simplifications. When only the area motions are of interest, all the dynamic equations for the intra area fast variables are reduced to equations (1.5) regardless of the fault location. A property observed in the example, which results in a different approximation, is that the fast phenomena from different areas are weakly coupled, while the coupling of slow phenomena is strong. This fact enables us to reduce the dynamic equations of intra area variables outside the study area to the static equations (1.5), while retaining the study area in detail. A further simplification is to neglect all the equations for the intra area variables outside the study area by assuming that the angles between machines are constant, which is the approach used in [4]. These three approximations are demonstrated on a 48 machine system.

2. ELECTROMECHANICAL MODEL

The well-known electromechanical model [8] of an n-machine power system is

$$\dot{\delta}_i = \Omega(\omega_i - 1), \tag{2.1}$$

$$2H_i\dot{\omega}_i = -D_i(\omega_i-1) + (P_{mi}-P_{ei}), \tag{2.2}$$

$i = 1, 2, \ldots, n,$

where δ_i, ω_i, P_{mi}, P_{ei}, H_i, D_i are the rotor angle, speed, mechanical input power, electrical output power, inertia constant, damping constants of machine i, respectively, and Ω is the base frequency. In this model the following assumptions are made.

(A1) Mechanical input power P_{mi} is constant.
(A2) The electrical output power is

$$P_{ei} = \sum_{\substack{j=1 \\ j\neq i}}^{n} V_iV_j[B_{ij}\sin(\delta_i-\delta_j)+G_{ij}\cos(\delta_i-\delta_j)]+V_i^2G_{ii},$$

$$i = 1, 2, \ldots, n, \tag{2.3}$$

where the per unit voltage V_i behind transient reactance is assumed to be constant and saliency is neglected. Loads are represented by passive impedances, and G and B are the real and imaginary parts of the reduced admittance matrix Y at the internal machine nodes.

Disturbances are represented by initial conditions, and in the case of structural changes, by changes in the Y matrix. The time scales are largely determined by the natural frequencies of the linearized electromechanical model around the equilibrium δ_i^* and $\omega_i^* = 1.0$,

$$\Delta\dot{\delta}_i = \Omega\Delta\omega_i, \tag{2.4}$$

$$2H_i\Delta\dot{\omega}_i = -D_i\Delta\omega_i - \sum_{j=1}^{n} k_{ij}\Delta\delta_j, \tag{2.5}$$

where $\Delta\delta_i = \delta_i-\delta_i^*$, $\Delta\omega_i = \omega_i-1$,

$$k_{ii} = \sum_{j=1}^{n} -k_{ij},\ j\neq i \tag{2.6}$$

$$k_{ij} = -V_iV_jB_{ij}\cos(\delta_i-\delta_j)_{\delta^*},\ j\neq i, \tag{2.7}$$

in which the terms involving G_{ij} are neglected.

At δ^* and ω^*, the eigenvalues of (2.4) and (2.5) are of the following three types:

1. a zero eigenvalue corresponding to the motion of all the machine angles,
2. a small negative real eigenvalue corresponding to the aggregate speed of all the machines, and
3. (n-1) pairs of lightly damped oscillatory modes which typically range in frequency from 1/2 to 2 Hz.

Models involving more details such as excitation systems and governors would still contain the above set of eigenvalues modified mostly in the damping and not in their frequencies [4]. Since the small damping constants D_i do not significantly affect the frequencies of the oscillatory modes they may be neglected. Thus, the linear model used in this paper is the second order system

$$\ddot{x} = -(1/2)\Omega H^{-1}K\,x \triangleq Ax, \tag{2.8}$$

where $x_i = \Delta\delta_i$, $H = \text{diag}\,(H_1,H_2,\ldots,H_n)$, and K is the matrix of k_{ij}. Therefore instead of dealing with a system of order 2n, we only need to deal with the nxn matrix A.

From (2.6) and (2.7), K is symmetric if Y is symmetric which is true for networks without phase shifters. Thus, A is diagonalizable because it is similar to the symmetric matrix

$$-(1/2)\Omega H^{-1/2}KH^{-1/2} \tag{2.9}$$

where $H^{1/2}$ is the square root of H. Thus, all the eigenvalues λ_i of A are real. For λ_i negative, the eigenvalues $\pm\sqrt{\lambda_i}$ of the second order system (2.8) are on the imaginary axis close to the slightly damped eigenvalues of (2.4) and (2.5). Thus, the low frequency modes of (2.4) and (2.5) are the slow modes of A.

3. SLOW COHERENCY

In this section we study systems in which it is possible to group the machines into r areas such that the difference $z_{ij}(t)$ in (1.1) contains none of the r slow modes. Such idealized systems, in which the slow coherency is exact and the number of coherent areas is equal to the number of slow modes, are called r-decomposable. In r-decomposable systems there exists a direct relationship between the time scales and the coherent areas. This relationship is established in this section and serves as a basis for the development of the grouping algorithm in the next section.

Let us first define a compact notation for areas by introducing a reference set of machines and a grouping matrix. In each area we pick an arbitrary machine as the reference machine. The reference machine angles are then considered as components of an r-vector x^1, while all other angles form the (n-r)-vector x^2. Equation (1.1) motivates the use of a grouping matrix L_g of dimension (n-r)xr to assign machines to areas. L_g has as many rows as the number of machines in x^2 and as many columns as the number of

In step 2 a basis for the 5-dimensional slow subspace is computed. In step 3 the Gaussian elimination is performed and the set of reference machines is found to be 5, 12, 14, 15, 16. In step 4 the dichotomic L_d corresponding to this reference set is computed and is given in Table 4.1. The largest element in each row of L_d which are underlined in Table 4.1, are used to identify the machines in each area. As a result the following area grouping of machines is obtained:

Area 1: machines 1-9
Area 2: machines 10-13
Area 3: machine 14
Area 4: machine 15
Area 5: machine 16.

Note that for machines 1, 2, 3, and 8, the entries in the column under machine 12 are not significantly smaller than those under machine 5. This can be interpreted that the responses of 1, 2, 3, and 8 are only slightly more coherent to 5 than to 12. Nevertheless, this area grouping gives quite favorable results as it will be demonstrated in the next section.

TABLE 4.1. Matrix L_d

Other Machines	Reference Machines				
	14	12	15	16	5
1	.0599	.411	-.0156	.0222	.522
2	.0335	.422	-.0135	.0014	.557
3	.0320	.387	-.0132	-.000466	.595
4	.0221	.178	-.00818	.00225	.806
6	.0217	.193	-.00971	-.00404	.799
7	.0227	.198	-.00987	-.00312	.793
8	.0585	.377	-.0170	.0186	.563
9	.0372	.215	-.0183	-.00352	.769
10	.100	.618	-.0179	.110	.189
11	.0720	.643	-.000447	.133	.152
13	.00140	.972	-.00116	.0197	.00794

INTERMACHINE AND AREA VARIABLES

Using the framework of singular perturbation theory we now show that an area decomposition is an essential step in two time scale and reduced order modeling. A model is singularly perturbed if some of its states are predominantly slow and others predominantly fast. The two time scale property can be exhibited by the different choices of the state variables. For linear time invariant models a possible choice are the modal variables. However, a frequent requirement is that the states be, or at least closely reflect, the actual variables of physical units in the system. We are therefore interested in a physically meaningful choice of state variables which in addition exhibits the time scale properties.

The state variables of the original electromechanical model (2.1), (2.2) and its linearization (2.4), (2.5) are physically meaningful but each of them contains mixed slow and fast parts. However, if the system is near-decomposable and the near-coherent areas have been found, then the model (3.6) exhibits the fast part $\dot{z}^2 = B_2 z^2$. The fast states z^2 are physically meaningful. They represent the intermachine oscillations $x_i - x_j$ of the machines i and j within an area, where machine j is the reference machine of the area. The other states x^1 of the model (3.7) still have "mixed" fast and slow parts and should be replaced by some predominantly slow states.

Knowing that the slow eigenvalues of A are in B_1, we now separate the slow subsystem using

$$\begin{bmatrix} z^1 \\ z^2 \end{bmatrix} = \begin{bmatrix} I & M \\ 0 & I \end{bmatrix} \begin{bmatrix} x^1 \\ z^2 \end{bmatrix} \tag{5.1}$$

The substitution of (5.1) into (3.6) yields

$$\begin{bmatrix} \dot{z}^1 \\ \dot{z}^2 \end{bmatrix} = \begin{bmatrix} B_1 & P(M) \\ 0 & B_2 \end{bmatrix} \begin{bmatrix} z^1 \\ z^2 \end{bmatrix} \tag{5.2}$$

which shows that, if M satisfies

$$P(M) = MB_2 - B_1 M + A_{12} = 0 \tag{5.3}$$

then $\dot{z}^1 = B_1 z^1$. Hence z^1 contains only slow modes and represents a possible choice of the slow variables. Is this choice physically meaningful? To answer this question we need the following result whose proof is given in the Appendix.

Lemma 5.1

Consider the matrix $A = -(1/2)\Omega H^{-1}K$ of (2.1) where K is symmetric and H is the diagonal matrix of machine inertias whose rxr and (n-r)x(n-r) diagonal blocks are H^1 and H^2, respectively. Then the solutions L of R(L)=0 and M of P(M)=0 are related by

$$M = (H^1 + L'H^2L)^{-1}L'H^2 . \tag{5.4}$$

A similar relationship can be obtained by modal methods (Saccomano, 1974a). Under the conditions of this Lemma the complete transformation from x to z variables (3.5) and (5.1) is

$$\begin{bmatrix} z^1 \\ z^2 \end{bmatrix} = \begin{bmatrix} H_a^{-1}H^1 & H_a^{-1}L'H^2 \\ -L & I \end{bmatrix} \begin{bmatrix} x^1 \\ x^2 \end{bmatrix} . \tag{5.5}$$

where

$$H_a = H^1 + L'H^2L. \tag{5.6}$$

If the system is r-decomposable, that is $L = L_g$ then the physical meaning of z^1 is readily recognized from the first row of (5.5) that is from

$$H_a z^1 = H^1 x^1 + L_g' H^2 x^2 \tag{5.7}$$

by noting the special structure of $L'H^2$ and H_a. Since the entries of L_g are zeros and ones and H^2 is diagonal, the nonzero entries of L_g' are replaced in $L'H^2$ by the entries of H^2. Furthermore, $L_g'H_2L_g$ is diagonal, because $L_g'L_g$ is diagonal. It can be seen that each entry of $L_g'H^2L_g$ is the sum of machine inertias in an area excluding the inertia of the reference machine in that area. Thus H_a is a diagonal matrix of the area inertias, that is

$$H_{ai} = \Sigma H_j, \text{ for all } j \text{ in area } i. \quad (5.8)$$

It follows that the i-th component z_i^1 of z^1 is

$$z_i^1 = \Sigma H_j x_j / H_{ai}, \text{ for all } j \text{ in area } i \quad (5.9)$$

and hence its physical meaning is the familiar notion of the "area center of inertia," which has been used in Marconato, Mariani and Saccomano (1973).

TABLE 5.1. Eigenvalue Approximations of the 16 Machine System

Sub-System	Accurate	Singular Perturbation Approximation
Slow	.0002318	.0002318
	-.1969	-.1969
	-.1063 ± j2.576	-.1058 ± j2.589
	-.09877 ± j3.498	-.09882 ± j3.496
	-.08970 ± j4.531	-.09097 ± j4.695
	-.09399 ± j5.068	-.09385 ± j5.075
Fast	-.1294 ± j5.997	-.1219 ± j5.975
	-.1162 ± j6.534	-.1222 ± j6.445
	-.1178 ± j7.159	-.1177 ± j7.156
	-.07192 ± j7.485	-.07207 ± j7.481
	-.1198 ± j7.962	-.1195 ± j7.962
	-.09360 ± j7.970	-.09425 ± j7.959
	-.08926 ± j8.405	-.08873 ± j8.259
	-.1350 ± j9.267	-.1351 ± j9.267
	-.1007 ± j9.650	-.1025 ± j9.646
	-.1264 ± j9.732	-.1264 ± j9.732
	-.2013 ± j11.419	-.1986 ± j11.378

In conclusion we emphasize that for an r-decomposable system an area decomposition results in physically meaningful slow and fast variables. For near-decomposable systems we still use the area variables (5.9) and the intermachine differences (3.1) as states. Although the time scale separation is not complete, the area variables z^1 will be predominantly slow and the intermachine variables z^2 will be predominantly fast. The same conclusion applies to models with damping (2.4), (2.5) and nonlinear models.

As an illustration we will consider the 2nx2n linearized model (2.4), (2.5) which includes damping. We define z to represent both angles and speeds: z^1 for the area variables and z^2 for the intermachine variables. Then the model (2.4), (2.5) becomes

$$\begin{bmatrix} \dot{z}^1 \\ \dot{z}^2 \end{bmatrix} = \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} \begin{bmatrix} z^1 \\ z^2 \end{bmatrix}. \quad (5.10)$$

The singular perturbation approach in Kokotovic and others, 1980) is to use

$$\dot{z}_s^1 = (F_{11} - F_{12}F_{22}^{-1}F_{21})\, z_s^1 \triangleq F_o\, z_s^1 \quad (5.11)$$

to approximate the slow subsystem and

$$\dot{z}_f^2 = F_{22} z_f^2 \quad (5.12)$$

to approximate the fast subsystem of (5.10).

We now examine this approximation on our 16-machine example, with the areas defined in the preceding section. As an indication of accuracy, we compare the eigenvalues of F_o in (5.11) and F_{22} in (5.12) with the accurate eigenvalues. As Table 5.1 shows, the worst error is 3.6% for the pair -.08790 + j4.531. In this example even though L_d is not very close to L_g, especially in rows 1, 2, 3, and 8, the eigenvalue approximation is excellent.

CONCLUSION

The concepts of slow coherency and r-decomposable systems have exhibited the time scale properties and retained the physical meaning of the fast and slow variables in electromechanical models of the power systems. The time scale interpretation of the notion of coherent areas has provided an analytical basis for the grouping algorithm proposed in this paper.

The algorithm uses the dichotomic solution of a lower order Riccati equation expressed in terms of a basis of the slow eigensubspace. First a basis is found, and then a particular dichotomic solution is obtained via Gaussian elimination. A grouping matrix, which is the closest approximation of the dichotomic solution, can be obtained and used to define areas. This algorithm is illustrated by a 16-machine example. Although the areas are determined on a simplified linearized model, it is expected they can be used to bring more detailed and nonlinear models to a singularly perturbed form. This opens new possibilities for obtaining nonlinear lower order equivalents by singular perturbation techniques.

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DEDICATION

This paper is dedicated to the memory of Giorgio Quazza who pioneered applications of modern control theory to power systems.

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APPENDIX

We start with the proof of Lemma 4.2 because the proof of Lemma 4.1 is easily obtained from the proof of Lemma 4.2.

Proof of Lemma 4.2

Since the columns of V form a basis of the slow eigensubspace, then there exists a matrix Λ whose eigenvalues are equal to the slow eigenvalues of A such that $AV=V\Lambda$ whose partitioned form is

$$A_{11}V_1 + A_{12}V_2 = V_1\Lambda \tag{A1}$$

$$A_{21}V_1 + A_{22}V_2 = V_2\Lambda, \tag{A2}$$

where A_{ij} are the submatrices of A of appropriate dimensions.

Premultiplying (A1) by V_1^{-1} and using it to eliminate Λ from (A2) yields

$$A_{21}V_1+A_{22}V_2-V_2V_1^{-1}(A_{11}V_1+A_{12}V_2)=0 \qquad (A3)$$

Thus, (3.10) follows from post-multiplying (A3) by V_1^{-1}, and L_d is identified as in (4.3). Furthermore,

$$B_1 = A_{11} + A_{12}L = V_1\Lambda V_1^{-1} \qquad (A4)$$

can be obtained from post-multiplying (A1) by V_1^{-1}, implying that the eigenvalues of B_1 are the slow eigenvalues. The uniqueness of L_d follows from Medanic (1979).

The proof of the second part of the lemma follows from the first part. Since V is of full rank, there are more than 1 combination of the rows of V forming a nonsingular matrix V_1, and hence the statement is meaningful.

Proof of Lemma 4.1

From the proof of Lemma 4.2, if u is an eigenvector of B_1, then

$$\underset{\sim}{v} = \begin{bmatrix} u \\ L_d u \end{bmatrix} \qquad (A5)$$

is an eigenvector of A. In particular, if $v=v_o$, then from (P1),

$$v = v_o = \begin{bmatrix} u_o \\ L_d u_o \end{bmatrix} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \qquad (A6)$$

Thus (4.1) is obtained by writing $L_d u_o$ in scalar form.

Proof of Lemma 5.1

Let us first rewrite R(L)=0 and P(M)=0 as

$$\begin{bmatrix} -L & I \end{bmatrix} A \begin{bmatrix} I \\ L \end{bmatrix} = 0 \qquad (A7)$$

$$\begin{bmatrix} I & M \end{bmatrix} B \begin{bmatrix} -M \\ I \end{bmatrix} = 0. \qquad (A8)$$

where

$$B = \begin{bmatrix} B_1 & A_{12} \\ 0 & B_2 \end{bmatrix}$$

Substituting $B=T_LAT_L^{-1}$ into (A8) yields

$$[I-ML \quad M] \quad A \begin{bmatrix} -M \\ I-LM \end{bmatrix}$$

$$= [I-ML \quad M] \, (-H^{-1}K)H^{-1}H \begin{bmatrix} -M \\ I-LM \end{bmatrix} \qquad (A9)$$

$$= [(I-ML)(H^1)^{-1} \quad M(H^2)^{-1}] \; A' \begin{bmatrix} -H^1 M \\ H^2(I-LM) \end{bmatrix} = 0.$$

Pre- and post-multiplying (A7) by $[H^2(I-LM)]'$ and $[(I-ML)(H^1)^{-1}]'$ and comparing to the transpose of (A9), we obtain

$$M'H^1 = (I-LM)'H^2L \qquad (A10)$$

which simplifies to (5.4).

machines in x^1. The (i,j) entry of L_g is 1 if machines x^1_j and x^2_i are in the same area and is zero otherwise.

As an illustration consider a three area five machine system with

$$x^1 = (x_2, x_3, x_5)^T \tag{3.1}$$

$$x^2 = (x_1, x_4)^T \tag{3.2}$$

$$L_g = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} . \tag{3.3}$$

The first row of L_g indicates that the first machine in x^2 belongs to the same area as the third machine in x^1, and the second row of L_g indicates that the second machine in x^2 belongs to the same area as the first machine in x^1. Thus the three areas composed of machines 5 and 1, machines 2 and 4, and machine 3 are uniquely defined. For a different choice of x^1 and x^2, such as

$$x^1 = (x_1, x_2, x_3)^T \tag{3.4}$$

$$x^2 = (x_4, x_5)^T \tag{3.5}$$

we need a different L_g that is

$$L_g = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \tag{3.6}$$

to define the same areas. Note that the zero column in L_g of (3.3) or (3.6) indicates the presence of a single machine area.

Using L_g, (1.1) is rewritten more compactly as

$$x^2(t) - L_g x^1(t) = z^2(t) \tag{3.7}$$

where the components of $z^2(t)$ are the corresponding functions $z_{ij}(t)$. For x^1, x^2 and L_g defined in (3.1) - (3.3),

$$z^2(t) = \begin{bmatrix} x_1 - x_5 \\ x_4 - x_2 \end{bmatrix} \tag{3.8}$$

Our procedure interprets (3.7) as a special case of the more general coordinate transformation

$$\begin{bmatrix} x^1 \\ z^2 \end{bmatrix} = \begin{bmatrix} I & 0 \\ -L & I \end{bmatrix} \begin{bmatrix} x^1 \\ x^2 \end{bmatrix} \tag{3.9}$$

where the (n-r)xr matrix L is not necessarily a grouping matrix. The application of the transformation to (2.8) results in

$$\begin{bmatrix} \ddot{x}^1 \\ \ddot{z}^2 \end{bmatrix} = \begin{bmatrix} B_1 & A_{12} \\ R(L) & B_2 \end{bmatrix} \begin{bmatrix} x^1 \\ z^2 \end{bmatrix} \tag{3.10}$$

where

$$B_1 = A_{11} + A_{12}L \ , \ B_2 = A_{22} - LA_{12} \tag{3.11}$$

$$R(L) = A_{22}L - LA_{11} - LA_{12}L + A_{21} \tag{3.12}$$

and A_{11}, A_{12}, A_{21}, A_{22} are the submatrices of A conformal with x^1 and x^2.

We are particularly interested in a so called dichotomic L which satisfies R(L) = 0 and $|\lambda(B_1)| < |\lambda(B_2)|$, that is, which groups the slow modes into the matrix B_1. It can be shown that the Riccati equation R(L)=0 in (3.12) can have at most one such dichotomic solution $L=L_d$ [9]. Moreover, for r-decomposable systems the dichotomic solution is also a grouping matrix, that is $L_d=L_g$ [9], and any choice of r machines, each from a different area, gives an L_g which solves $R(L_g) = 0$. If two machines from the same area were in the reference set, then the corresponding Riccati equation would not have a dichotomic solution.

This grouping approach has a geometric representation in terms of a basis of the eigensubspace of the slow modes of A for a particular ordering of the machine angles in x. Considering such a basis as the columns of an nxr matrix V, we see that if the machines i and j are slowly coherent, the ith and jth rows of V must be identical. If no machines from the same area are in x^1, the rxr submatrix V_1 of the basis matrix

$$V = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \tag{3.13}$$

is nonsingular and the dichotomic solution of R(L) = 0 is given by [9]

$$L_d = L_g = V_2 V_1^{-1} . \tag{3.14}$$

If two machines from the same area are in x^1, then V_1 is singular since it has two identical rows. Hence L_d does not exist.

The transformation of (2.8) into (3.10) with R(L)=0 decouples the fast subsystem, but the slow subsystem is still coupled through A_{12}. Knowing that the slow eigenvalues of A are in B_1, we now separate the slow subsystem from the fast. Applying the transformation

$$\begin{bmatrix} z^1 \\ z^2 \end{bmatrix} = \begin{bmatrix} I & M \\ 0 & I \end{bmatrix} \begin{bmatrix} x^1 \\ z^2 \end{bmatrix} \tag{3.15}$$

to (3.10), with R(L)=0, we obtain

$$\begin{bmatrix} \ddot{z}^1 \\ \ddot{z}^2 \end{bmatrix} = \begin{bmatrix} B_1 & P(M) \\ 0 & B_2 \end{bmatrix} \begin{bmatrix} z^1 \\ z^2 \end{bmatrix} . \tag{3.16}$$

For M in (3.15) we use the solution of

$$P(M) = MB_2 - B_1 M + A_{12} = 0 \tag{3.17}$$

which completely separates (3.16) into the slow and fast subsystems. It has been shown [9] that the solutions L_g of $R(L_g) = 0$ and M of P(M) = 0 are related by

$$M = (H^1 + L_g^T H^2 L_g)^{-1} L_g^T H^2 \tag{3.18}$$

where H^1 and H^2 are the rxr and (n-r)x(n-r) diagonal matrices of the x^1 and x^2 machine inertias respectively.

Thus, the complete transformation from the x to z variables using (3.9) and (3.15) is

$$\begin{bmatrix} z^1 \\ z^2 \end{bmatrix} = \begin{bmatrix} H_a^{-1}H^1 & H_a^{-1} L_g^T H^2 \\ -L_g & I \end{bmatrix} \begin{bmatrix} x^1 \\ x^2 \end{bmatrix} \tag{3.19}$$

where

$$H_a = H^1 + L_g^T H^2 L_g . \tag{3.20}$$

Since the entries of the grouping matrix L_g are zeros and ones and H^2 is diagonal matrix of machine inertias, each entry in $L_g^T H^2 L_g$ is the sum of machine inertias in an area excluding the inertia of the

reference machine in that area. Thus, H_a is a diagonal matrix of the area inertias, that is, the ith diagonal element, H_{ai}, of H_a is the sum of all the inertias of machines in area i.

For an r-decomposable system the physical meaning of z^1 and z^2 can be readily determined by examining (3.19). We see that the fast states z^2 represent the fast intermachine oscillations $x_i - x_j$ of machines i and j within an area, where machine j is the reference machine of the area. From (3.19) we also see that

$$H_a z^1 = H^1 x^1 + L_g^T H^2 x^2 \qquad (3.21)$$

and it follows that the slow states of z^1 represent the familiar center of inertia variables, that is component wise

$$z_i^1 = \Sigma H_j x_j / H_{ai}, \text{ for all j in area i.} \qquad (3.22)$$

These area variables are the weighted sums of the machine angles in the areas. They can be regarded as the angles of equivalent machines for the areas [1, 2,4]. As a consequence of our separating the time scales, these equivalent machines have larger inertias and hence exhibit slower motions.

4. THE GROUPING ALGORITHM

For idealized r-decomposable systems the determination of areas amounts to finding an x^1 vector of reference machines for which a dichotomic solution of R(L) = 0 exists. The difficulty with realistic models, which are not exactly r-decomposable, is that in general for a given x^1 a dichotomic solution of R(L) = 0 exists, but is not a grouping matrix. For such realistic situations, we present a grouping algorithm to determine the areas.

We begin by considering the case when the slow coherency definition (1.1) can only be approximately satisfied. Machines "i" and "j" are said to be near-coherent if in (1.1) the contribution of the slow modes in $z_{ij}(t)$ is small in some prespecified sense. Then a near-coherent area is an area composed of all machines which are near-coherent to each other. Following the approach for r-decomposable systems, we need to first find the reference machines and then approximate L_d by an L_g, since now L_d is different from L_g.

For near-coherent areas, the row vectors of any slow eigensubspace basis matrix V corresponding to machines in the same area are not the same. However, they are close in the sense that they are of approximately the same length and are clustered in a narrow cone. There are r such nonintersecting cones, one for each area.

To identify the areas, we find the r "most linearly independent vectors", one from each cone, and use them as the reference row vectors. After that, V is reordered such that V_1 consists of the reference row vectors, see (3.13). Recalling that $L_d = V_2 V_1^{-1}$, we see from

$$VV_1^{-1} = \begin{bmatrix} I \\ L_d \end{bmatrix} \qquad (4.1)$$

that the entries of L_d are the projections of other rows vectors onto the reference vectors. Therefore, in each row of L_d the entry close to 1 corresponds to the projection of the vector on the corresponding reference vector, and the entries close to zero are projections of the vector onto the other reference vectors.

An important property of L_d is that it is independent of the scaling of V. Given a basis matrix V, any other basis can be obtained as

$$VS = \begin{bmatrix} V_1 S \\ V_2 S \end{bmatrix} \qquad (4.2)$$

where S is an rxr nonsingular matrix. The matrix L_d is invariant to this change in basis, that is

$$L_d = V_2 S \, (V_1 S)^{-1} = V_2 V_1^{-1}. \qquad (4.3)$$

To find a set of the r "most linearly independent" row vectors to be used as the reference row vectors, we apply Gaussian elimination with complete pivoting to V. During the elimination, the rows and columns of V are permuted such that the (1,1) entry of the resulting V is the largest in magnitude. Note that permuting the rows of V is equivalent to changing the ordering of the machines. This (1, 1) entry of V is the pivot for performing the first step of the Gaussian elimination. Then the largest entry from the remaining (n-1)x(r-1) submatrix is used as the pivot for the next elimination step. The elimination terminates in r steps and the machines corresponding to the first r rows of the final reduced V matrix, are designated as the reference machines. In this Gaussian elimination process, rows having small entries will not be used as the pivoting row because these small entries are the result of elimination with almost identical rows already used as pivoting rows. Thus, the algorithm does not put two near-coherent machines into the reference set.

For the set of reference machines found by the algorithm the corresponding L_d is readily computed from

$$V_1^T L_d^T = V_2^T \qquad (4.4)$$

using the LU decomposition of V_1 already obtained from the Gaussian elimination. The next step is to find an L_g approximating L_d, that is to find the machines belonging to each area. To do this we examine each row of L_d. If the largest positive entry in row i is the jth entry, then in the matrix L_g entry (i, j) is 1 and all other entries in the ith row are 0.

Summarizing, the grouping algorithm consist of four steps:

a. computation of a basis V for the slow subspace of A,
b. Gaussian elimination of V,
c. computation of L_d by (4.4),
d. approximation of L_d by an L_g.

With the reference machines and L_g known, the areas are determined.

The algorithm is efficient because its most time consuming part the basis calculation in step a is only for r modes where r<<n and is carried out on the symmetric matrix (2.9). There are special purpose programs available in EISPACK which make use of these properties for handling large scale systems.

5. APPLICATION TO A NONLINEAR MODEL

The grouping algorithm and slow coherency properties are now examined on a 48 machine NPCC test system [10]. This model is of particular interest

Figure 6.2 are an approximation to the slow dynamics present in the intermachine variables in the faulted area. The smaller the magnitude of these slow dynamics the closer the system is to an r-decomposable system. Figure 6.3 illustrates the response of individual machine angles in area 1 which is adjacent to the faulted area. The close agreement between the exact curves and the approximation implies that the fast dynamics in area 1 is small even though the fast dynamics in the faulted area are substantial.

This localized nature of the fast dynamics can be used to improve approximation A1 by including the differential equations of the fast difference variables in the study area. Thus, for this approximation (A2) we have a set of differential equations for both the area variables and the intermachine difference variables in the study area. The external intermachine difference variables are modeled with $\varepsilon=0$, that is with a set of static equations. Figures 6.4, 6.5, and 6.6 show the close agreement between E and A2. The differences between these curves are due to the neglected fast dynamics in the external areas. Thus, A2 curves are more accurate than those of A1. However, both approximations provide the correct steady state value.

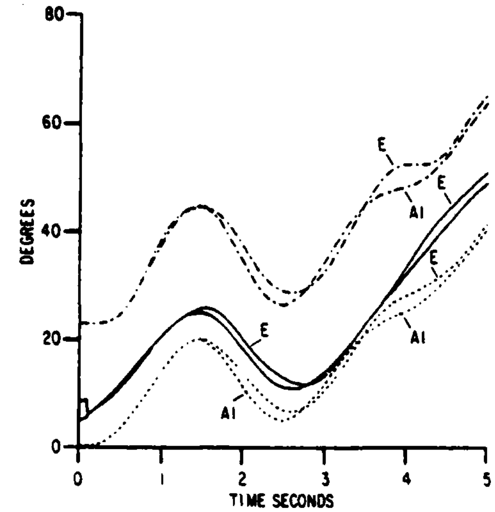

Figure 6.3: Individual machine angles area 1, exact and approximation A1

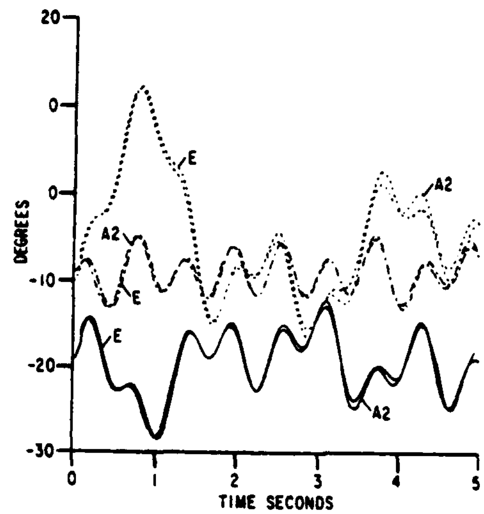

Figure 6.5: Angular difference variables area 5, exact and approximation A2

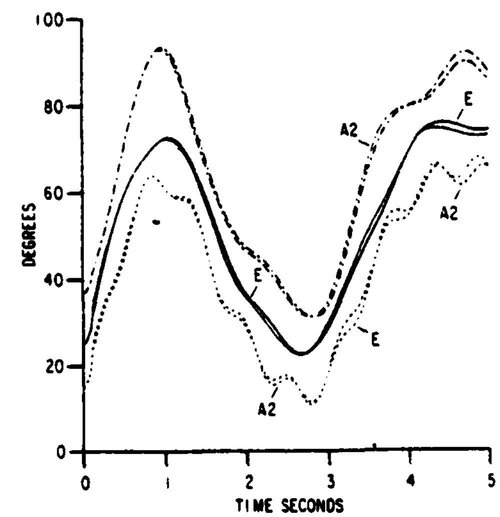

Figure 6.4: Individual machine angles area 5, exact and approximation A2

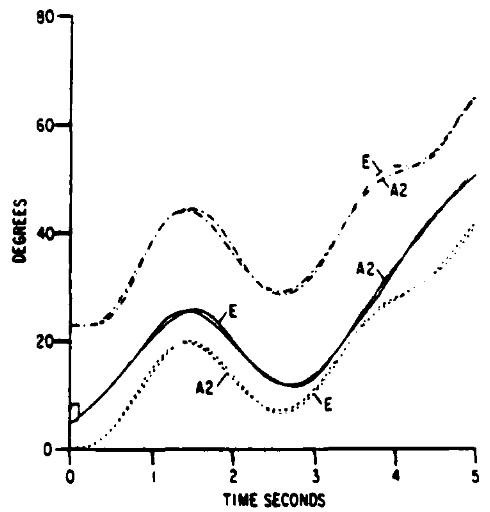

Figure 6.6: Individual machine angles area 1, exact and approximation A2

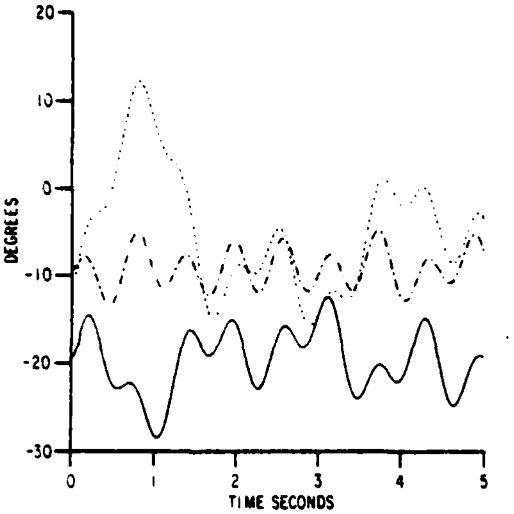

Figure 5.2: Angular difference variables, area 5

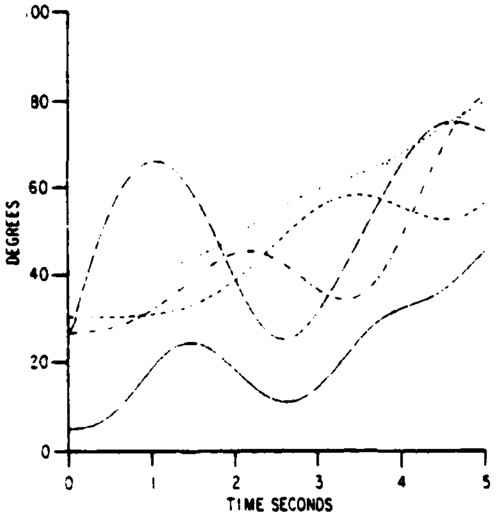

Figure 5.3: Area variables for areas 1-5

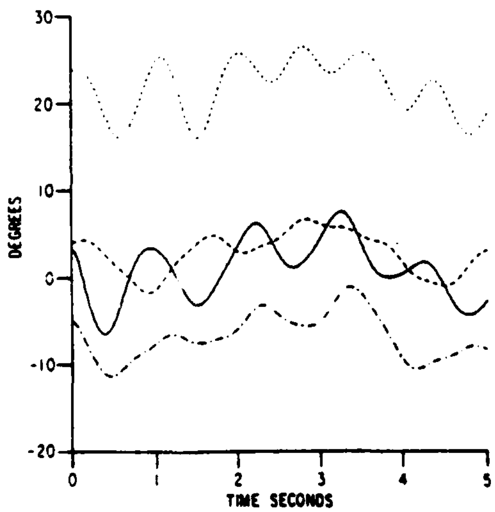

Figure 5.4: Angular difference variables, area 9

6. APPROXIMATE SIMULATIONS

From the above discussion we have shown that the time scale properties are preserved in the nonlinear model. Having expressed the system dynamics in the form of (1.2) and (1.3) we present three different approximate simulations of the Medway fault.

After the fault was cleared, we set $\varepsilon = 0$ for the entire fast subsystem and obtain a set of equations in the form of (1.4) and (1.5). With this approximation (A1) the slow dynamics are in differential equation form (1.4) and the slow part of the fast dynamics are represented by a set of static equations (1.5). Figures 6.1, 6.2 and 6.3 show the close aggreement between the exact solution (E) and the approximation A1 for selected machines in the faulted as well as adjacent area. The error introduced by this approximation is only in the fast dynamics and there is no steady state error between A1 and E. The A1 curves in

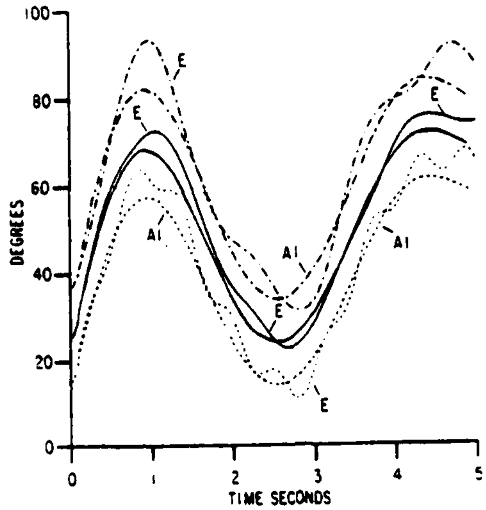

Figure 6.1: Individual machine angles area 5, exact and approximation A1

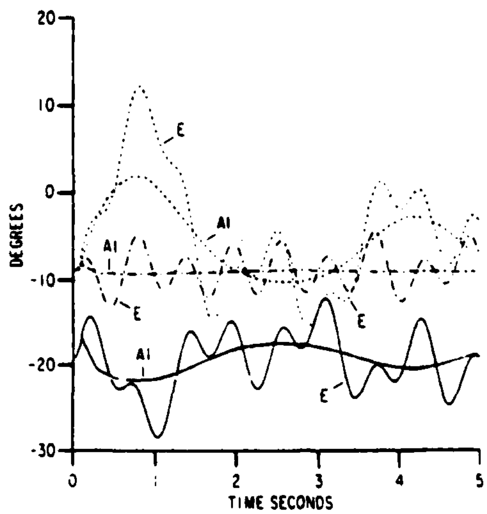

Figure 6.2: Angular difference variables area 5, exact and approximation A1

If the test system were truly an r-decomposable system the difference variables would contain no slow area motions. Figure 6.2 indicates how close our system is to an r-decomposable system. If we make the assumption that the test system is r-decomposable and that the fast dynamics are strictly local to the study area then the intermachine difference variables remain constant outside the faulted area. This approximation is basically similar to the equivalencing technique used in [4]. Errors introduced by this approximation (A3) will be both in the fast variables, for the same reasons as discussed above, and in the slow variables, which will have a steady state error. This steady state error is due to the fact that we have constrained the angular differences between machines outside the study area to be the same as the pre-fault equilibrium conditions. These angular differences are represented as phase shifters in [4]. In approximation A2, these angles are allowed to vary with time. Depending on the post-fault equilibrium this may or may not be a significant error. The agreement between approximation A3 and the exact solution E is shown in Figures 6.7, 6.8 and 6.9. Within the study area, Figures 6.7, 6.8, the agreement remains good. However, A3 curves for area 1 (Figure 6.9) do not compare as well to the exact curves as in previous cases. This is due to the approximation of the intermachine difference variables as constants.

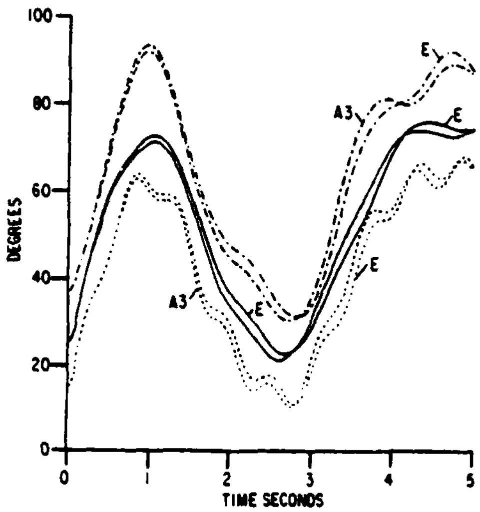

Figure 6.7: Individual machine angles area 5, exact and approximation A3

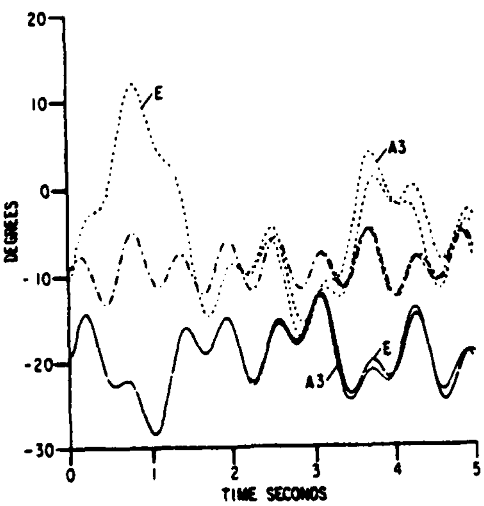

Figure 6.8: Angular difference variables area 5, exact and approximation A3

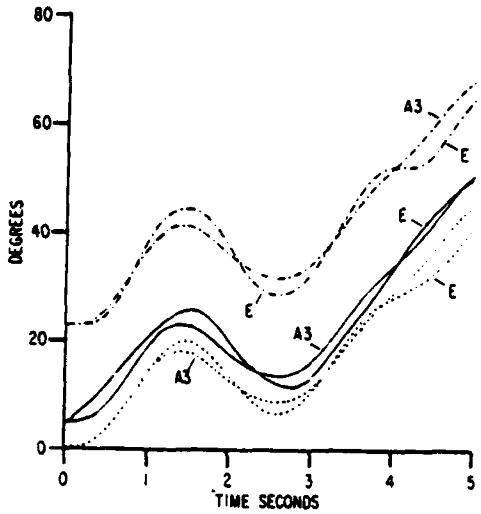

Figure 6.9: Individual machine angles area 1, exact and approximation A3

7. CONCLUSIONS

The concepts of slow coherency and r-decomposable systems have exhibited the time scale properties in electromechanical models by introducing a set of physically meaningful fast and slow variables. These variables are obtained through the dichotomic transformation which is the solution of a lower order Riccati equation. A grouping algorithm is formulated to find a particular dichotomic solution, from which the grouping matrix can be obtained and used to define the areas.

Through the 48 machine system, we show that the areas obtained from linear analysis are valid for nonlinear simulations and are fault location independent. Three types of approximations using the singular perturbation technique are illustrated. By neglecting all the intermachine variables, the slow variables reproduce the area motions. The accuracy of the simulation is improved by including also the intermachine dynamics in the study area. Without much loss in accuracy, the intermachine variables in the external area can be kept constant. These approximations offer new approaches to reduced simulations for power system studies.

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The authors would like to thank Mr. Lester Fink, formerly of the Department of Energy, and Dr. Kjell Carlsen of the General Electric Company for their encouragement and support in this work. This research has been supported by the U.S. Department of Energy, Division of Electric Energy Systems, through Contract Number DE-AC05-77 ET29104.

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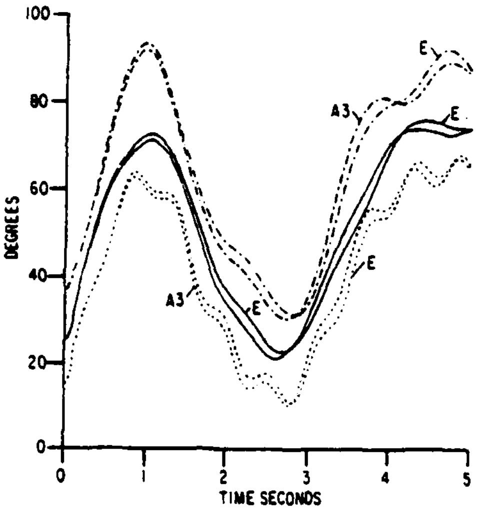

Figure 6.7: Individual machine angles area 5, exact and approximation A3

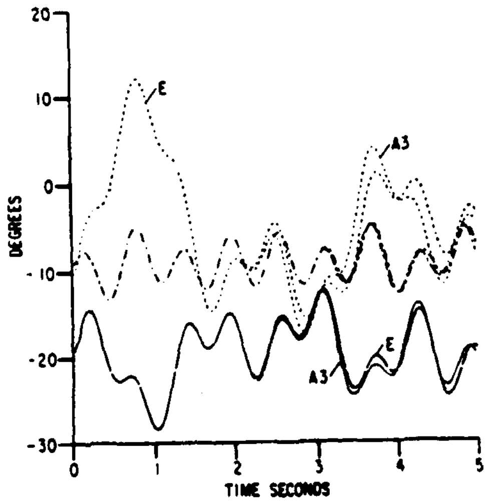

Figure 6.8: Angular difference variables area 5, exact and approximation A3

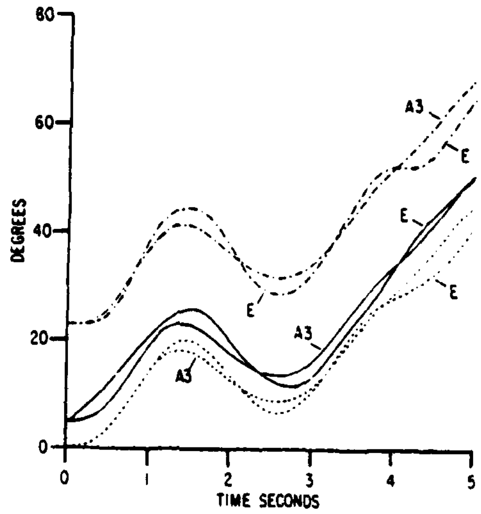

Figure 6.9: Individual machine angles area 1, exact and approximation A3

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SECTION 4

DESIGN OF LINEAR REGULATORS

Singular Perturbation of Linear Regulators: Basic Theorems

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***Abstract*—The behavior of the solution of the Riccati equation for the linear regulator problem with a parameter whose perturbation changes the order of the system is analyzed. Sufficient conditions are given under which the solution of the original problem tends to the solution of a low-order problem. This result can be used for the decomposition of a high-order problem into two low-order problems.**

Introduction

THE DEPENDENCE of the solution of a linear regulator problem on a parameter whose small perturbation can change the order of the system is analyzed. The system considered is

$$\frac{dx}{dt} = \bar{A}_1x + \bar{A}_2z + \bar{B}_1u, \qquad x(t_0) = x^0 \tag{1a}$$

$$\lambda\frac{dz}{dt} = \bar{A}_3x + \bar{A}_4z + \bar{B}_2u, \qquad z(t_0) = z^0 \tag{1b}$$

where λ is a small positive scalar, x and z are n- and m-dimensional states, respectively, and u is an r-dimensional control. The performance index to be minimized is

$$J = \tfrac{1}{2}y'(t_f)\bar{F}y(t_f) + \tfrac{1}{2}\int_{t_0}^{t_f}(y'\bar{D}y + u'\bar{R}u)\,dt \tag{2}$$

where $y = \bar{C}_1x + \bar{C}_2z = \bar{C}X$ is an s-dimensional output, X is the $(n+m)$-dimensional state of (1) and $\bar{C} = [\bar{C}_1\bar{C}_2]$.

In physical systems our parameter λ represents small time constants, masses, moments of inertia, etc.[1] Following his intuition and experience a designer usually neglects these small parameters during the design of a regulator system. He has at least two strong practical reasons for this simplification. An evident reason is that the presence of these "parasitic" parameters can make the dimensionality of a dynamic system prohibitively high. Another, less apparent, reason is that equations describing systems with small parameters multiplying derivatives belong to a class of "stiff" differential equations, which are difficult to solve even when the dimensionality of the system is not high [1].

In the method of this paper the small parameter λ is not neglected in the state equation (1), but rather in an appropriately formulated Riccati system for the regulator problem (1), (2). At $\lambda = 0$ this "full" Riccati system is decomposed into two smaller Riccati systems, one corresponding to the variable z, and the other corresponding to the variable x. An efficient decomposition is achieved since the z Riccati system does not depend on the x Riccati system and thus can be solved separately. It is of practical importance that the z Riccati system is algebraic rather than differential, even for a finite time interval problem. In contrast to the full Riccati system (9), the two smaller Riccati systems (11c) and (16) constitute a "reduced" system. In Theorems 1 and 2 conditions are formulated under which the full solution tends to the reduced solution as $\lambda \to 0^+$. Hence, for λ sufficiently small, the reduced solution can be used as an approximation of the full solution. In a future paper [2] an asymptotic expansion method is developed which improves this approximation.

To appreciate the nontriviality of the perturbation problem considered, note that at $\lambda = 0$ the matrices

$$\bar{A} = \begin{bmatrix}\bar{A}_1 & \bar{A}_2\\ \bar{A}_3/\lambda & \bar{A}_4/\lambda\end{bmatrix}, \qquad \bar{B} = \begin{bmatrix}\bar{B}_1\\ \bar{B}_2/\lambda\end{bmatrix} \tag{3}$$

of the state equation (1), that is, of the system

$$\frac{dX}{dt} = \bar{A}X + \bar{B}u, \qquad X(t_0) = X^0, \tag{4}$$

may be unbounded at $\lambda = 0$. To analyze this singular perturbation problem a "boundary layer" concept is introduced in the Riccati system. The "thickness" of this layer is a short time interval $[t_1, t_f]$ during which a rapid transient of the z Riccati system decays. Asymptotic stability of this transient is a crucial condition in most theorems of singular perturbation theory [3], [4]. Readers unfamiliar with singular perturbation theory are referred to theorems of Levin and Levinson [5] and Hoppensteadt [6] in the Appendix.

This paper is organized as follows. After preliminary notation and definitions the main result is presented in Theorem 1. This result is then extended to the infinite time interval ($t_f = \infty$) problem, Theorem 2. It should be noted that in Theorem 2 the existence and uniqueness of the full solution is established via controllability and observability test for the reduced system, thus avoiding the difficulty with the unboundedness of matrices $\bar{A}$ and $\bar{B}$

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[1] For example, if T is a small time constant and M is a small mass, then we can write $T = \alpha_1\lambda$ and $M = \alpha_2\lambda$, where α_1 and α_2 are appropriate coefficients.

as $\lambda \to 0$. Singular perturbation of linear regulators was first considered in [7]. This result now appears as a special case of Corollary 2.

Preliminary Definitions

In the regulator problem (1), (2) the following usual conditions are set for $t \in [t_0, t_f]$ and $\lambda \in [0, \lambda^0]$.

1) $\tilde A_1$, $\tilde A_2$, $\tilde A_3$, $\tilde A_4$, $\tilde B_1$, $\tilde B_2$, and $\tilde C$ are continuous in t and λ.
2) $\tilde R$ and $\tilde D$ are positive definite and continuous in t and λ.
3) $\tilde F$ is time invariant, positive semidefinite, and continuous in λ.

It is well known [8] that under these conditions and for $\lambda > 0$ and t_f finite the optimum control is

$$u = -\tilde R^{-1}\tilde B'\tilde K X \tag{5}$$

where $\tilde K$ is the solution of

$$\frac{d\tilde K}{dt} = -\tilde K\tilde A - \tilde A'\tilde K + \tilde K\tilde B\tilde R^{-1}\tilde B'\tilde K - \tilde C'\tilde D\tilde C, \qquad \tilde K(t_f) = \tilde C'\tilde F\tilde C \tag{6}$$

and the optimum regulator system is

$$\frac{dX}{dt} = (\tilde A - \tilde B\tilde R^{-1}\tilde B'\tilde K)X. \tag{7}$$

Partitioning $\tilde K$, $\tilde Q = \tilde C'\tilde D\tilde C$, and $\tilde \Pi = \tilde C'\tilde F\tilde C$ into n by n, n by m, and m by m arrays,

$$\tilde K = \begin{bmatrix} \tilde K_1 & \lambda\tilde K_2 \\ \lambda\tilde K_2' & \lambda\tilde K_3 \end{bmatrix}, \qquad \tilde Q = \begin{bmatrix} \tilde Q_1 & \tilde Q_2 \\ \tilde Q_2' & \tilde Q_3 \end{bmatrix},$$

$$\tilde \Pi = \begin{bmatrix} \tilde \Pi_1 & \lambda\tilde \Pi_2 \\ \lambda\tilde \Pi_2' & \lambda\tilde \Pi_3 \end{bmatrix} \tag{8}$$

and denoting $\tilde S_1 = \tilde B_1\tilde R^{-1}\tilde B_1'$, $\tilde S_2 = \tilde B_2\tilde R^{-1}\tilde R_2'$, and $\tilde S = \tilde B_1\tilde R^{-1}\tilde B_2'$, we rewrite (6) as

$$\frac{d\tilde K_1}{dt} = -\tilde K_1\tilde A_1 - \tilde A_1'\tilde K_1 - \tilde K_2\tilde A_3 - \tilde A_3'\tilde K_2' + \tilde K_1\tilde S_1\tilde K_1 + \tilde K_1\tilde S\tilde K_2' + \tilde K_2\tilde S'\tilde K_1 + \tilde K_2\tilde S_2\tilde K_2' - \tilde Q_1 \tag{9a}$$

$$\lambda\frac{d\tilde K_2}{dt} = -\tilde K_1\tilde A_2 - \tilde K_2\tilde A_4 - \lambda\tilde A_1'\tilde K_2 - \tilde A_3'\tilde K_3 + \lambda\tilde K_1\tilde S_1\tilde K_2 + \tilde K_1\tilde S\tilde K_3 + \lambda\tilde K_2\tilde S'\tilde K_2 + \tilde K_2\tilde S_2\tilde K_3 - \tilde Q_2 \tag{9b}$$

$$\lambda\frac{d\tilde K_3}{dt} = -\lambda\tilde K_2'\tilde A_2 - \lambda\tilde A_2'\tilde K_2 - \tilde K_3\tilde A_4 - \tilde A_4'\tilde K_3 + \lambda^2\tilde K_2'\tilde S_1\tilde K_2 + \lambda\tilde K_2'\tilde S\tilde K_3 + \lambda\tilde K_3\tilde S'\tilde K_2 + \tilde K_3\tilde S_2\tilde K_3 - \tilde Q_3 \tag{9c}$$

with the end condition

$$\tilde K_1(t_f) = \tilde\Pi_1, \qquad \tilde K_2(t_f) = \tilde\Pi_2, \qquad \tilde K_3(t_f) = \tilde\Pi_3. \tag{10}$$

The preceding form of $\tilde K$ makes the Riccati system (9) suitable for singular perturbation analysis. This system is called the full system. If λ is zero, the full system formally reduces to a system of one differential and two algebraic equations, called the degenerate system,

$$\frac{dK_1}{dt} = -K_1(A_1 - SK_2') - (A_1 - SK_2')'K_1 + K_1S_1K_1 - K_2A_3 - A_3'K_2' + K_2S_2K_2' - Q_1, \qquad K_1(t_f) = \Pi_1 \tag{11a}$$

$$0 = K_2(S_2K_3 - A_4) - K_1A_2 - A_3'K_3 + K_1SK_3 - Q_2 \tag{11b}$$

$$0 = -K_3A_4 - A_4'K_3 + K_3S_2K_3 - Q_3 \tag{11c}$$

where the absence of a tilde above a matrix denotes the evaluation of that matrix at $\lambda = 0$.

Note that as $\lambda \to 0$ in (9) the derivatives $d\tilde K_2/dt$ and $d\tilde K_3/dt$ may tend to infinity since, in general, $K_2 = \Pi_2$ and $K_3 = \Pi_3$ do not satisfy (11b) and (11c) and thus the right-hand sides of (9b) and (9c) are not zero at $\lambda = 0$ and $t = t_f$. Hence, in an interval about t_f the solution (9) differs markedly from a solution of (11) and, since $d\tilde K_2/dt$ and $d\tilde K_3/dt$ are large, $\tilde K_2$ and $\tilde K_3$ rapidly change in this interval. This interval is called the boundary layer because of an analogy with problems in fluid dynamics [9]. To analyze this boundary layer phenomenon the following boundary layer system is introduced:

$$\frac{dL_2(\tau)}{d\tau} = L_2(\tau)[S_2(t)L_3(\tau) - A_4(t)] + [K_1(t)S(t) - A_3'(t)]L_3(\tau) - K_1(t)A_2(t) - Q_2(t) \tag{12a}$$

$$\frac{dL_3(\tau)}{d\tau} = -L_3(\tau)A_4(t) - A_4'(t)L_3(\tau) + L_3(\tau)S_2(t)L_3(\tau) - Q_3(t) \tag{12b}$$

where the independent variable is τ, and t is considered as a fixed parameter, $t \in [t_0, t_f]$. The variable τ is often referred to as "fast time" since (12) can be viewed as being obtained from (9) by the use of the "stretching" transformation $t = \lambda\tau + t_f$ and allowing $\lambda \to 0$, see [3, p. 254]. Using (12b) we now introduce two important definitions.

1) The system (1) is called boundary layer controllable if for each fixed $t \in [t_0, t_f]$

$$\text{rank } [B_2, A_4B_2, \cdots, A_4^{m-1}B_2] = m. \tag{13}$$

2) The system (1) is called boundary layer observable if for each fixed $t \in [t_0, t_f]$

$$\text{rank } [C_2', A_4'C_2', \cdots, (A_4')^{m-1}C_2'] = m. \tag{14}$$

Next note that a solution of the degenerate system (11) is not unique, since (11c) has several roots. It will be shown in Lemma 1 that a unique positive definite root K_3 of (11c) exists such that $[A_4 - S_2K_3]$ has an inverse for all $t \in [t_0, t_f]$. Then the root K_2 of (11b) is uniquely defined

$$K_2 = (K_1SK_3 - K_1A_2 - A_3'K_3 - Q_2)\cdot(A_4 - S_2K_3)^{-1}. \tag{15a}$$

The stability of the root is discussed in Lemma 2. Given K_3, the root K_2 is a linear function of K_1

$$K_2 = K_1E_1 - E_2 \tag{15b}$$

where

$$E_1 = (SK_3 - A_2)(A_4 - S_2K_3)^{-1}$$
$$E_2 = (A_3'K_3 + Q_2)(A_4 - S_2K_3)^{-1}. \tag{15c}$$

Substituting K_2 into (11a) gives

$$\frac{dK_1}{dt} = -K_1\hat{A} - \hat{A}'K_1 + K_1\hat{B}R^{-1}\hat{B}'K_1 - \hat{Q},$$
$$K_1(t_f) = \Pi_1 \tag{16}$$

where

$$\hat{A} = A_1 + E_1A_3 + SE_2' + E_1S_1E_2' \tag{17a}$$
$$\hat{B} = B_1 + E_1B_2 \tag{17b}$$
$$\hat{Q} = -E_2A_3 - A_3'E_2' - E_2S_2E_2' + Q_1. \tag{17c}$$

The system (16), along with K_2 and K_3 defined previously, is called the reduced system. The dimensionality of the reduced system (16) is n by n, while the dimensionality of the full system (9) is $n + m$ by $n + m$. The existence and uniqueness of the solution of (16) will be established later in Lemma 3.

Main Theorem

Theorem 1

Let conditions 1)–3) be satisfied and assume that the system (1) is

4) boundary layer controllable;
5) boundary layer observable.

Then for $t \in [t_0, t_f]$ and $\lambda \in [0, \lambda^0]$ the unique solution $\bar{K}(t, \lambda)$ of (9) with $\bar{K}(t_f, \lambda) = \bar{\Pi}$ exists on the interval $[t_0, t_f]$ and

$$\lim_{\lambda \to 0} \bar{K}_1(t, \lambda) = K_1(t), \qquad t \in [t_0, t_f] \tag{18}$$
$$\lim_{\lambda \to 0} \bar{K}_2(t, \lambda) = K_2(t), \qquad t \in [t_0, t_f) \tag{19a}$$
$$\lim_{\lambda \to 0} \bar{K}_3(t, \lambda) = K_3(t), \qquad t \in [t_0, t_f) \tag{19b}$$

where $K_3(t)$ is the unique positive definite root of (11c), $K_2(t)$ is defined by (15) and $K_1(t)$ is the unique solution of the reduced system (16). The limit (18) is uniform in t on the interval $[t_0, t_f]$ and the limits (19) are uniform in t on any interval $[t_0, t_1]$, where t_1 is arbitrarily close to t_f, $t_1 < t_f$.

The proof of this theorem is carried out in four lemmas. In lemma 1 it is shown that boundary layer controllability and observability (13), (14) insure that the solution $L_3(\tau)$ of the boundary layer equation (12b) will be attracted to the asymptotically stable positive definite root $K_3(t)$ of (11c) for each fixed $t \in [t_0, t_f]$. Lemma 2 gives a similar result for (12a) and (11b) at $t = t_f$. In Lemma 3 these facts assure the existence and uniqueness of the solution to the reduced system (16). Finally, Lemma 4 guarantees that the asymptotic stability of the boundary layer system (12) is uniform with respect to t which is the essential condition for the application of Theorem L in the Appendix.

Lemma 1

Let conditions 1)–5) be satisfied. Then for each fixed $t \in [t_0, t_f]$, first, there exists a unique positive definite root $K_3(t)$ of (11c); second, this root is an asymptotically stable equilibrium of (12b) as $\tau \to -\infty$; third, Π_3 belongs to the domain of attraction[2] of this root; and fourth,

$$\alpha(t) = A_4(t) - S_2(t)K_3(t) \tag{20}$$

is a stable matrix.[2]

Proof: The lemma follows directly from well-known results of the output regulator theory for completely controllable and observable plants [8], [10] since for each fixed $t \in [t_0, t_f]$ conditions 1)–5) insure that

$$\frac{dz}{d\tau} = A_4(t)z(\tau) + B_2(t)u(\tau) \tag{21a}$$

$$J = \tfrac{1}{2}\int_0^\infty [z'(\tau)Q_3(t)z(\tau) + u'(\tau)R(t)u(\tau)]\, d\tau \tag{21b}$$

is a well-defined regulator problem.

Lemma 2

If conditions 1)–5) are satisfied then the root of (11b) at $t = t_f$

$$K_2(t_f) = [\Pi_1S(t_f)K_3(t_f) - \Pi_1A_2(t_f) - A_3'(t_f)K_3(t_f) - Q_2(t_f)]\alpha^{-1}(t_f) \tag{22}$$

is an asymptotically stable equilibrium of (12a) as $\tau \to -\infty$.

Proof: Rewrite (1a) as

$$\frac{dL_2(\tau)}{d\tau} = -L_2(\tau)[\alpha(t_f) + \beta(\tau, t_f)] + \gamma(\tau, t_f) \tag{23}$$

where

$$\beta(\tau, t_f) = S_2(t_f)[K_3(t_f) - L_3(\tau)] \tag{24a}$$
$$\gamma(\tau, t_f) = -\Pi_1[A_2(t_f) - S(t_f)L_3(\tau)] - A_3'(t_f)L_3(\tau) - Q_2(t_f). \tag{24b}$$

In view of known results for the stability of linear systems, see [12, p. 70, theorem 9], it follows that if a) $\beta(\tau, t_f) \to 0$ as $\tau \to -\infty$, and b) $L_2(\tau) = 0$ is an asymptotically stable equilibrium of

$$\frac{dL_2(\tau)}{d\tau} = -L_2(\tau)\alpha(t_f) \tag{25}$$

then $K_2(t_f)$ is an asymptotically stable equilibrium of (16). Condition a) is satisfied by (24a). To prove b) let

[2] Following Hahn [11] a matrix is called stable if all its eigenvalues have negative real parts. The domain of attraction is also defined in [11].

$l_i(\tau)$ be the ith row of L_2 and consider (25)

$$\frac{dl_i}{d\tau} = -[A_4(t_f) - S_2(t_f)K_3(t_f)]l_i, \qquad i = 1, 2, \ldots, n. \tag{26}$$

Since by Lemma 1 $[A_4(t_f) - S_2(t_f)K_3(t_f)]$ is a stable matrix, the null solution of (26) is an asymptotically stable equilibrium as $\tau \to -\infty$.

With Lemmas 1 and 2 we satisfy condition L2 in the Appendix. To satisfy the condition L3 we establish the existence and uniqueness of the solution $K_1(t)$ of the reduced system (23).

Lemma 3

If conditions 1)–5) are satisfied, then $\hat{Q}$ is positive semidefinite and therefore the solution $K_1(t)$ of the reduced system (16) exists and is unique on the interval $[t_0, t_f]$.

Proof: It follows from [8] that if R is symmetric positive definite, and $\hat{Q}$ and Π_1 are symmetric positive semidefinite; then the solution K_1 exists and is unique on the interval $[t_0, t_f]$. To show that Q is symmetric positive semidefinite we apply a matrix identity from (11c) to (17c) to obtain

$$\begin{aligned}\hat{Q} = {} & (Q_2 + A_3'K_3)(A_4'K_3 + Q_3)^{-1}Q_3(A_4'K_3 + Q_3)'^{-1} \\ & \cdot(Q_2 + A_3'K_3)' - Q_2(A_4'K_3 + Q_3)'^{-1}(Q_2' \\ & + K_3A_3) - (Q_2' + K_3A_3)'(A_4'K_3 + Q_3)^{-1}Q_2' \\ & + Q_1.\end{aligned} \tag{27}$$

For an arbitrary vector θ let

$$v = \theta \text{ and } w = -(A_4'K_3 + Q_3)'^{-1}(Q_2 + A_3'K_3)'\theta. \tag{28}$$

Since by condition 2) Q is positive semidefinite,

$$\theta'\hat{Q}\theta = v'Q_1v + v'Q_2w + w'Q_2'v + w'Q_3w \geq 0. \tag{29}$$

Thus $\hat{Q}$ is positive semidefinite. Since condition 3) implies the positive semidefiniteness of Π_1, and R is positive definite by assumption, all the conditions of [8] for the existence and uniqueness of $K_1(t)$ are satisfied.

Finally we show in Lemma 4 that the Jacobian of the system (9b) and (9c) evaluated along the reduced solution is a stable matrix as required by the condition L4 in the Appendix. Map the matrices $\bar{K}_1$, $\bar{K}_2$, and $\bar{K}_3$ into nn, nm, and mm-vectors $\boldsymbol{K}_1$, $\boldsymbol{K}_2$, and $\boldsymbol{K}_3$, respectively, and rewrite (9) in vector form

$$\frac{d\boldsymbol{K}_1}{dt} = f_1(\boldsymbol{K}_1, \boldsymbol{K}_2, t, \lambda) \tag{30}$$

$$\lambda\frac{d\boldsymbol{K}_2}{dt} = f_2(\boldsymbol{K}_1, \boldsymbol{K}_2, \boldsymbol{K}_3, t, \lambda) \tag{31a}$$

$$\lambda\frac{d\boldsymbol{K}_3}{dt} = f_3(\boldsymbol{K}_2, \boldsymbol{K}_3, t, \lambda). \tag{31b}$$

The Jacobian $\Gamma(\lambda)$ of $[f_2'f_3']'$ is then given by

$$\Gamma(\lambda) = \begin{bmatrix}\Gamma_{22}(\lambda) & \Gamma_{23}(\lambda)\\ \Gamma_{32}(\lambda) & \Gamma_{33}(\lambda)\end{bmatrix} \tag{32}$$

where

$$\Gamma_{i,j}(\lambda) = \frac{\partial f_i}{\partial \boldsymbol{K}_j}(\boldsymbol{K}_1, \boldsymbol{K}_2, \boldsymbol{K}_3, t, \lambda), \qquad i, j = 2, 3. \tag{33}$$

Let $\Gamma(0)$ denote $\Gamma(\lambda)$ evaluated along the reduced solution $K_1(t)$, $K_2(t)$, $K_3(t)$.

Lemma 4

If conditions 1)–5) are satisfied then all the eigenvalues of $\Gamma(0)$ have positive real parts for $t \in [t_0, t_f]$.

Proof: Since $\Gamma_{32}(0) = 0$ the eigenvalues of $\Gamma(0)$ consist of the eigenvalues of $\Gamma_{22}(0)$ and those of $\Gamma_{33}(0)$. Note that $\Gamma_{22}(0)$ and $\Gamma_{33}(0)$ can be expressed in terms of Kronecker products [13],

$$\Gamma_{22}(0) = -(A_4 - S_2K_3)' \times I_N \tag{34a}$$

$$\Gamma_{33}(0) = -I_M \times (A_4 - S_2K_3)' - (A_4 - S_2K_3)' \times I_M \tag{34b}$$

where I_N and I_M are $n \times n$ and $m \times m$ identity matrices. Therefore, the mm eigenvalues of $\Gamma_{33}(0)$ are $\mu_i + \mu_j$, $i, j = 1, \cdots, m$, and the nm eigenvalues of $\Gamma_{22}(0)$ are μ_i, $i = 1, \cdots, m$, each one of which is of multiplicity n, where μ_i, $i = 1, \cdots, m$ are the eigenvalues of $-(A_4 - S_2K_3)$, see [14]. Since by Lemma 1 all the eigenvalues μ_i, $i = 1, \cdots, n$, have positive real parts, all the eigenvalues of Γ have positive real parts.

With Lemmas 1–4 we satisfy all the conditions of Theorem L in the Appendix. This completes the proof of Theorem 1.

When $A_4(t)$ is a staple matrix, Theorem 1 can be extended to systems which are not boundary layer controllable or observable. Corollaries 1 and 2 deal with two extreme situations.

Corollary 1

Let conditions 1)–3) be satisfied and instead of conditions 4) and 5) assume that for all fixed $t \in [t_0, t_f]$

6) $A_4(t)$ is stable;
7) $B_2(t) \equiv 0$.

Then the results of Theorem 1 still hold.

Proof: Lemma 1 holds since (12b) is an asymptotically stable Lyapunov equation, and Lemmas 2, 3, and 4 hold since $S_2(t) \equiv 0$ and $\alpha(t) = A_4(t) - S_2(t)K_3(t) = A_4(t)$ is a stable matrix for all fixed $t \in [t_0, t_f]$.

Corollary 2

Let conditions 1)–3) be satisfied and instead of conditions 4) and 5) assume that for all fixed $t \in [t_0, t_f]$

7) $A_4(t)$ is stable;
8) $Q_3(t) \equiv 0$.

Then using the root $K_3(t) = 0$ of (11c) the results of Theorem 1 still hold.

Proof: The result of Lemma 1 for the unique positive definite root $K_3(t)$ of (11c) now applies to the isolated[3] root $K_3(t) = 0$ of (11c) with $Q_3(t) = 0$. This result for the Riccati equation (12b) with $Q_3(t) = 0$ is known from [15], [16]. Since $\alpha(t) = A_4(t) - S_2(t)K_3(t) = A_4(t)$ is a stable matrix for all fixed $t \in [t_0, t_f]$, Lemmas 2–4 still hold.

A special case of Corollary 2, when $A_4(t)$ is negative definite and $\Pi_2 = 0$ was considered in [7].

As a further extension it may be shown that Theorem 1 will still hold if conditions 4) and 5) are violated, but uncontrollable or unobservable modes of (21a) are asymptotically stable.

Time-Invariant Problem

An important class of regulator problems occurs when $t_f = \infty$ and the system (1) and $\bar{C}$, $\bar{D}$, and $\bar{R}$ in (2) are time invariant. For the finite time interval problem the existence of the solution of the full system (9) is assured by the conditions 1)–3) which are easily checked. The existence problem for (9) when $t_f = \infty$ is harder since the controllability and observability of $\bar{A}$, $\bar{B}$, and $\bar{C}$ must be checked for all $\lambda \in [0, \lambda^0]$. This is particularly difficult for λ very small, since $\bar{A}$ and $\bar{B}$ contain terms A_3/λ, A_4/λ, and B_2/λ. This difficulty is avoided in Theorem 2 where controllability and observability conditions of the reduced system guarantee the existence and uniqueness of the solution of the full problem for λ sufficiently small. A second result of Theorem 2 is that the reduced solution approximates the full solution for λ sufficiently small.

For the time-invariant problem, conditions 1)–3) are modified as follows:

1*) $\bar{A}_1$, $\bar{A}_2$, $\bar{A}_3$, $\bar{A}_4$, $\bar{B}_1$, $\bar{B}_2$, and $\bar{C}$ are time invariant and continuous in λ for $\lambda \in [0, \lambda^0]$;

2*) $\bar{R}$ and $\bar{D}$ are symmetric positive definite, time invariant, and continuous in λ for $\lambda \in [0, \lambda^0]$;

3*) $F \equiv 0$.

Theorem 2

Let conditions 1)–3*) and 4) and 5) be satisfied. Also assume that the matrices of the reduced system (17) satisfy

9) rank $[\hat{C}', \hat{A}'\hat{C}', \cdots, (\hat{A}')^{n-1}\hat{C}'] = n$ where $\hat{C}$ is a solution of $\hat{C}'\hat{C} = \hat{Q}$;

10) rank $[\hat{B}, \hat{A}\hat{B}, \cdots, (\hat{A})^{n-1}\hat{B}] = n$.

Then for sufficiently small λ the asymptotically stable equilibrium $\bar{K}^{\infty}(\lambda)$ of the full system (9) exists. Moreover,

$$\lim_{\lambda\to 0} \bar{K}_i^{\infty}(\lambda) = K_i^{\infty}, \qquad i = 1, 2, 3 \tag{35}$$

where K_i^{∞} is the asymptotically stable equilibrium of the reduced system (11).

Proof: The structure of the proof is to divide the time interval into two parts $t^* \le t < t_f$ and $-\infty < t \le t^*$. We apply the result of Theorem 1 to the first interval since it is finite and then show that the hypotheses of a theorem by Hoppensteadt (Theorem H in the Appendix) are satisfied on the second interval.

If conditions 1*)–3*) and 4) and 5) are satisfied then by Theorem 1 there exists a $\lambda^* > 0$ and a $t^* < t_f$ such that[4] $\|\bar{K}(t^*, \lambda^*) - K(t^*)\| < \epsilon$ for an ϵ which satisfies the ϵ closeness requirement of Theorem H. Furthermore, (9) has the unique solution $\bar{K}(t, \lambda^*)$ on $[t^*, t_f]$ satisfying the end condition $\bar{K}(t_f, \lambda^*) = 0$ and $\bar{K}(t, \lambda) \to K(t)$ as $\lambda \to 0$ on $[t^*, t_1]$ where $t^* < t_1 < t_f$.

Hypotheses H1, 2, 4, 5 of Theorem H are evidently satisfied by the form of (9) and conditions 1)–3). From the results of the linear regulator theory, conditions 1*)–3*) and 9) and 10) insure the existence and asymptotic stability of the solution $K_1(t)$ of (16) on $(-\infty, t^*]$, as required by H3 and H6. The crucial hypothesis H7 is that the solution of the boundary layer equation (12) be asymptotically stable uniformly in initial conditions and the parameters t and K_1 for $t \in (-\infty, t^*]$ and K_1 positive definite. By 1*) and 3*) the system (12) does not depend on t, nor does (12b) contain K_1. Hence by Lemma 1 the solution L_3 of (12b) satisfies H7. By Lemma 2 the solution L_2 of (12a) is asymptotically stable uniformly in K_1. This completes the proof of Theorem 2. Extensions similar to Corollaries 1 and 2 are immediate.

Two-Stage Design

Using the results of Theorems 1 and 2, the linear regulator design can be decomposed into a two-stage procedure. At the first stage the algebraic system (11c) is solved for $K_3(t)$. At the second stage the differential system (16) is solved for $K_1(t)$, and $K_2(t)$ is evaluated using the explicit formula (15).

This decomposition and reduction of dimensionality is particularly efficient in finite time interval problems with time-invariant systems. In this case the accurate design requires solution of the full $1/2(n + m + 1)(n + m)$-dimensional differential system (6), and the whole regulator matrix $\bar{K}(t, \lambda)$ is time-varying, while in the two-stage design the reduced differential system (16) is $1/2\,(n + 1)n$ dimensional. The time-invariant K_3 is easily obtained by an algebraic method and is less expensive to implement than $\bar{K}_3(t, \lambda)$.

A familiar speed control problem for a small dc motor is used to demonstrate the two-stage regulator design. The motor state equation is

$$\frac{d\omega}{dt} = (D/G)i \tag{36a}$$

$$\lambda L\frac{di}{dt} = -C\omega - R_a i + v \tag{36b}$$

where ω, i, and v are speed, current, and voltage deviations from their respective nominal values 400 rad/s, 0.25 A, 11.8 V. The motor constants are $R_a = 7{,}9\Omega$, $L = 0.0136$ H,

[3] That $K_3(t) = 0$ is not a multiple root follows from the nonsingularity of the Jacobian Γ_{33} in Lemma 4.

[4] The norm of a matrix is taken to be the sum of the absolute values of its elements.

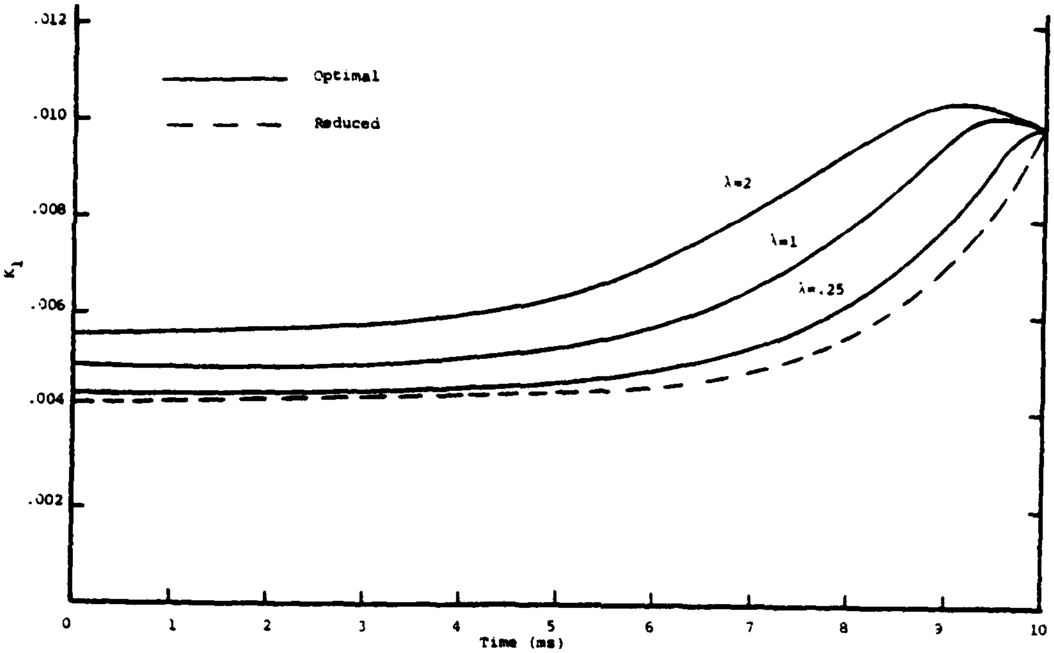

Fig. 1. Optimal and reduced Riccati gains $\bar{K}_1$ and K_1.

$C = 0.0246$ V·s/rad, $G = 1.32 \times 10^{-6}$ kg·m². In (36) the armature inductance, being a small parameter, is multiplied by a factor λ. Let the performance index be

$$J = \frac{0.01\omega^2}{2} + \frac{1}{2}\int_0^{t_f} (\omega^2 + 4600i^2 + 30v^2)\, dt. \quad (37)$$

Even in this extremely simple problem the full system

$$\frac{d\bar{K}_1}{dt} = 3.6\,\bar{K}_2 + 180\bar{K}_2^2 - 1,\ \bar{K}_1(t_f) = 0.01 \quad (38a)$$

$$\lambda\frac{d\bar{K}_2}{dt} = -18654\bar{K}_1 + 581\bar{K}_2 + 1.8\bar{K}_3 + 180\bar{K}_2\bar{K}_3 \quad (38b)$$

$$\lambda\frac{d\bar{K}_3}{dt} = -\lambda 37308\bar{K}_2 + 1162\bar{K}_3 + 180\bar{K}_3^2 - 4600 \quad (38c)$$

with $\bar{K}_2$ and $\bar{K}_3$ zero at $t = t_f$, must be solved on a computer. Note that, at $t = t_f$, $d\bar{K}_3/dt = -4600/\lambda$, and hence for $\lambda = 1$ the change of $\bar{K}_3$ is 4600 times faster than the change of $\bar{K}_1$, so that (38) can be considered a stiff system. In the degenerate system

$$\frac{dK_1}{dt} = 3.6K_2 + 180K_2^2 - 1, \qquad K_1(t_f) = 0.01 \quad (39a)$$

$$0 = -18654K_1 + 581K_2 + 1.8K_3 + 180\,K_3K_2 \quad (39b)$$

$$0 = 1162K_3 + 180K_3^2 - 4600 \quad (39c)$$

the positive definite root of (39c) is $K_3 = 2.77$. Solving (39b) for $K_2 = 17.3\,K_1 - 0.0046$, and substituting it in (39a) we obtain the reduced system

$$\frac{dK_1}{dt} = 33K_1 + 53683K_1^2 - 1.013, \qquad K_1(t_f) = 0.01 \quad (40)$$

which can be solved analytically. The reduced solution $K_1(t)$, $K_2(t)$, and K_3 is shown as dashed curves in Figs. 1–3. The solid curves represent the full solution $\bar{K}_1(t, \lambda)$, $\bar{K}_2(t, \lambda)$, and $\bar{K}_3(t, \lambda)$ for different values of λ. From the family of curves in Fig. 1 it is apparent that the limiting process $\bar{K}_1(t, \lambda) \to K_1(t)$ is uniform on the whole interval. The families of curves in Figs. 2 and 3 show that the limiting process for $\bar{K}_2(t, \lambda)$ and $\bar{K}_3(t, \lambda)$ takes place for $t < t_f = 10$ ms. The convergence is not uniform on the whole interval due to the boundary layer phenomenon near t_f.

A shortcoming of the two-stage design method presented here is that it does not give an estimate of a range $\lambda \in [0, \lambda^0]$ in which the reduced solution can be used as a "good" approximation of the full solution. Although theoretically important, this shortcoming does not seem to be critical in regulator design practice. Recall that even when the accurate $\bar{K}$ is designed several trials for weighting matrices $\bar{R}$ and $\bar{Q}$ are made, and the resulting system is tested before an acceptable control is found. Thus no loss of system performance will occur when these trials are carried out with the reduced solution K_1, K_2, and K_3.

The two-stage design has been introduced using full and reduced Riccati equations. This procedure can now be interpreted by an analysis of the full and reduced state equations of the resulting regulator system. In an implementation of the control law (5), K_1, K_2, and K_3 are used instead of $\bar{K}_1$, $\bar{K}_2$, and $\bar{K}_3$

$$u = -\bar{R}^{-1}[(\bar{B}_1'K_1 + \bar{B}_2'K_2')x + (\lambda\bar{B}_1'K_2 + \bar{B}_2'K_3)z] \quad (41)$$

and the full state equation of the regulator system becomes

$$\dot{x} = (\bar{A}_1 - \bar{S}_1K - \bar{S}K_2')x + (\bar{A}_2 - \bar{S}K_3 - \lambda\bar{S}_1K_2)z \quad (41a)$$

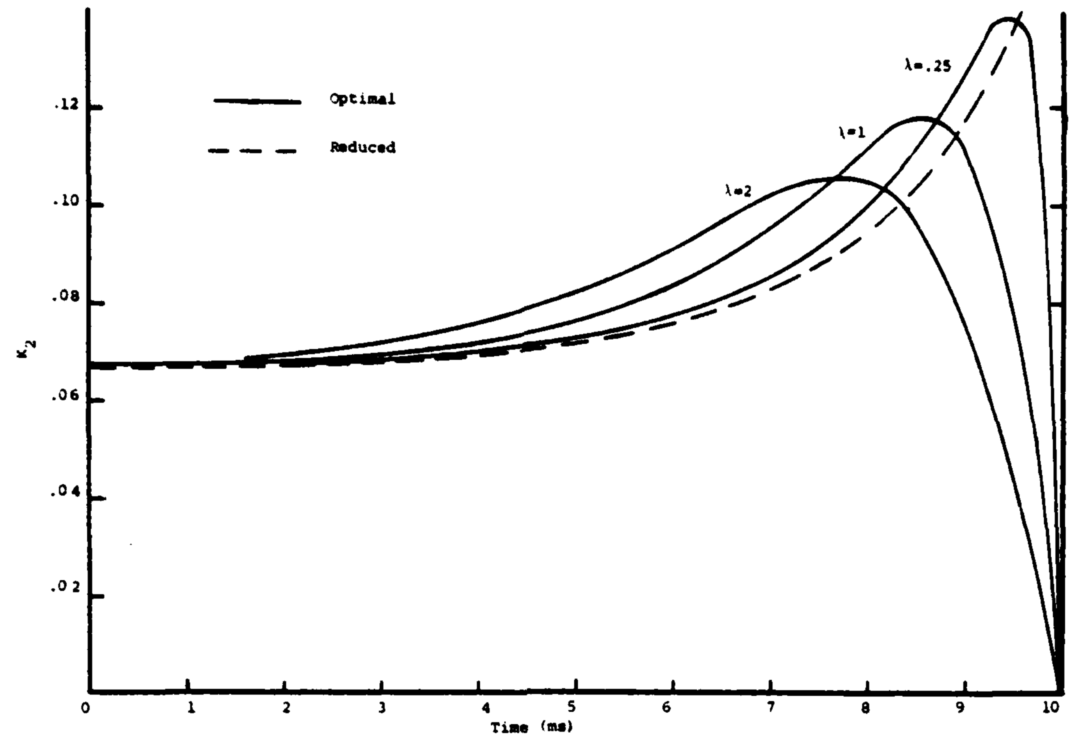

Fig. 2. Optimal and reduced Riccati gains $\bar{K}_2$ and K_2.

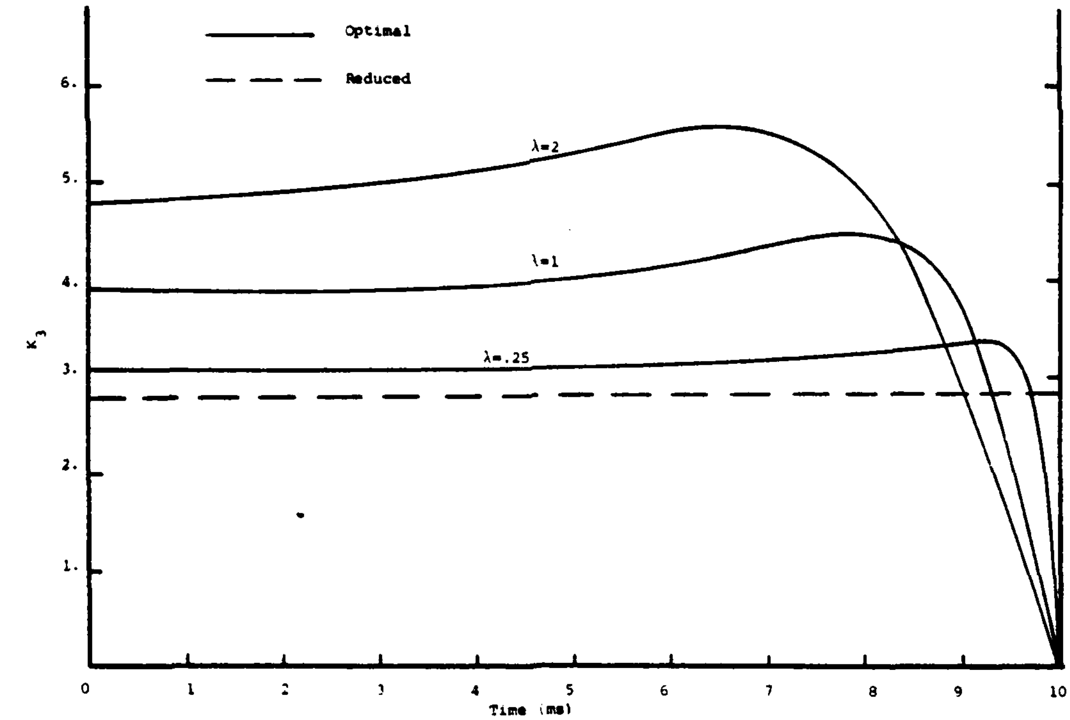

Fig. 3. Optimal and reduced Riccati gains $\bar{K}_3$ and K_3.

$$\lambda \dot{z} = (\bar{A}_3 - \bar{S}'K_1 - \bar{S}_2K_2')x + (\bar{A}_4 - \bar{S}_2K_3 - \lambda \bar{S}'K_2)z. \tag{41b}$$

The reduced state equation is unique since $(A_4 - S_2K_3)^{-1}$ exists and

$$0 = (A_3 - S'K_1 - S_2K_2')x + (A_4 - S_2K_3)z \tag{42}$$

can be uniquely solved for z. The state equation (41) satisfies all the hypotheses of Theorem L in the Appendix, and therefore the full solution will tend to the reduced solution as $\lambda \to 0^+$. Note that z exhibits a boundary layer near $t = t_0$ and that this layer is controlled by the solution of the regulator problem (21) in Lemma 1. Thus the first stage of the two-stage design procedure is used to

design a boundary layer regulator for the m-dimensional state z. To interpret the second stage the root of (42) is substituted into (41a) with $\lambda = 0$ and, using the notation (15b) and (17a), (17b), the reduced state equation can be written in compact form

$$\dot{x} = \hat{A}x + \hat{B}\hat{u} \tag{43}$$

where

$$\hat{u} = -R^{-1}\hat{B}'K_1x. \tag{44}$$

It is apparent from (16), (43), and (44) that in the second stage the reduced regulator system (43) is optimized with respect to the performance index defined by R and $\hat{Q}$.

Conclusion

Notions of boundary layer controllability and observability have been introduced to establish sufficient conditions for the solution of a higher order linear regulator problem to tend to the solution of a lower order problem as a parameter λ tends to zero. For λ sufficiently small a two-stage procedure greatly simplifies the design of a linear regulator. The results of this paper are also applicable to other problems involving matrix Riccati equations such as filtering, estimation, and nonlinear trajectory optimization via second variation techniques.

Appendix

Consider the initial value problem

$$\frac{d\xi}{dt} = f(t, \xi, \eta, \lambda), \qquad \xi(t_0) = \xi_0 \tag{45}$$

$$\lambda\frac{d\eta}{dt} = g(t, \xi, \eta, \lambda), \qquad \eta(t_0) = \eta_0 \tag{46}$$

where λ is a small positive parameter and ξ and η are n- and m-dimensional vectors, respectively. Formally setting $\lambda = 0$ in the full system (45) and (46) gives the degenerate system

$$\frac{d\bar{\xi}}{dt} = f(t, \bar{\xi}, \bar{\eta}, 0), \qquad \bar{\xi}(t_0) = \xi_0 \tag{47}$$

$$0 = g(t, \bar{\xi}, \bar{\eta}, 0). \tag{48}$$

Since (48) may have several roots, suppose that a particular root $\bar{\eta} = \varphi(t, \bar{\xi})$ is of interest and substitute it in (47). The n-dimensional system

$$\frac{d\bar{\xi}}{dt} = f[t, \bar{\xi}, \varphi(t, \bar{\xi}), 0], \qquad \bar{\xi}(t_0) = \xi_0 \tag{47'}$$

is a reduced system of (45), (46).

Introducing a new time variable τ (46) is rewritten in the form of a boundary layer system,

$$\frac{d\sigma}{d\tau} = g(\alpha, \beta, \sigma, 0), \qquad \sigma(0) = \eta_0 \tag{49}$$

where $\alpha = t_0$ and $\beta = \bar{\xi}_0$ are fixed parameters. In the space of variables ξ, η, λ, t we define a region $\mathcal{R}$: $\|\xi - \bar{\xi}(t)\| < r$, $\|\eta - \bar{\eta}(t)\| < r$, $0 \le \lambda \le \lambda^0$, $t_0 \le t \le t_f$ where $r > |\eta_0 - \bar{\eta}(t_0)| \ge 0$.

Theorem L [4]

Let the following conditions be satisfied.

L1: f, $\partial f/\partial\xi$, $\partial f/\partial\eta$, g, $\partial g/\partial\xi$, $\partial g/\partial\eta$ are of class C^0 in $(\xi, \eta, t, \lambda) \in \mathcal{R}$.

L2: The solution $\sigma(\tau)$ of (50) exists on $\tau \in [0, \infty)$, is unique, and is asymptotically stable with respect to the root $\varphi(t_0, \xi_0)$ of (48).

L3: The solution $\bar{\xi}(t)$ of the reduced system (47') exists and is unique on $t \in [t_0, t_f]$.

L4: The real parts of the eigenvalues of the Jacobian matrix

$$\partial g/\partial\eta(t, \bar{\xi}, \bar{\eta}, 0) \tag{50}$$

are negative on $[t_0, t_f]$, for $\bar{\eta} = \varphi(t, \bar{\xi})$.

Then for sufficiently small λ, the full system (45), (46) has a unique solution $\xi(t, \lambda)$, $\eta(t, \lambda)$ on $t \in [t_0, t_f]$ satisfying the initial conditions $\xi(t_0, \lambda) = \bar{\xi}_0$, $\eta(t_0, \lambda) = \eta_0$. Furthermore,

$$\lim_{\lambda\to 0} \xi(t, \lambda) = \bar{\xi}(t) \text{ on } [t_0, t_f] \tag{51}$$

$$\lim_{\lambda\to 0} \eta(t, \lambda) = \bar{\eta}(t) \text{ on } (t_0, t_f] \tag{52}$$

where the limit (51) is uniform in t on $[t_0, t_f]$ and the limit (52) is uniform in t on any interval $[t_1, t_f]$, where $t_0 < t_1 < t_f$.

Theorem H [5]

Let $t_f = \infty$ in the definition of $\mathcal{R}$, and denote by Ω its $(n + 1)$-dimensional subspace in ξ and t. Let the following conditions be satisfied.

H1: f, g, $\partial f/\partial\xi$, $\partial f/\partial\eta$, $\partial g/\partial t$, $\partial g/\partial\xi$, $\partial g/\partial\eta$ are of class C^1 for all $(t, \xi, \eta, \lambda) \in \mathcal{R}$.

H2: There exists a bounded function $\eta = \varphi(\xi, t)$ of class C^2 which is an isolated root of $g(t, \xi, \varphi(t, \xi), 0) = 0$ for all $(\xi, t) \in \Omega$.

H3: The solution $\bar{\xi}(t)$ of the reduced system (47'), corresponding to the isolated root $\bar{\eta}(t)$ exists on $[t_0, \infty)$.

H4: The function f is of class C^0 at $\eta(t) = \bar{\eta}(t)$, $\lambda = 0$ uniformly in $(\xi, t) \in \Omega$ and $f(t, \xi, \bar{\eta}(t), 0)$ and $\partial f/\partial\xi(t, \xi, \bar{\eta}(t), 0)$ are bounded in Ω.

H5: The function g is of class C^0 at $\lambda = 0$ uniformly in $(t, \xi, \eta) \in \mathcal{R}$ and g, $\partial g/\partial t$, $\partial g/\partial\xi$, $\partial g/\partial\eta$ at $\lambda = 0$ are bounded on $\mathcal{R}$.

H6: The solution of the reduced system is uniformly asymptotically stable.

H7: The solution of the boundary layer system is uniformly asymptotically stable uniformly in the parameters $(\alpha, \beta) \in \Omega$.

If the ϵ closeness requirement $\|\eta_0 - \varphi(t_0, \xi_0)\| < \epsilon$ is satisfied for a sufficiently small ϵ, then for a sufficiently small λ the solution of the full system exists for $t_0 \le t < \infty$.

Furthermore, the limits (51) and (52) hold uniformly on all closed subsets of $t_0 < t < \infty$.

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A Decomposition of Near-Optimum Regulators for Systems with Slow and Fast Modes

J. H. CHOW, MEMBER, IEEE, AND P. V. KOKOTOVIĆ

Abstract—For systems with slow and fast subsystems a near-optimum state regulator is composed of two subsystem regulators. Conditions for complete separation of slow and fast regulator designs are formulated. A second-order approximation of the optimal performance is achieved without the knowledge of the small singular perturbation parameter.

I. Introduction

Linear time-invariant models of many physical systems contain slow and fast modes. Control problems for such models are often ill-conditioned and have motivated several model-simplification approaches. Simplified models obtained via aggregation [1] and dominant modes [2], [3] approaches neglect fast modes and some of the poorly controllable and observable slow modes. In the singular perturbation method [4], [5] both slow and fast modes are retained, but analysis and design problems are solved in two stages, first for the fast and then for the slow modes. However the separation in [5] was not complete since the slow regulator design depended on the fast feedback gain matrix.

This short paper presents a new procedure for a complete separation of slow and fast regulator designs. Furthermore the performance achieved by the composition of a slow and a fast regulator proposed here is a second-order approximation of the optimal performance. It is significant that, in contrast to previous designs, this near-optimal regulator does not require the knowledge of the singular perturbation parameter μ. Hence this regulator is applicable to systems where μ represents small uncertain parameters. In addition to the presentation of these new results, another purpose of this short paper is to give a self-contained development of the two-time-scale method based on singular perturbations [8]–[10]. The paper assumes only the knowledge of standard facts of linear control theory. No familiarity with the singular perturbation literature is required.

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II. Slow and Fast Subsystems

We consider a singularly perturbed linear time-invariant system

$$\dot{x}_1 = A_{11}x_1 + A_{12}x_2 + B_1u, \qquad x_1(0) = x_{10} \tag{1a}$$

$$\mu\dot{x}_2 = A_{21}x_1 + A_{22}x_2 + B_2u, \qquad x_2(0) = x_{20} \tag{1b}$$

$$y = C_1x_1 + C_2x_2 \tag{1c}$$

where μ is a small positive scalar, the state x is formed by the n_1 and n_2 vectors x_1, x_2, the control u is an m vector and the output y a k vector. As shown in [11], system (1) possesses a two-time-scale property, that is, it has n_1 small eigenvalues of magnitude $O(1)$ and n_2 large eigenvalues of magnitude $O(1/\mu)$. Preliminary to a separation of slow and fast designs, system (1) is approximately decomposed into a slow subsystem with n_1 small eigenvalues and a fast subsystem with n_2 large eigenvalues. In an asymptotically stable system the fast modes corresponding to the large eigenvalues are important only during a short initial period. After that period they are negligible and the behavior of the system can be described by its slow modes. Neglecting the fast modes is equivalent to assuming that they are infinitely fast, that is letting $\mu \to 0$ in (1). Without the fast modes the system (1) reduces to

$$\dot{\bar{x}}_1 = A_{11}\bar{x}_1 + A_{12}\bar{x}_2 + B_1\bar{u}, \qquad \bar{x}_1(0) = x_{10} \tag{2a}$$

$$0 = A_{21}\bar{x}_1 + A_{22}\bar{x}_2 + B_2\bar{u} \tag{2b}$$

$$\bar{y} = C_1\bar{x}_1 + C_2\bar{x}_2 \tag{2c}$$

where a bar indicates that $\mu = 0$. Assuming that A_{22} is nonsingular, we express $\bar{x}_2$ as

$$\bar{x}_2 = -A_{22}^{-1}(A_{21}\bar{x}_1 + B_2\bar{u}) \tag{3}$$

and, substituting it into (2), we define the slow subsystem of (1) as

$$\dot{x}_s = A_0x_s + B_0u_s, \qquad x_s(0) = x_{10} \tag{4a}$$

$$y_s = C_0x_s + D_0u_s \tag{4b}$$

where

$$\begin{aligned} A_0 &= A_{11} - A_{12}A_{22}^{-1}A_{21}, \quad & B_0 &= B_1 - A_{12}A_{22}^{-1}B_2 \\ C_0 &= C_1 - C_2A_{22}^{-1}A_{21}, \quad & D_0 &= -C_2A_{22}^{-1}B_2. \end{aligned} \tag{4c}$$

Thus $\bar{x}_1 = x_s$, $\bar{y} = y_s$, $\bar{u} = u_s$, and $\bar{x}_2$ are the slow parts of the corresponding variables in (1).

To derive the fast subsystem, we assume that the slow variables are constant during fast transients, that is, $\dot{\bar{x}}_2 = 0$ and $\bar{x}_1 = x_s =$ constant. From (1b) and (3), we then obtain

$$\mu(\dot{x}_2 - \dot{\bar{x}}_2) = A_{22}(x_2 - \bar{x}_2) + B_2(u - u_s). \tag{5}$$

Letting $x_f = x_2 - \bar{x}_2$, $u_f = u - u_s$, $y_f = y - y_s$, the fast subsystem of (1) is defined as

$$\mu\dot{x}_f = A_{22}x_f + B_2u_f, \qquad x_f(0) = x_{20} - \bar{x}_2(0) \tag{6a}$$

$$y_f = C_2x_f. \tag{6b}$$

Suppose now that $u_s = G_0 x_s$ and $u_f = G_2 x_f$ are designed such that x_s and x_f meet some specifications. In view of

$$\bar{x}_2 = -A_{22}^{-1}(A_{21}+B_2G_0)x_s \tag{7}$$

which follows from (3), the "composite control"

$$u_s + u_f = G_0 x_s + G_2 x_f \tag{8}$$

can be rewritten as

$$u_s + u_f = \left[(I+G_2A_{22}^{-1}B_2)G_0 + G_2A_{22}^{-1}A_{21}\right]x_s + G_2\left[-A_{22}^{-1}(A_{21}+B_2G_0)x_s + x_f\right]. \tag{9}$$

The following lemma establishes properties of the feedback system (1) with a composite control of the form (9), but with x_1 replacing x_s and x_2 replacing $\bar{x}_2 + x_f$.

Lemma 1

If the controls

$$u_s = G_0 x_s \tag{10a}$$

$$u_f = G_2 x_f \tag{10b}$$

$$u = \left[(I+G_2A_{22}^{-1}B_2)G_0 + G_2A_{22}^{-1}A_{21}\right]x_1 + G_2 x_2 \tag{11}$$

are applied to systems (4), (6), and (1), respectively, and if $A_{22}+B_2G_2$ is stable, then

$$x_1(t) = x_s(t) + O(\mu) \tag{12a}$$

$$x_2(t) = -A_{22}^{-1}(A_{21}+B_2G_0)x_s(t) + x_f(t) + O(\mu) \tag{12b}$$

$$u(t) = u_s(t) + u_f(t) + O(\mu) \tag{12c}$$

$$y(t) = y_s(t) + y_f(t) + O(\mu) \tag{12d}$$

hold for all finite $t>0$. If $A_0 + B_0G_0$ is also stable, then (12a)–(12d) hold for all $t \in [0,\infty)$.

Proof: The feedback system (1), (11) is

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix} A_{11}+B_1(I+G_2A_{22}^{-1}B_2)G_0 + B_1G_2A_{22}^{-1}A_{21} & A_{12}+B_1G_2 \\ (A_{22}+B_2G_2)A_{22}^{-1}(A_{21}+B_2G_0)/\mu & (A_{22}+B_2G_2)/\mu \end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}. \tag{13}$$

Following [11], we construct a transformation

$$T = \begin{bmatrix} I_1-\mu HL & -\mu H\\ L & I_2\end{bmatrix}, \qquad T^{-1} = \begin{bmatrix} I_1 & \mu H\\ -L & I_2-\mu LH\end{bmatrix} \tag{14}$$

where I_1 and I_2 are $n_1 \times n_1$ and $n_2 \times n_2$ identity matrices, respectively, and

$$L = A_{22}^{-1}(A_{21}+B_2G_0) + \mu N \tag{15a}$$

$$N = (A_{22}+B_2G_2)^{-1}A_{22}^{-1}(A_{21}+B_2G_0)(A_0+B_0G_0) + O(\mu) \tag{15b}$$

$$H = (A_{12}+B_1G_2)(A_{22}+B_2G_2)^{-1} + O(\mu). \tag{15c}$$

Neglecting the $O(\mu^2)$ terms, we obtain

$$T\mathcal{Q}T^{-1} = \begin{bmatrix}\mathcal{Q}_0 & 0\\ 0 & \mathcal{Q}_2\end{bmatrix} \tag{16}$$

where $\mathcal{Q}$ is the system matrix of (13), $\mathcal{Q}_0 = (A_0+B_0G_0) - \mu(A_{12}+B_1G_2)N$, and $\mathcal{Q}_2 = (A_{22}+B_2G_2) - \mu L(A_{12}+B_1G_2)$. If $A_{22}+B_2G_2$ is stable, the solution of (13) is approximated for all finite $t>0$ by

$$x_1(t) = \exp[(A_0+B_0G_0)t]x_s(0) + O(\mu) \tag{17a}$$

$$x_2(t) = -A_{22}^{-1}(A_{21}+B_2G_0)\exp[(A_0+B_0G_0)t]x_s(0) + \exp[(A_{22}+B_2G_2)t/\mu]x_f(0) + O(\mu) \tag{17b}$$

where $x_s(0)$, $x_f(0)$ are given by (4a), (6a). Note that the effects of the continuous perturbations of the matrices in (13)–(16) are incorporated in the $O(\mu)$ terms in (17). If in addition $A_0 + B_0G_0$ is also stable, (17) holds for all $t \in [0,\infty)$. Then (12) follows directly from (17), (4), and (6).

Lemma 1 suggests that G_0 and G_2 be separately designed according to the slow and fast mode performance specifications, and implemented as the composite control defined by (11). This idea has been applied to the pole placement design [12]. A similar separation of designs is now developed for the optimal linear regulator problem.

III. Subsystem Regulator Problems

In this section we decompose the optimum state regulator problem for the system (1) and the performance index

$$J = \frac{1}{2}\int_0^\infty (y'y + u'Ru)\,dt, \qquad R>0 \tag{18}$$

into two subsystem regulator problems. Our approach is to extract from J two quadratic performance indices, one for the variables of the slow subsystem (4), and the other for the variables of the fast subsystem (6). We formulate and analyze two separate regulator problems, denoted by s for the slow, and f for the fast subsystems. From the subsystem optimal controls u_s and u_f we then form a composite control $u_c = u_s + u_f$ which is to be implemented on the original system (1). Its performance is analyzed in the next section.

Problem s

Find u_s to minimize

$$J_s = \frac{1}{2}\int_0^\infty (y_s'y_s + u_s'Ru_s)\,dt, \qquad R>0 \tag{19}$$

for the slow subsystem (4).

In terms of x_s and u_s, (19) becomes

$$J_s = \frac{1}{2}\int_0^\infty [x_s'C_0'C_0x_s + 2u_s'D_0'C_0x_s + u_s'R_0u_s]\,dt \tag{20}$$

where $R_0 = R + D_0'D_0$. From [13], [14], we know that if the Riccati equation

$$0 = -K_s(A_0 - B_0R_0^{-1}D_0'C_0) - (A_0 - B_0R_0^{-1}D_0'C_0)'K_s + K_sB_0R_0^{-1}B_0'K_s - C_0'(I - D_0R_0^{-1}D_0')C_0 \tag{21}$$

has a positive semidefinite stabilizing solution K_s, then the optimal control for (4) and (19) is

$$u_s = -R_0^{-1}(D_0'C_0 + B_0'K_s)x_s. \tag{22}$$

A sufficient condition for the existence and uniqueness of K_s is given in the following theorem.

Theorem 1

Condition a: If the triple (A_0, B_0, C_0) is stabilizable-detectable, then (21) has a unique positive semidefinite stabilizing solution K_s.

Proof: From [13], [14], the stabilizing solution K_s exists if the triple $(A_0 - B_0R_0^{-1}D_0'C_0, B_0, \hat{C}_0)$ is stabilizable-detectable, where $\hat{C}_0'\hat{C}_0 = C_0'(I - D_0R_0^{-1}D_0')C_0$. Note that $(A_0 - B_0R_0^{-1}D_0'C_0, B_0)$ is stabilizable if and only if (A_0, B_0) is stabilizable. From the well known[1] identity $I - D_0R_0^{-1}D_0' = (I + D_0RD_0')^{-1} > 0$ which holds when $R>0$, it follows that there exists a nonsingular Q_0 such that $Q_0'Q_0 = I - D_0R_0^{-1}D_0'$. Hence $(A_0 - B_0R_0^{-1}D_0'C_0, Q_0C_0)$ is detectable if and only if (A_0, C_0) is detectable. This proves that the stabilizability-detectability of the triple $(A_0 - B_0R_0^{-1}D_0'C_0, B_0, \hat{C}_0)$ is equivalent to Condition a.

Problem f

Find u_f to minimize

$$J_f = \frac{1}{2}\int_0^\infty (y_f'y_f + u_f'Ru_f)\,dt \tag{23}$$

for the fast subsystem (6).

[1] See, for example [15, pp. 190–191].

It is well known that the optimal control for (6), (23) is

$$u_f = -R^{-1}B_2'K_fx_f \tag{24}$$

where K_f is the positive semidefinite stabilizing solution of the Riccati equation

$$0 = -K_fA_{22} - A_{22}'K_f + K_fB_2R^{-1}B_2'K_f - C_2'C_2. \tag{25}$$

Condition b: This solution exists and is unique if the triple (A_{22}, B_2, C_2) is stabilizable-detectable.

The controls u_s and u_f defined by (22) and (24) are only subsystem optimal. It is much easier to compute u_s and u_f than the optimal control for the complete system (1). Lemma 1 indicated how a composite control $u_c = u_s + u_f$ can be obtained in terms of x_1 and x_2. It is of interest to investigate the performance of the system (1) controlled by u_c. With $G_0 = -R_0^{-1}(D_0'C_0 + B_0'K_s)$ and $G_2 = -R^{-1}B_2'K_f$, the composite feedback control (11) is

$$u_c = -\Big[\big(I - R^{-1}B_2'K_fA_{22}^{-1}B_2\big)R_0^{-1}(D_0'C_0 + B_0'K_s) + R^{-1}B_2'K_fA_{22}^{-1}A_{21}\Big]x_1 - R^{-1}B_2'K_fx_2. \tag{26}$$

The complete separation property of the composite control (26) lies in the fact that the Conditions a and b are mutually independent. This contrasts the conditions in [5],[8] where the existence of the solution for the slow regulator problem depended on K_f.

Another practically important property of the control law (26) is that it does not explicitly depend on μ. It is now shown that even without the knowledge of μ, the control (26) results in an $O(\mu^2)$ approximation of the optimal performance index.

IV. Near-Optimality of the Composite Control

For a comparison with the exact optimal control, we express u_c in a more convenient form. Since $A_{22} - B_2R^{-1}B_2'K_f$ is nonsingular, manipulation of the expression R_0 using (25) yields the identity

$$\big(I - R^{-1}B_2'K_fA_{22}^{-1}B_2\big)R_0^{-1} = R^{-1}\Big[I + B_2'\big(A_{22} - B_2R^{-1}B_2'K_f\big)'^{-1}K_fB_2R^{-1}\Big] \tag{27}$$

and hence (26) can be rewritten as

$$u_c = -R^{-1}\big(B_1'K_sx_1 + B_2'K_m'x_1 + B_2'K_fx_2\big) = -R^{-1}B'\begin{bmatrix} K_s & 0 \\ \mu K_m' & \mu K_f \end{bmatrix}x = -R^{-1}B'M_cx \tag{28}$$

where $B' = [B_1' \; B_2'/\mu]$, $x' = (x_1' x_2')$, and

$$K_m = \Big[K_s\big(B_1R^{-1}B_2'K_f - A_{12}\big) - \big(A_{21}'K_f + C_1'C_2\big)\Big]\big(A_{22} - B_2R^{-1}B_2'K_f\big)^{-1}. \tag{29}$$

On the other hand the exact optimal control for the complete problem (1), (18) is $u_{opt} = -R^{-1}B'Kx$, where K is the stabilizing solution of the Riccati equation

$$0 = -KA - A'K + KSK - C'C \tag{30}$$

with $C = [C_1 C_2]$, $S = BR^{-1}B'$, and

$$A = \begin{bmatrix} A_{11} & A_{12} \\ \frac{1}{\mu}A_{21} & \frac{1}{\mu}A_{22} \end{bmatrix}.$$

The expression (30) defines an implicit dependence $K(\mu)$ of K on the parameter μ. Our objective is to analyze the relationship between $K(\mu)$ and M_c in (28) for μ small and positive. For this purpose we construct a power series expansion of $K(\mu)$.

Theorem 2

If the Conditions a and b are satisfied, then the positive semidefinite stabilizing solution $K = K(\mu)$ of (30) possesses a power series expansion at $\mu = 0$, that is,

$$K = \begin{bmatrix} K_1 & \mu K_2 \\ \mu K_2' & \mu K_3 \end{bmatrix} + \sum_{i=1}^{\infty} \frac{\mu^i}{i!}\begin{bmatrix} K_1^{(i)} & \mu K_2^{(i)} \\ \mu K_2^{(i)\prime} & \mu K_3^{(i)} \end{bmatrix}. \tag{31}$$

Furthermore, the matrices K_1, K_2, K_3 satisfy the identities

$$K_1 = K_s, \quad K_2 = K_m, \quad K_3 = K_f \tag{32}$$

where K_s, K_f, and K_m are defined by (21), (25), and (29), respectively.

Proof: The substitution of (31) into (30) yields at $\mu = 0$ the equations

$$0 = -K_1(A_{11} - S_{12}K_2') - (A_{11} - S_{12}K_2')'K_1 + K_1S_1K_1 - K_2A_{21} - A_{21}'K_2' + K_2S_2K_2' - C_1'C_1 \tag{33a}$$

$$0 = K_2(S_2K_3 - A_{22}) - K_1A_{12} - A_{21}'K_3 + K_1S_{12}K_3 - C_1'C_2 \tag{33b}$$

$$0 = -K_3A_{22} - A_{22}'K_3 + K_3S_2K_3 - C_2'C_2 \tag{33c}$$

where $S_1 = B_1R^{-1}B_1'$, $S_2 = B_2R^{-1}B_2'$, and $S_{12} = B_1R^{-1}B_2'$. Under Condition b, (25) and (33c) imply that $K_3 = K_f$. Hence (33b) can be solved for K_2 and substituted into (33a). This results in the Riccati equation

$$0 = -K_1\hat{A} - \hat{A}'K_1 + K_1\hat{B}R^{-1}\hat{B}'K_1 - \hat{C}'\hat{C} \tag{34}$$

where

$$\hat{A} = A_0 - B_0R_0^{-1}D_0'C_0 \tag{35a}$$

$$\hat{B}R^{-1}\hat{B}' = B_0R_0^{-1}B_0' \tag{35b}$$

$$\hat{C}'\hat{C} = C_0'\big(I - D_0R_0^{-1}D_0'\big)C_0. \tag{35c}$$

The derivation of (34), (35) involves simple but lengthy calculations which are found in [6]. Under Condition a, (21) and (34) imply that $K_1 = K_s$. Hence from (29) and (33b), $K_2 = K_m$. The existence of the series (31) then follows from the implicit function theorem [8].

When the linear singularly perturbed regulator problem was treated in [5],[8], the fast regulator (33c) was designed first and only then the slow regulator (33a). We point out that in [5],[8] it was unclear whether the stabilizability-detectability of the triple $(\hat{A}, \hat{B}, \hat{C})$ depended on K_f, since there $\hat{A}$, $\hat{B}$, and $\hat{C}$ appeared as explicit functions of K_f. This question is now answered by Theorem 2, which results in the separation of designs of K_s and K_f.

Theorem 2 establishes that the composite feedback control u_c in (26) is $O(\mu)$ close to u_{opt}. This implies that J_c, the value of the performance index J of system (1) with u_c, is at least $O(\mu)$ close to the value of the optimal performance index J_{opt}. The following analysis of J_c and J_{opt} reveals that u_c in fact yields an $O(\mu^2)$ approximation of J_{opt}.

Since $J_{opt} = \frac{1}{2}x_0'Kx_0$ and $J_c = \frac{1}{2}x_0'P_cx_0$, where P_c is the positive definite solution of the Lyapunov equation

$$P_c(A - SM_c) + (A - SM_c)'P_c = -M_c'SM_c - C'C \tag{36}$$

the following theorem holds.

Theorem 3

The first two terms of the power series of J_c and J_{opt} at $\mu = 0$ are the same, that is,

$$J_c = J_{opt} + O(\mu^2) \tag{37}$$

and hence the composite feedback control (26) is an $O(\mu^2)$ near-optimal solution to the complete regulator problem (1), (18).

Proof: Adding (30) to (36) and rearranging, we obtain a Lyapunov equation for $P_c - K = W$:

$$W(A - SM_c) + (A - SM_c)'W + (K - M_c')S(K - M_c) = 0. \tag{38}$$

By an application of the implicit function theorem to (36) we can show

that P_c possesses a power series at $\mu=0$. Thus W can also be expanded as follows:

$$W=\sum_{i=0}^{\infty}\frac{\mu^i}{i!}\begin{bmatrix} W_1^{(i)} & \mu W_2^{(i)} \\ \mu W_2^{(i)\prime} & \mu W_3^{(i)}\end{bmatrix}. \tag{39}$$

From (28), (31), and (32) we have $(K-M_c')S(K-M_c)=0(\mu^2)$ and, since the matrices $A_0-B_0R_0^{-1}(D_0'C_0+B_0'K_s)$ and $A_{22}-B_2R^{-1}B_2'K$ are stable, the substitution of (39) into (38) yields $W_j^{(0)}=0$ and $W_j^{(1)}=0$, $j=1,2,3$. Hence $W=0(\mu^2)$, which proves (37).

We have therefore shown that the composite control (26), even though it does not contain μ explicitly, guarantees an $0(\mu^2)$ approximation of the optimal performance. Thus this design eliminates the need to know μ, required in previous designs [5], [8] for a second-order approximation. As before μ would be needed for a higher order approximation which is not considered here.

V. Reduced Control

Now we consider the approximation achieved by optimizing only the slow subsystem.

If $u_f=G_2x_f$ is designed such that $A_{22}+B_2G_2$ is stable, from (11), the control optimizing the slow subsystem is

$$u=-\left[\left(I+G_2A_{22}^{-1}B_2\right)R_0^{-1}\left(D_0'C_0+B_0'K_s\right)-G_2A_{22}^{-1}A_{21}\right]x_1+G_2x_2. \tag{40}$$

In particular, if A_{22} is stable and $G_2=0$, then (40) becomes the "reduced control"

$$u_r=-R_0^{-1}\left(D_0'C_0+B_0'K_s\right)x_1=Fx. \tag{41}$$

The value J_r of the performance index J with u_r in (41) as the feedback is $J_r=\frac{1}{2}x_0'P_rx_0$, where P_r is the positive definite solution of the Lyapunov equation

$$P_r(A-BF)+(A-BF)'P_r=-F'R^{-1}F-C'C. \tag{42}$$

Even though u_r in (41) is not $0(\mu)$ close to u_{opt}, its performance J_r does approximate J_{opt} to $0(\mu)$ order.

Theorem 4

If A_{22} is stable, then the constant terms of the power series of J_r and J_{opt} at $\mu=0$ are equal, that is,

$$J_r=J_{opt}+0(\mu) \tag{43}$$

and hence the feedback control u_r in (41) is an $0(\mu)$ near-optimal solution to the complete regulator problem (1), (18).

Proof: Using (25) and (27), u_r in (41) can be expressed as

$$u_r=-R^{-1}B'\begin{bmatrix} K_s & 0 \\ K_r' & 0\end{bmatrix}x=-R^{-1}B'M_rx \tag{44}$$

where

$$K_r=\left[K_m+(K_sS_{12}-A_{21}')A_{22}^{\prime -1}K_f\right]\left[I+S_2\left(A_{22}-S_2K_f\right)^{\prime -1}K_f\right]. \tag{45}$$

Hence (42) can be rewritten as

$$P_r(A-SM_r)+(A-SM_r)'P_r=-M_r'SM_r-C'C \tag{46}$$

and P_r possesses a power series in μ. Adding (30) to (46) and rearranging, we obtain a Lyapunov equation for $P_r-K=V$:

$$V(A-SM_r)+(A-SM_r)'V+(K-M_r')S(K-M_r)=0. \tag{47}$$

Substituting the power series

$$V=\begin{bmatrix} V_1 & \mu V_2 \\ \mu V_2' & \mu V_3\end{bmatrix}+\sum_{i=1}^{\infty}\frac{\mu^i}{i!}\begin{bmatrix} V_1^{(i)} & \mu V_2^{(i)} \\ \mu V_2^{(i)\prime} & \mu V_3^{(i)}\end{bmatrix} \tag{48}$$

into (47) and evaluating at $\mu=0$ yields

$$V_1\bar{A}_{11}+\bar{A}_{11}'V_1+V_2\bar{A}_{21}+\bar{A}_{21}'V_2+(K_m-K_r)S_2(K_m-K_r)'=0 \tag{49a}$$

$$V_1A_{12}+V_2A_{22}+\bar{A}_{21}'V_3+(K_m-K_r)S_2K_f=0 \tag{49b}$$

$$V_3A_{22}+A_{22}'V_3+K_fS_2K_f=0 \tag{49c}$$

where

$$\bar{A}_{11}=A_{11}-S_1K_s-S_{12}K_r' \tag{50a}$$

$$\bar{A}_{21}=A_{21}-S_{12}'K_s-S_2K_r'. \tag{50b}$$

Since A_{22} is stable, there exists a unique positive semidefinite solution V_3 of (49c). Expressing V_2 in terms of V_1 and V_3 and substituting into (49a), we obtain from (49c)

$$V_1\left(\bar{A}_{11}-A_{12}A_{22}^{-1}\bar{A}_{21}\right)+\left(\bar{A}_{11}-A_{12}A_{22}^{-1}\bar{A}_{21}\right)'V_1$$

$$=-\left(K_m-K_r-\bar{A}_{21}'A_{22}^{\prime -1}K_f\right)S_2\left(K_m-K_r-\bar{A}_{21}'A_{22}^{\prime -1}K_f\right)'. \tag{51}$$

Rearrangement of (45) yields

$$K_m-K_r-\bar{A}_{21}'A_{22}^{\prime -1}K_f=0 \tag{52}$$

and hence the right-hand side of (51) is identically zero. Furthermore $\bar{A}_{11}-A_{12}A_{22}^{-1}\bar{A}_{22}=A_0-B_0R_0^{-1}(D_0'C_0+B_0'K_s)$, which is stable. Thus the solution (51) is $V_1=0$ implying $V=0(\mu)$ and (43).

Comparing Theorem 4 to Theorem 3, it is obvious that the major portion of J is contributed by the slow subsystem, while the contribution of the fast subsystem is $0(\mu)$. Another inference from Theorem 4 is the insensitivity of the slow subsystem to change in the fast subsystem, provided that A_0, B_0, C_0, and D_0 remain unchanged. Frequently only the slow subsystem is modeled. For changes in the fast subsystem that do not affect A_0, B_0, C_0, and D_0, the feedback control (41) remains near-optimal.

VI. Design Procedure and Example

Summarizing the preceding sections, we propose the following design procedure.

For an $0(\mu^2)$ near-optimum regulator solve (21) for K_s and (25) for K_f. The composite control to be implemented is given by (26).

For an $0(\mu)$ near-optimum regulator when A_{22} is stable solve (21) for K_s and implement the reduced control given by (41).

As an illustration we consider a system with fast and slow modes in the form (1),

$$A_{11}=\begin{bmatrix} 0 & 0.4 \\ 0 & 0\end{bmatrix},\qquad A_{12}=\begin{bmatrix} 0 & 0 \\ 0.345 & 0\end{bmatrix}$$

$$A_{21}=\begin{bmatrix} 0 & -0.524 \\ 0 & 0\end{bmatrix},\qquad A_{22}=\begin{bmatrix} -0.465 & 0.262 \\ 0 & -1\end{bmatrix} \tag{53a}$$

$$B_1=\begin{bmatrix} 0 \\ 0\end{bmatrix},\qquad B_2=\begin{bmatrix} 0 \\ 1\end{bmatrix} \tag{53b}$$

$$C_1=\begin{bmatrix} 1 & 0 \\ 0 & 0\end{bmatrix},\qquad C_2=\begin{bmatrix} 0 & 0 \\ 1 & 0\end{bmatrix}. \tag{53c}$$

Let the performance index be

$$J=\frac{1}{2}\int_0^{\infty}(y'y+u'u)\,dt. \tag{54}$$

Conditions a and b are satisfied. Solving problems s and f, the composite control (26) is

$$u_c=[-1\ \ -0.86123]x_1+[-0.18036\ \ -0.046187]x_2 \tag{55}$$

whereas the reduced control (41) is

$$u_r=[-0.87122\ \ -0.57325]x_1. \tag{56}$$

For $\mu=0.1$, the optimal feedback solution is

$$u_{opt}=[-1\ \ -0.89202]x_1+[-0.24396\ \ -0.061996]x_2. \tag{57}$$

For the initial condition $x_0'=(1.0, 1.0)$, the values of the performance

index are

$$J_{opt} = 4.2406 \tag{58a}$$

$$J_c = 4.2428 \tag{58b}$$

$$J_r = 4.2506. \tag{58c}$$

Hence the performance loss with u_c is less than 0.052 percent and with u_r less than 0.2 percent.

VII. Conclusion

The proposed reduced and composite feedback controls are independent of μ and yet achieve, respectively, $O(\mu)$ and $O(\mu^2)$ approximations of optimal performance. The new existence Conditions a and b are mutually independent and establish the complete separation of slow and fast regulator designs. The implementation of the regulators involves solving lower order mutually independent Riccati equations. These results make the design of linear regulators for singularly perturbed systems considerably simpler than in [5],[8] and applicable to systems with small unmeasurable parameters.

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Note on Singular Perturbation of Linear State Regulators

Abstract—**This correspondence examines the relationship between the results obtained when the dimensionality of a regulator problem is reduced in the statement of the problem and when this reduction is made in the Riccati equation for the nonreduced problem.**

Problem Statement

Consider the optimal state regulator problem

$$\dot{x} = A_1 x + A_2 z + B_1 u, \qquad x(t_0) = x^0 \tag{1a}$$

$$\lambda \dot{z} = A_3 x + A_4 z + B_2 u, \qquad z(t_0) = z^0 \tag{1b}$$

$$J = \tfrac{1}{2} y'(t_f)\begin{bmatrix} \Pi_1 & \lambda\Pi_2 \\ \lambda\Pi_2' & \lambda\Pi_3 \end{bmatrix} y(t_f) + \tfrac{1}{2}\int_{t_0}^{t_f}\left\{ y'\begin{bmatrix} Q_1 & Q_2 \\ Q_2' & Q_3 \end{bmatrix} y + u' R u \right\} dt \tag{2}$$

where x, z, and u are n-, m-, and r-dimensional vectors, $y' = [x' \vdots z']$, and λ is a small positive parameter. For $\lambda > 0$ the optimal control is

$$u = -R^{-1}\left[B_1', \ \frac{1}{\lambda} B_2'\right] K y \tag{3}$$

where

$$K = \begin{bmatrix} K_1 & \lambda K_2 \\ \lambda K_2' & \lambda K_3 \end{bmatrix} \tag{4}$$

is the solution of an $(n + m) \times (n + m)$ matrix Riccati equation. Sufficient conditions are given[1] such that when $\lambda \to 0^+$, then

$$K_1 \to K_1, \qquad \forall t \in [t_0, t_f] \tag{5a}$$

$$K_2 \to K_2 \qquad K_3 \to K_3, \qquad \forall t \in [t_0, t_2] \tag{5b}$$

where $t_2 < t_f$ is arbitrarily close to t_f. The limits K_1, K_2, and K_3 are obtained from a system of one differential and two algebraic equations,

$$\dot{K}_1 = -(K_1 A_1 + A_1' K_1 + A_3' K_2' + K_2 A_3 + Q_1) + N R^{-1} N',$$

$$K_1(t_f) = \Pi_1 \tag{6a}$$

$$0 = -(K_2 A_4 + K_1 A_2 + A_3' K_3 + Q_2) + N R^{-1} B_2' K_3 \tag{6b}$$

$$0 = -(K_3 A_4 + A_4' K_3 + Q_3) + K_3 B_2 R^{-1} B_2' K_3 \tag{6c}$$

where $N = K_1 B_1 + K_2 B_2$, and K_3 is the positive definite root of (6c). Under the same conditions, when $\lambda \to 0^+$ then

$$u \to \bar{u} = -R^{-1}(N' x + B_2' K_3 z), \qquad \forall t \in [t_0, t_2]. \tag{7}$$

In the approach[1] it is essential that the limiting process $\lambda \to 0^+$ is performed in the Riccati equation for K. The purpose of this correspondence is to analyze the relationship between the limits (5) and (7), and the solution of the "reduced" problem

$$\dot{\hat{x}} = A_1 \hat{x} + A_2 \hat{z} + B_1 \hat{u}, \qquad \hat{x}(t_0) = x^0 \tag{8a}$$

$$0 = A_3 \hat{x} + A_4 \hat{z} + B_2 \hat{u} \tag{8b}$$

$$J = \tfrac{1}{2}\hat{x}'(t_f)\Pi_1 \hat{x}(t_f) + \tfrac{1}{2}\int_{t_0}^{t_f}\left\{ [\hat{x}' \vdots \hat{z}']\begin{bmatrix} Q_1 & Q_2 \\ Q_2' & Q_3 \end{bmatrix}\begin{bmatrix} \hat{x} \\ \hat{z} \end{bmatrix} + \hat{u}' R \hat{u} \right\} dt. \tag{9}$$

This problem is obtained by formally neglecting λ in the original problem (1), (2) and thus reducing dimensionality of the state space from $n + m$ to n.

Under the conditions stated[1] and if A_4^{-1} exists, the "reduced" problem (8), (9) becomes

$$\dot{\hat{x}} = \hat{A}\hat{x} + \hat{B}\hat{u}, \qquad \hat{x}(t_0) = x^0 \tag{10}$$

$$J = \tfrac{1}{2}\hat{x}'(t_f)\Pi_1 \hat{x}(t_f) + \tfrac{1}{2}\int_{t_0}^{t_f} [\hat{x}'\hat{Q}_1 \hat{x} + \hat{u}'\hat{R}\hat{u} - 2\hat{x}'\hat{Q}_2 C \hat{u}]\, dt \tag{11}$$

where

$$\hat{A} = A_1 - A_2 A_4^{-1} A_3 \qquad \hat{B} = B_1 - A_2 A_4^{-1} B_2 \qquad C = A_4^{-1} B_2$$

$$\hat{Q}_1 = Q_1 - Q_2 A_4^{-1} A_3 - (Q_2 A_4^{-1} A_3)' + (A_4^{-1} A_3)' Q_3 A_4^{-1} A_3$$

$$\hat{Q}_2 = Q_2 - (A_4^{-1} A_3)' Q_3 \qquad \hat{R} = R + C' Q_3 C. \tag{12}$$

The solution of this problem is

$$\hat{u} = -\hat{R}^{-1}[\hat{K}_1 \hat{B} - \hat{Q}_2 C]'\hat{x} \triangleq -\hat{R}^{-1}\hat{N}'\hat{x} \tag{13}$$

where $\hat{K}_1$ is obtained from the $n \times n$ Riccati equation

$$\dot{\hat{K}}_1 = -(\hat{K}_1 \hat{A} + \hat{A}'\hat{K}_1 + \hat{Q}_1) + \hat{N}\hat{R}^{-1}\hat{N}', \qquad \hat{K}_1(t_f) = \Pi_1. \tag{14}$$

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Equivalence of Riccati Gains

The following result establishes a relationship between the solutions of (14) and (6).

Lemma 1: Under the conditions stated[1] and if A_4^{-1} exists, then

$$\hat K_1 = K_1, \qquad \forall t \in [t_0, t_f]. \tag{15}$$

Proof: From (6b) we have

$$K_2 = (NR^{-1}B_2'K_3 - K_1A_2 - A_3'K_3 - Q_2)A_4^{-1}, \tag{16}$$

which when substituted in (6a) results in

$$\dot K_1 = -[K_1\hat A + \hat A'K_1 + Q_1 - Q_2A_4^{-1}A_3 - (Q_2A_4^{-1}A_3)'] + PR^{-1}P' + (A_4^{-1}A_3)'[K_3A_4 + A_4'K_3 - K_3B_2R^{-1}B_2'K_3]A_4^{-1}A_3 \tag{17}$$

where

$$P = N - (A_4^{-1}A_3)'K_3B_2 = K_1B_1 + K_2B_2 - (A_4^{-1}A_3)'K_3B_2. \tag{18}$$

[1] P. V. Kokotović and R. A. Yackel, "Singular perturbation theory of linear state regulators," in *Proc. 8th Annu. Allerton Conf. Circuit and System Theory*, Oct. 1970.

The substitution of (6c) in the last term of (17) and the substitution of (16) in (18) yield, respectively,

$$\dot K_1 = -[K_1\hat A + \hat A'K_1 + \hat Q_1] + PR^{-1}P' \tag{19}$$

$$P = K_1\hat B + NR^{-1}B_2'K_3C - Q_2C - (A_4^{-1}A_3)'(A_4'K_3 + K_3A_4)A_4^{-1}B_2. \tag{20}$$

The use of (6c) to replace the last term of (20) reduces P to the following expression

$$P = K_1\hat B - \hat Q_2C + PR^{-1}B_2'K_3C = \hat N + PR^{-1}B_2'K_3C. \tag{21}$$

which implies

$$P = \hat N(I - R^{-1}B_2'K_3C)^{-1} \triangleq \hat N M. \tag{22}$$

The expression for M can be reduced further as follows with the use of (6c) so that

$$\begin{aligned} M^{-1} &= [I - R^{-1}B_2'(K_3A_4^{-1})B_2] \\ &= I - R^{-1}B_2'A_4'^{-1}(K_3B_2R^{-1}B_2'K_3A_4^{-1} - Q_3A_4^{-1} - K_3)B_2 \\ &= R^{-1}(R + C'Q_3C) + R^{-1}C'K_3B_2(I - R^{-1}B_2'K_3C) \\ &= R^{-1}\hat R + R^{-1}C'K_3B_2M^{-1}, \end{aligned} \tag{23}$$

which yields

$$\begin{aligned} M &= \hat R^{-1}R(I - R^{-1}C'K_3B_2) \\ &= \hat R^{-1}(M')^{-1}R \end{aligned} \tag{24}$$

and finally

$$MR^{-1}M' = \hat R^{-1}. \tag{25}$$

Consequently (25), (22), and (19) yield the same equation for K_1 as (14), which proves the lemma.

Equivalence of Controls

We now establish the control $\tilde u$ defined by (7) and the control $\hat u$ defined by (13) are identical on the open interval (t_0, t_f).

Lemma 2: Under the conditions stated[1] and if A_4^{-1} exists, then

$$\tilde u = \hat u, \qquad \forall t \in [t_1, t_2] \tag{26}$$

where $t_1 > t_0$ is arbitrarily close to t_0.

Proof: Substitute $\tilde u$ for u in (1) and note that $A_4 - B_2R^{-1}B_2'K_3$ is a stable matrix for all $t \in [t_0, t_2]$. Therefore, by a theorem,[2] when $\lambda \to 0^+$ then

$$z \to -(A_4 - B_2R^{-1}B_2'K_3)^{-1}(A_3 - B_2R^{-1}N')x, \qquad \forall t \in [t_1, t_2] \tag{27}$$

and hence

$$\begin{aligned} \tilde u \to v &= -R^{-1}[N' - B_2'K_3(A_4 - B_2R^{-1}B_2'K_3)^{-1}(A_3 - B_2R^{-1}N')]x \\ &\triangleq -Tx, \qquad \forall t \in [t_1, t_2]. \end{aligned} \tag{28}$$

This expression for v is now shown to be identical to the expression (13) for $\hat u$. We have from (28) the equality chain

$$\begin{aligned} T &= R^{-1}\{N' + B_2'K_3A_4^{-1}(I - B_2R^{-1}B_2'K_3A_4^{-1})^{-1}B_2R^{-1}N' \\ &\quad - B_2'K_3A_4^{-1}(I - B_2R^{-1}B_2'K_3A_4^{-1})^{-1}A_3\} \\ &= R^{-1}\{N' + B_2'K_3CMR^{-1}N' - RMR^{-1}B_2'K_3A_4^{-1}A_3\} \\ &= \{R^{-1}(I + B_2'K_3CMR^{-1})N' - MR^{-1}B_2'K_3A_4^{-1}A_3\} \\ &= \{R^{-1}(RM^{-1} + B_2'K_3C)MR^{-1}N' - MR^{-1}B_2'K_3A_4^{-1}A_3\} \\ &= MR^{-1}(N' - B_2'K_3A_4^{-1}A_3) = MR^{-1}P' = MR^{-1}M'\hat N' = \hat R^{-1}\hat N', \end{aligned} \tag{29}$$

which is identical to (13) and proves the lemma.

[2] J. Levin and N. Levinson, "Singular perturbation of nonlinear systems of differential equations and an associated boundary layer equation," *J. Rational Mech. Anal.*, vol. 3, 1954, pp. 247–270.

Concluding Remarks

In practical design the dimensionality of the state regulator problem (1), (2) can be reduced by neglecting λ either in (1) and (2), or in the Riccati equation for $\hat K$. It is shown in this correspondence that both approaches result in the same Riccati gain K_1 and that the corresponding controls $\tilde u$ and $\hat u$ are identical on the open interval (t_0, t_f). A question that needs further clarification is under what conditions will the application of either $\tilde u$ or $\hat u$ result in the same limiting behavior of the system (1) as $\lambda \to 0^+$.

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Lower Order Control for Systems with Fast and Slow Modes*

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Key Words—Perturbation techniques; system order reduction; approximation theory; models.

Abstract—Given a stabilizing control for a linear system with fast and slow subsystems, a lower order control as a function of slow states alone is designed to give a first-order approximation of the original quadratic performance index, which is not necessarily minimized by the given stabilizing full-order control.

1. *Introduction*

SINGULAR perturbation theory for linear regulators is well known (Kokotovic, O'Malley and Sannuti, 1976; Haddad and Kokotovic, 1971; Chow and Kokotovic, 1976; Kokotovic and Haddad, 1975). However, in all cases the objective has been to design a suboptimal control to approximate some characteristic of the system due to an optimal control. This paper extends the linear regulator theory to the case of designing a control with *partial* state feedback to approximate the performance cost of a given *full* state feedback control. The given control need not be optimal as was the case in (Chow and Kokotovic, 1976) and the reduced control is based on both fast and slow subsystems which is not the case with the reduced control in (Chow and Kokotovic, 1976). Reduced order controls do not necessarily lead to well-posed formulations (Gardner and Cruz, 1978) so that the study of the asymptotic behavior of reduced controls is important.

2. *Slow and fast subsystems*

Consider a singularly perturbed linear time-invariant system

$$\dot{\mathbf{x}}_1 = A_{11}\mathbf{x}_1 + A_{12}\mathbf{x}_2 + B_1\mathbf{u}; \quad \mathbf{x}_1(0)=\mathbf{x}_{10} \tag{1a}$$

$$\mu\dot{\mathbf{x}}_2 = A_{21}\mathbf{x}_1 + A_{22}\mathbf{x}_2 + B_2\mathbf{u}; \quad \mathbf{x}_2(0)=\mathbf{x}_{20} \tag{1b}$$

where μ is a small positive scalar, $\mathbf{x}_1$ and $\mathbf{x}_2$ and n_1 and n_2 dimension vectors respectively and $\mathbf{u}$ is an m dimension vector. The matrices have appropriate dimension. It has been shown (Kokotovic and Haddad, 1975) that system equation (1) has n_1 slow eigenvalues and n_2 fast eigenvalues of order $0(1)$ and $(1/\mu)$ respectively. The usual approach to the approximate design of a control is to decompose the system equation (1) into 'fast' and 'slow' subsystems. The fast subsystem approximately models the behavior of the fast modes of equation (1) and the slow subsystem approximately models the slow modes. This decomposition is well known (Chow and Kokotovic, 1976) and it is outlined below.

We assume that we have a control as a feedback function of $\mathbf{x}_1$ and $\mathbf{x}_2$ which we will apply to equation (1). This control is

$$u = G_1\mathbf{x}_1 + G_2\mathbf{x}_2. \tag{2}$$

It is necessary that this control results in a stable system when applied to equation (1). G_1 and G_2 which result in a stable feedback system are not guaranteed to exist. However, (Chow and Kokotovic, 1976) gives conditions under which they can be designed from fast and slow subsystems performance specifications. In this paper we assume that G_1 and G_2 are given. Substituting equation (2) into equation (1), we find the slow subsystem by neglecting the fast modes of equation (1), that is letting $\mu=0$ in equation (1). This gives, assuming $A_{22}+B_2G_2$ is nonsingular,

$$\bar{\mathbf{x}}_2 = -[A_{22}+B_2G_2]^{-1}[A_{21}+B_2G_1]\bar{\mathbf{x}}_1 \tag{3}$$

where the bar indicates that $\mu=0$. If equation (3) is substituted into equation (1a) we get the slow subsystem

$$\dot{\mathbf{x}}_s = A_0\mathbf{x}_s; \qquad \mathbf{x}_s(0)=\mathbf{x}_{10} \tag{4}$$

where

$$A_0 = A_{11}+B_1G_1-[A_{12}+B_1G_2][A_{22}+B_2G_2]^{-1}[A_{21}+B_2G_1].$$

To find the fast subsystem of equation (1) we assume that the slow variables are constant during the fast transients, that is, $\dot{\bar{\mathbf{x}}}_2=0$ and $\bar{\mathbf{x}}_1=\mathbf{x}_s=$ constant. Then from equations (1b) and (3) we have

$$\mu(\dot{\mathbf{x}}_2-\dot{\bar{\mathbf{x}}}_2)=(A_{22}+B_2G_2)(\mathbf{x}_2-\bar{\mathbf{x}}_2). \tag{5}$$

Letting $\mathbf{x}_f=\mathbf{x}_2-\bar{\mathbf{x}}_2$, the fast subsystem of equation (1) is

$$\mu\dot{\mathbf{x}}_f=[A_{22}+B_2G_2]\mathbf{x}_f; \quad \mathbf{x}_f(0)=\mathbf{x}_{20}-\bar{\mathbf{x}}_2(0). \tag{6}$$

The slow and fast modes of equation (1) when equation (2) is applied are approximated by equation (4) and equation (6) respectively.

3. *Approximate control formulation*

It is desired to find a control as a function of x_1 only which yields a close approximation of the performance of that of the control as a function of x_1 and x_2 equation (2) when applied to system equation (1). To compare the performance of the two controls we select the quadratic performance index

$$J=\tfrac{1}{2}\int_0^T[\mathbf{x}'Q\mathbf{x}+\mathbf{u}'R\mathbf{u}]\,dt \tag{7}$$

where

$$Q=\begin{bmatrix}Q_1 & Q_2\\ Q_2' & Q_3\end{bmatrix}, \quad \mathbf{x}=\begin{bmatrix}\mathbf{x}_1\\ \mathbf{x}_2\end{bmatrix}, \quad Q\geqq 0,\ R>0.$$

We define the notation

$$A=\begin{bmatrix}A_{11} & A_{12}\\ \dfrac{A_{21}}{\mu} & \dfrac{A_{22}}{\mu}\end{bmatrix}, \qquad B=\begin{bmatrix}B_1\\ B_2/\mu\end{bmatrix}.$$

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Then, if the feedback control equation (2) is applied to equation (1) for performance index equation (7) a cost results which we denote by

$$J = \tfrac{1}{2}x_0' P x_0 \tag{8}$$

where P is the positive semidefinite solution of the Lyapunov equation

$$P\{A + B[G_1 \vdots G_2]\} + \{A + B[G_1 \vdots G_2]\}' P + Q + \begin{bmatrix} G_1' \\ G_2' \end{bmatrix} R [G_1 \vdots G_2] = 0 \tag{9}$$

The system and performance index: equations (1), (7) are of interest only if the control of equation (2) is stabilizing. That is, if the feedback matrix when equation (2) is applied to equation (1) is not stable, then the cost is infinite and is of no interest. Hence, we assume that equation (2) is stabilizing, in which case $A + B[G_1 \vdots G_2]$ is stable. Moreover,

$$Q = \begin{bmatrix} G_1' \\ \cdots \\ G_2' \end{bmatrix} R [G_1 \vdots G_2] \geqq 0.$$

Thus, there exists a unique positive semidefinite solution to equation (9). Note, however that since A and B contain μ, equation (9) is numerically 'stiff'.

Theorem 1. If the fast state feedback matrix $[A_{22} + B_2 G_2]$ and the slow state feedback matrix A_0 are stable, then P possesses a power series expansion at $\mu = 0$, that is

$$P(\mu) = \begin{bmatrix} P_1 & \mu P_2 \\ \mu P_2' & \mu P_3 \end{bmatrix} + \sum_{j=1}^{r} \frac{\mu^j}{j!} \begin{bmatrix} P_1^{(j)} & \mu P_2^{(j)} \\ \mu P_2'^{(j)} & \mu P_3^{(j)} \end{bmatrix}. \tag{10}$$

Proof. The proof consists of showing that each term in the series expansion of P exists and is unique. Then, clearly, there is a $\mu^* > 0$ small enough to guarantee convergence of the series for all $0 < \mu < \mu^*$.

The substitution of equation (10) into equation (9) at $\mu = 0$ yields

$$\begin{aligned} 0 = P_1[A_{11} + B_1 G_1] + P_2[A_{21} + B_2 G_1] + [A_{11} + B_1 G_1]' P_1 \\ + [A_{21} + B_2 G_1]' P_2' + Q_1 + G_1' R G_1 \end{aligned} \tag{11a}$$

$$\begin{aligned} 0 = P_1[A_{12} + B_1 G_2] + P_2[A_{22} + B_2 G_2] \\ + [A_{21} + B_2 G_1]' P_3 + Q_2 + G_1' R G_2 \end{aligned} \tag{11b}$$

$$0 = P_3[A_{22} + B_2 G_2] + [A_{22} + B_2 G_2]' P_3 + Q_3 + G_2' R G_2 \tag{11c}$$

If $[A_{22} + B_2 G_2]$ is stable equation (11c) possesses a unique positive semidefinite solution. Solving for P_2 from equation (11b) and substituting into equation (11a) gives

$$\begin{aligned} 0 = P_1 A_0 + A_0' P_1 + \begin{bmatrix} I \\ -[A_{22} + B_2 G_2]^{-1}[A_{21} + B_2 G_1] \end{bmatrix}' \\ \cdot \begin{bmatrix} Q_1 & Q_2 \\ Q_2' & Q_3 \end{bmatrix} \begin{bmatrix} I \\ -[A_{22} + B_2 G_2]^{-1}[A_{21} + B_2 G_1] \end{bmatrix} \\ + [G_1' \vdots -[A_{21} + B_2 G_1]'([A_{22} + B_2 G_2]^{-1})' G_2'] \\ \cdot R \begin{bmatrix} G_1 \\ -G_2[A_{22} + B_2 G_2]^{-1}[A_{21} + B_2 G_1] \end{bmatrix}. \end{aligned} \tag{12}$$

If A_0 is stable, equation (12) possesses a unique solution.

To find the second term in the expansion we substitute equation (10) into equation (9) and take the first partial with respect to μ at $\mu = 0$. This gives

$$\begin{aligned} 0 = P_1^{(1)}[A_{11} + B_1 G_1] + [A_{11} + B_1 G_1]' P_1^{(1)} \\ + P_2^{(1)}[A_{21} + B_2 G_1] + [A_{21} + B_2 G_1]' P_2'^{(1)} \end{aligned} \tag{13a}$$

$$\begin{aligned} 0 = P_1^{(1)}[A_{12} + B_1 G_2] + P_2^{(1)}[A_{22} + B_2 G_2] \\ + [A_{11} + B_1 G_1]' P_2 + [A_{21} + B_2 G_1]' P_3^{(1)} \end{aligned} \tag{13b}$$

$$\begin{aligned} 0 = P_3^{(1)}[A_{22} + B_2 G_2] + [A_{22} + B_2 G_2]' P_3^{(1)} \\ + [A_{12} + B_1 G_2]' P_2 + P_2'[A_{12} + B_1 G_2]. \end{aligned} \tag{13c}$$

If $[A_{22} + B_2 G_2]$ is stable equation (13c) has a unique solution.

$P_2^{(1)}$ can be found from equation (13b) and substituted into equation (13a) to give

$$P_1^{(1)} A_0 + A_0' P_1^{(1)} + \xi = 0 \tag{14}$$

where ξ is some known matrix. Thus, if A_0 is stable equation (14) possesses a unique solution. Higher order terms follow in a similar manner. Thus, Theorem 1 is proved.

We now desire to approximate the control equation (2) by a control which is a function of x_1 only. To do this we substitute equation (3) for x_2 in equation (2) to give our reduced control

$$\begin{aligned} u_r = \{G_1 - G_2[A_{22} + B_2 G_2]^{-1}[A_{21} + B_2 G_1]\} x_1 \\ = S x_1. \end{aligned} \tag{15}$$

For μ sufficiently small, the eigenvalues of the system using this reduced control are close to the eigenvalues of A_0 and A_{22} [4]. If equation (15) is applied to equation (1) for the performance index in equation (7) the resulting cost is

$$J^* = \tfrac{1}{2} x_0' \Gamma x_0 \tag{16}$$

where Γ satisfies

$$\Gamma\{A + B[S \vdots 0]\} + \{A + B[S \vdots 0]\}' \Gamma + Q + \begin{bmatrix} S' \\ \cdots \\ 0 \end{bmatrix} R [S \vdots 0] = 0. \tag{17}$$

If $A + B[S \vdots 0]$ is stable then equation (17) possesses a unique positive semidefinite solution. If A_0, A_{22}, and $A_{22} + B_2 G_2$ are stable, then for sufficiently small μ, $A + B[S \vdots 0]$ is stable.

Theorem 2. If A_{22} and A_0 are stable, then Γ possesses a power series expansion at $\mu = 0$, that is

$$\Gamma(\mu) = \begin{bmatrix} \Gamma_1 & \mu \Gamma_2 \\ \mu \Gamma_2' & \mu \Gamma_3 \end{bmatrix} + \sum_{j=1}^{r} \frac{\mu^j}{j!} \begin{bmatrix} \Gamma_1^{(j)} & \mu \Gamma_2^{(j)} \\ \mu \Gamma_2'^{(j)} & \mu \Gamma_3^{(j)} \end{bmatrix}. \tag{18}$$

Proof. The proof is similar to the proof of Theorem 1 and is omitted for brevity.

We now desire to compare the performance costs that result when the original control, equation (2) and the reduced control, equation (15) are applied to equation (1). By subtracting equation (9) from equation (17) and letting

$$W = \Gamma - P \tag{19}$$

we get a new Lyapunov equation in W given by

$$\begin{aligned} W\{A + B[G_1 \vdots G_2]\} + \{A + B[G_1 \vdots G_2]\}' W \\ + \Gamma B[S - G_1 \vdots -G_2] + [S - G_1 \vdots -G_2]' B' \Gamma + \begin{bmatrix} S' \\ \cdots \\ 0 \end{bmatrix} R [S \vdots 0] \\ - \begin{bmatrix} G_1' \\ G_2' \end{bmatrix} R [G_1 \vdots G_2] = 0. \end{aligned} \tag{20}$$

Since Γ qnd P possess power series expansions at $\mu = 0$ it is easy to show that W also possesses a power series expansion at $\mu = 0$.

Theorem 3. If A_{22}, A_0, and $[A_{22} + B_2 G_2]$ are stable, then the constant terms of the power series are Γ and P are equal, that is

$$J^* = J + 0(\mu). \tag{21}$$

Proof. Substituting the power series

$$W = \begin{bmatrix} W_1 & \mu W_2 \\ \mu W_2' & \mu W_3 \end{bmatrix} + \sum_{j=1}^{r} \frac{\mu^j}{j!} \begin{bmatrix} W_1^{(j)} & \mu W_2^{(j)} \\ \mu W_2'^{(j)} & \mu W_3^{(j)} \end{bmatrix} \tag{22}$$

into equation (20) and evaluating at $\mu = 0$ yields

$$\begin{aligned} 0 = W_1[A_{11} + B_1 G_1] + W_2[A_{21} + B_2 G_1] + [A_{11} + B_1 G_1]' W_1 \\ + [A_{21} + B_2 G_1]' W_2' + \Gamma_1 B_1 [S - G_1] + \Gamma_2 B_2 [S - G_1] \\ + [S - G_1]' B_1' \Gamma_1 + [S - G_1]' B_2' \Gamma_2' + S' R S - G_1' R G_1 \end{aligned} \tag{23}$$

$$0 = W_1[A_{12} + B_1G_2] + W_2[A_{22} + B_2G_2] + [A_{21} + B_2G_1]'W_3 - \Gamma_1B_1G_2 - \Gamma_2B_2G_2 + [S - G_1]'B_2'\Gamma_3 - G_1'RG_2 \tag{24}$$

$$0 = W_3[A_{22} + B_2G_2] + [A_{22} + B_2G_2]'W_3 - \Gamma_3B_2G_2 - G_2'B_2'\Gamma_3 - G_2RG_2. \tag{25}$$

To show that equation (21) is true we must demonstrate that $W_1 = 0$. If $[A_{22} + B_2G_2]$ is stable, we may solve for W_2 uniquely in equation (24). When this is substituted into equation (23) we get

$$\begin{aligned} 0 = {} & W_1A_0 + A_0'W_1 - [A_{21} + B_2G_1]'([A_{22} + B_2G_2]^{-1})' \\ & \times \{[A_{22} + B_2G_2]'W_3 + W_3[A_{22} + B_2G_2] \\ & - \Gamma_3'B_2G_2 - G_2'B_2'\Gamma_3\}[A_{22} + B_2G_2]^{-1}[A_{21} + B_2G_1] \\ & + G_1'RG_2[A_{22} + B_2G_2]^{-1}[A_{21} + B_2G_1] \\ & + [A_{21} + B_2G_1]'([A_{22} + B_2G_2]^{-1})'G_2'RG_1 \\ & + S'RS - G_1'RG_1. \end{aligned} \tag{26}$$

Substitution for S from equation (15) and then using equation (25) yields

$$0 = W_1A_0 + A_0'W_1. \tag{27}$$

If A_0 is stable then the unique solution of equation (27) is

$$W_1 = 0. \tag{28}$$

This implies that $W = 0(\mu)$ and so Theorem 3 is proved.

Notice that for the original control in equation (2), it is assumed that G_1 and G_2 are chosen so that $A_{22} + B_2G_2$ and A_0 are stable. In order to use the reduced control in equation (15), we further assume that A_{22} is stable.

4. *Conclusions*

We have shown that given any linear stabilizing control which is a function of fast and slow states that it is possible to design a lower order control as a function of slow states alone which gives an $0(\mu)$ approximation to the quadratic performance cost. However, it should be noted that the state trajectories will not be close except outside the usual initial boundary layer after the fast transients have died out. Knowledge of the value of the small parameter, μ, is not necessary to achieve the results derived in this paper.

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A Two-Stage Design of Linear Feedback Controls

R. G. PHILLIPS

Abstract—A two-stage design is based on an explicitly invertible block-diagonalizing transformation. Earlier singular perturbation results appear as special cases of this approach.

INTRODUCTION

In this paper we decompose the feedback design of the n-dimensional system

$$\dot{x} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} x + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u \tag{1}$$

into two reduced-order subsystems designs. In (1) the control u is an m-vector and A_{ii} are $(n_i \times n_i)$ matrices; $i = 1,2$; $n_1 + n_2 = n$. In most applications a desired effect of feedback is to move the eigenvalues farther left in the complex plane, which also increases their magnitudes. In the proposed two-stage design procedure the n_1 eigenvalues, which were larger in the open-loop system (1), also become the n_1 larger eigenvalues of the designed closed-loop system. In other words, the requirement

$$|\lambda_i| > |\lambda_j|, \; i = 1, \cdots, n_1; \qquad j = n_1 + 1, \cdots, n \tag{2}$$

is satisfied at each design stage. Similar two-stage designs can be based on any block-diagonalizing transformation. The main tool of the design proposed here is an explicitly invertible transformation. Its properties are reviewed first.Then the design procedure is presented. Finally, we show that earlier proposed decompositions such as [1]–[3], applicable to singularly perturbed systems, are special cases of this procedure when eigenvalue separation (2) is sufficiently large.

AN EXPLICITLY INVERTIBLE TRANSFORMATION

It can easily be verified that the transformation

$$T = \begin{bmatrix} I_1 - ML & -M \\ L & I_2 \end{bmatrix} \tag{3}$$

where I_i is the $n_i \times n_i$ identity, $i = 1,2$; M is $n_1 \times n_2$; and L is $n_2 \times n_1$ has the following properties.

1) For any L and M the inverse of T is

$$T^{-1} = \begin{bmatrix} I_1 & M \\ -L & I_2 - LM \end{bmatrix}. \tag{4}$$

2) If L satisfies

$$(A_{21} + LA_{11} - A_{22}L - LA_{12}L = 0 \tag{5}$$

and M satisfies

$$A_{11} - A_{12}LM - M(A_{22} + LA_{12}) + A_{12} = 0, \tag{6}$$

then $y = Tx$ transforms (1) into

$$\dot{y} = \begin{bmatrix} F_1 & 0 \\ 0 & F_2 \end{bmatrix} y + \begin{bmatrix} G_1 \\ G_2 \end{bmatrix} u \tag{7}$$

where

$$F_1 = A_{11} - A_{12}L, \; F_2 = A_{22} + LA_{12}$$
$$G_1 = (I_1 - ML)B_1 - MB_2, \; G_2 = LB_1 + B_2. \tag{8}$$

Note that $|\lambda(F_1)|>|\lambda(F_2)|$ as required by (2) assures that M is uniquely determined by (6). It is also useful to note that $L=0$ when $A_{21}=0$ and $M=0$ when $A_{12}=0$. Rapidly convergent iterative methods are available [4] for computing L and M. Note also that, in general, a permutation of states in (1) is necessary to isolate fast and slow states. An algorithm to accomplish this is given in [4].

The Two-Stage Design

For clarity we assume that complete controllability of (1), which implies the same for the subsystems in (7), that is, for the pairs (F_1, G_1) and (F_2, G_2), and consider the pole placement problem for (1) using a state feedback control $u=Kx$.

In the first stage we design an $m\times n_1$ feedback matrix H_1 to place the eigenvalues of $F_1+G_1H_1$ at the desired n_1 locations. The substitution of

$$u=u_1+u_2=[H_1 0]y+u_2 \tag{10}$$

into (3) yields

$$\dot{y}=\begin{bmatrix} F_1+G_1H_1 & 0 \\ G_2H_1 & F_2 \end{bmatrix}y+\begin{bmatrix} G_1 \\ G_2 \end{bmatrix}u_2. \tag{11}$$

Now we apply to (11) the transformation (3) and denote it by $\hat{T}$. In (11) the block "1,2" is zero; hence, $\hat{M}=0$ and only

$$\hat{L}(F_1+G_1H_1)-F_2\hat{L}+G_2H_1=0$$

needs to be solved for $\hat{L}$. The transformation $z=\hat{T}y$ yields

$$\dot{z}=\begin{bmatrix} F_1+G_1H_1 & 0 \\ 0 & F_2 \end{bmatrix}z+\begin{bmatrix} G_1 \\ G_1+\hat{L}G_1 \end{bmatrix}u_2. \tag{12}$$

Again we note that the pair $(F_2, G_2+\hat{L}G_1)$ is completely controllable and we proceed to the second stage in which we design an $m\times n_2$ feedback matrix H_2 to place the eigenvalues of $F_2+(G_2+\hat{L}G_1)H_2$ at the desired n_2 locations. Substituting the feedback control

$$u_2=[0 \quad H_2]z \tag{13}$$

into (10) results in

$$u=([H_1 \quad 0]T+[0 \quad H_2]\hat{T}T)x. \tag{14}$$

Expressing T and $\hat{T}$ according to (4) we obtain the final form of the feedback matrix K in $u=Kx$:

$$K=\left[(H_1+H_2(L-\hat{L}))(I-ML)\vdots -H_1M+H_2(I+\hat{L}M)\right]. \tag{15}$$

Although this two-stage design is presented in terms of a full state feedback, it also can be used in output feedback design.

Approximate Design

The form (14) of the final result is valid for any transformations T and $\hat{T}$ block diagonalizing (1) and (11), respectively. However, the usefulness of this approach depends on the computational effort needed to obtain these transformations and their inverses. The advantages of (3) are, first, its explicit inverse (4) and, second, the recursive determination of L and M [5]. First

$$L_{k+1}=(A_{22}L_k+L_kA_{12}L_k-A_{21})A_{11}^{-1} \tag{16}$$

$$L_0=-A_{21}A_{11}^{-1}$$

and then M from

$$M_{k+1}=A_{11}^{-1}M_k(A_{22}+LA_{12})+A_{11}^{-1}A_{12}LM_k-A_{11}^{-1}A_{12} \tag{17}$$

$$M_0=-A_{11}^{-1}A_{12}.$$

A similar recursion is used for $\hat{L}$. Convergence rate of these iterations is known [4] to be of order ϵ^k where ϵ is the "separation ratio"

$$\epsilon=\frac{\sup|\lambda(F_2)|}{\inf|\lambda(F_1)|}. \tag{18}$$

When this ratio is small, as in singularly perturbed systems, then a few iterations are typically sufficient.

This property is illustrated by a power system control problem [6] whose system matrices are

$$A=\begin{bmatrix} -5 & 0 & 0 & 0 & 4.75 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 & -2 & 0 \\ -0.08 & -0.11 & -3.99 & -0.93 & 0 & -0.07 & 10 \\ 0 & 0 & 1.32 & -1.39 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.2 & 0 & 0 \\ 0.17 & 0 & 0 & 0 & 0 & -0.17 & 0 \\ 0 & 0 & 0.2 & 0 & 0 & 0 & -0.5 \\ 0.01 & 0.01 & -0.06 & 0.12 & 0 & 0.01 & 0 \end{bmatrix} \tag{19}$$

$$B^T=\begin{bmatrix} 0 & 0 & 10 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 \end{bmatrix} \tag{20}$$

with open-loop eigenvalues of

$$\lambda_1=-4.35 \qquad \lambda_{5,6}=-0.13\pm .21j$$
$$\lambda_2=-5.00 \qquad \lambda_7=-0.17$$
$$\lambda_3=-2.00 \qquad \lambda_8=-0.2$$
$$\lambda_4=-1.39$$

The minimum open-loop separation ratio is about 1:5 for $|\lambda_5|:|\lambda_4|$, that is, with $n_1=n_2=4$. After two iterations L is approximated by

$$L_1=\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0.034 & 0 & 0 & 0 \\ -0.001 & -0.001 & 0.048 & -0.001 \\ 0.002 & 0.007 & 0.014 & 0.092 \end{bmatrix} \tag{21}$$

and M by

$$M.=\begin{bmatrix} 0.989 & 0 & 0 & 0 \\ -0.137 & 1.091 & 0 & 0 \\ -0.006 & -0.012 & 2.359 & -1.966 \\ 0.006 & 0.032 & 2.48 & -1.625 \end{bmatrix} \tag{22}$$

resulting in

$$F_1=\begin{bmatrix} 5 & 0 & 0 & 0 \\ 0.068 & -2 & 0 & 0 \\ 0.056 & -0.033 & -4.355 & -0.011 \\ 0 & 0.002 & 1.323 & -1.363 \end{bmatrix} \tag{23}$$

$$G_1^T=\begin{bmatrix} -3.958 & 0.548 & 0.025 & -0.024 \\ 0 & 0 & 9.123 & -0.981 \end{bmatrix}. \tag{24}$$

For desired eigenvalues

$$\lambda_{1,2}(\mathrm{des})=-8\pm 2j$$
$$\lambda_3(\mathrm{des})=-6$$
$$\lambda_4(\mathrm{des})=-4.$$

A feedback gain H_1 is obtained as

$$H_1=\begin{bmatrix} -0.615 & -20.865 & 0 & 0 \\ 0.009 & -0.016 & -0.777 & -1.843 \end{bmatrix}. \tag{25}$$

To design a feedback H_2 for the pair $(F_2, G_2+\hat{L}G_1)$, we first find $\hat{L}_1$

$$\hat{L}_1=\begin{bmatrix} -0.020 & 6.178 & 0 & 0 \\ 0.007 & -0.033 & 0 & 0 \\ -0.001 & -0.00\text{[illegible]} & \text{[illegible]}057 & 0.132 \\ 0 & -0.002 & 0.015 & 0.035 \end{bmatrix} \tag{26}$$

resulting in

$$F_2=\begin{bmatrix} -0.2 & 0 & 0 & 0 \\ 0.164 & -0.167 & 0 & 0 \\ -0.003 & -0.006 & -0.015 & -0.439 \\ 0.008 & 0.023 & 0.137 & -0.262 \end{bmatrix} \tag{27}$$

$$\left[G_2-\hat{L}G_1\right]^T=\begin{bmatrix} 0.533 & 0.045 & -0.001 & 0.002 \\ 0 & 0 & 0.088 & 0.031 \end{bmatrix}. \tag{28}$$

For desired eigenvalues of

$$\lambda_{5,6}(\text{des}) = -2 \pm 1j$$
$$\lambda_7(\text{des}) = -1$$
$$\lambda_8(\text{des}) = -0.5$$

a feedback gain H_2 is obtained:

$$H_2 = \begin{bmatrix} -2.615 & -49.15 & 0 & 0 \\ -0.234 & -1.142 & -2.266 & -39.916 \end{bmatrix}. \tag{29}$$

The actual closed-loop eigenvalues using the composite feedback (14) are

$$\lambda_{1,2}(\text{CL}) = -7.99 \pm 1.99j \qquad \lambda_{5,6}(\text{CL}) = -2 \pm 1j$$
$$\lambda_3(\text{CL}) = -6.09 \qquad \lambda_7(\text{CL}) = -1.03$$
$$\lambda_4(\text{CL}) = -3.93 \qquad \lambda_8(\text{CL}) = -0.5$$

and are close to their desired locations, the worst error being

$$\frac{(\lambda_7(\text{CL}) - \lambda_7(\text{des}))}{\lambda_7(\text{des})} = 3 \text{ percent}.$$

Applications to Singular Perturbations

The approach presented here offers a simpler way to obtain some known results for singularly perturbed systems [1], [2].

Consider the system

$$\epsilon \dot{x} = \begin{bmatrix} A_{11} & A_{12} \\ \epsilon A_{21} & \epsilon A_{22} \end{bmatrix} x + \begin{bmatrix} B_1 \\ \epsilon B_2 \end{bmatrix}. \tag{30}$$

If in this case L and M are approximated by

$$L = -\epsilon A_{21} A_{11}^{-1} \tag{31}$$

$$M = -A_{11}^{-1} A_{12} \tag{32}$$

then the transformed system

$$\epsilon \dot{y} = \begin{bmatrix} A_{11} + \epsilon A_{12} A_{21} A_{11}^{-1} & -\epsilon A_{12} A_{21} A_{11}^{-2} A_{12} \\ \epsilon^2 A_0 A_{21} A_{11}^{-1} & \epsilon A_0 - \epsilon^2 A_0 A_{21} A_{11}^{-2} A_{12} \end{bmatrix} y + \begin{bmatrix} B_1 + \epsilon A_{11}^{-1} A_{12} B_0 \\ \epsilon B_0 \end{bmatrix} u \tag{33}$$

is $O(\epsilon)$ close to the block diagonal form (7) where

$$A_0 = A_{22} - A_{21} A_{11}^{-1} A_{12} \quad B_0 = B_2 - A_{21} A_{11}^{-1} B_1.$$

H_1 can now be designed to place the eigenvalues of $A_{11} + B_1 H_1$ at λ_{des} locations. The resulting closed-loop eigenvalues will be of the form $(\lambda_{(\text{des})} + O(\epsilon))/\epsilon$. The partially closed-loop system of (33) is now given as

$$\epsilon \dot{y} = \begin{bmatrix} A_{11} + B_1 H_1 + \epsilon E_{11} & -\epsilon A_{12} A_{21} A_{11}^{-2} A_{12} \\ \epsilon B_0 H_1 + \epsilon^2 A_0 A_{21} A_{11}^{-1} & \epsilon A_0 - \epsilon^2 A_0 A_{21} A_{11}^{-2} A_{12} \end{bmatrix} y + \begin{bmatrix} B_1 + \epsilon A_{11}^{-1} A_{12} B_0 \\ \epsilon B_0 \end{bmatrix} u_2 \tag{34}$$

where

$$E_{11} = A_{12} A_{21} A_{11}^{-1} + A_{11}^{-1} A_{12} B_0 H_1.$$

Next, approximating $\hat{L}$ in by

$$\hat{L} = \epsilon B_0 H_1 (A_{11} + B_1 H_1)^{-1} + O(\epsilon^2) \tag{35}$$

and using $y = \hat{T} z$, (34) becomes

$$\epsilon \dot{z} = \begin{bmatrix} A_{11} + B_1 H_1 + \epsilon E_{11} + O(\epsilon^2) & \epsilon E_{12} \\ O(\epsilon^2) & \epsilon A_0 + O(\epsilon^2) \end{bmatrix} z + \begin{bmatrix} B_1 + \epsilon A_{11}^{-1} A_{12} B_0 \\ \epsilon B_0 - \hat{L} B_1 + O(\epsilon^2) \end{bmatrix} u_2 \tag{36}$$

where

$$E_{12} = -A_{12} A_{21} A_{11}^{-2} A_{12}.$$

The feedback H_2 is designed for the pair $(A_0, B_0 - \hat{L}(B_1/\epsilon))$. The resulting closed-loop eigenvalues corresponding to this subsystem will be of the form $\lambda_{(\text{des})} + O(\epsilon)$. Substitution of (31), (32), and (35) into (15) gives the composite feedback matrix

$$K = \left[H_1 \vdots H_1 A_{11}^{-1} A_{12} + H_2 \right] + O(\epsilon). \tag{37}$$

This parallels the results of [1] where H_1 and H_2 were the results of two reduced-order linear optimal regulator problems.

In [2] and $O(\epsilon)$ independence of the two stages was accomplished using a particular form of the control u_2 in the design of H_2, namely,

$$u_2 = (I + H_1 A_{11}^{-1} B_1) v \tag{38}$$

resulting in the control matrix of (36):

$$\begin{bmatrix} (B_1 + \epsilon A_{11}^{-1} A_{12} B_0)(I + H_1 A_{11}^{-1} B_1) \\ \epsilon B_0 (I - H_1 (A_{11} + B_1 H_1)^{-1} B_1)(I + H_1 A_{11}^{-1} B_1) + O(\epsilon^2) \end{bmatrix} v. \tag{39}$$

Using a known matrix identity, it can be shown that the matrix multiplying ϵB_0 in (39) is I, and hence the subsystem pair for the second stage is $(A_0 + O(\epsilon),\ B_0 + O(\epsilon))$. Since it is independent of H_1 up to $O(\epsilon)$ terms, the need for taking H_1 into account in the second stage has been eliminated. The composite feedback matrix of (15) now becomes

$$K = \left[H_1 \vdots H_1 A_{11}^{-1} A_{12} + H_2 (I + H_1 A_{11}^{-1} B_1) \right] \tag{40}$$

which parallels the results of [2] and [3].

Conclusions

An explicitly invertible transformation enables a reduced-order eigenvalue placement problem to be solved in two stages. The only requirement on the open-loop system is that the spectrum be disjoint. Earlier decompositions applicable to singularly perturbed systems appear as special cases of this two-stage procedure when the eigenvalue separation was sufficiently large. A similar design procedure for discrete-time systems is developed in [7]. Finally, the decomposition is applicable to design criteria other than eigenvalue locations and output feedback problems.

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A Singular Perturbation Analysis of High-Gain Feedback Systems

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Abstract—In this paper a singular perturbation approach is used to unify a class of classical and recent results on high-gain systems and to show their relationships with multivariable transmission zero analysis, cheap control problems, and sliding mode in variable structure systems. A new pole placement method and a decomposition of near optimal high-gain regulator problems are presented.

Introduction

High-gain feedback has been a classical tool for reduction of effects of disturbances, parameter variations, and distortions. Although limited to single input–single output feedback systems, the early investigations of structures permitting high gains [1], the rules for root locus asymptotes [2] and the results on sensitivity and return difference [3], [4] had greatly deepened the intuition of control engineers in the 1950's. Recent developments in the multivariable system theory have revived the interest in high-gain systems. First, in the works on disturbance rejection [5], parameter uncertainty [6], and decoupling of large scale systems [7], high-gain coefficients are either purposely introduced in the problem statement or they implicitly appear in the resulting feedback structures. Second, in a class of so called variable structure systems [8], [9], the sliding mode which occurs on switching surfaces can be analyzed using high-gain methodologies. Third, feedback implementations of optimal controls when only small penalities are made on control variables (the "cheap" control problem) result in loops with high gain [10], [11]. Fourth, various recent studies of multivariable system transmission zeros [12]–[14] and root locus asymptotes [15] also exploit a high-gain analogy.

A fundamental property of high-gain systems, which brings us to the subject of this paper, is their relationship with singularly perturbed systems. It has been observed in [16] that a dynamic loop whose gain tends to infinity causes change in the system order, characteristic of singular perturbation. In fact, all singularly perturbed systems can be represented as high-gain systems and all high-gain systems can be analyzed as singularly perturbed systems. The class of linear time-invariant high-gain systems dealt with in this paper and in [13], [14] is of the form

$$\dot{x}_0 = A_0 x_0 + B_0 u \tag{1}$$

$$u = g C_0 x_0 \tag{2}$$

where g is the large scalar gain factor, the state x is an n-vector, and the control u is an m-vector. The relationship between the system (1), (2) and a standard singularly perturbed linear time-invariant system,

$$\dot{z} = F_{11} z + F_{12} y \tag{3}$$

$$\mu \dot{y} = F_{21} z + F_{22} y \tag{4}$$

where μ is a small scalar parameter will be established by considering that

$$\mu = \frac{1}{g}, \qquad g \to \infty, \quad \mu \to 0. \tag{5}$$

In this paper we demonstrate how the singular perturbation methodology can unify and simplify known facts about high-gain systems and give new interpretations of their properties. A new pole placement method and a decomposition of near optimal high-gain regulator problems are presented. Furthermore, connections among apparently unrelated notions such as transmission zeros, cheap controls, and sliding modes in variable structure systems are clarified.

Fast and Slow Modes and Transmission Zeros

In this section we reveal the mode separation property of high-gain systems and relate it to the notion of transmission zeros. The feedback system (1), (2) with $\mu \cong 1/g$ becomes

$$\mu \dot{x}_0 = (\mu A_0 + B_0 C_0) x_0. \tag{6}$$

It will now be shown that if

$$\operatorname{rank} B_0 C_0 = \operatorname{rank} C_0 B_0 = m, \tag{7}$$

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and if the nonzero eigenvalues of B_0C_0 have negative real parts

$$\mathrm{Re}\lambda_i(B_0C_0)<0, \qquad i=1,\cdots,m, \tag{8}$$

then the motion of (6) consists of a fast transient to a $0(\mu)$ neighborhood of $C_0x_0=0$, followed by a slow motion in this neighborhood. When assumption (7) is not satisfied, the limiting phenomena as $g\to\infty$, that is, $\mu\to 0$, are more involved and will not be analyzed here. Assumption (8) assures the asymptotic stability of the fast transient. To convert the system (6) into the standard singularly perturbed form (3), (4) let $\tilde{x}=Tx_0$ where

$$T=\left[\begin{array}{c|c} I_{n-m} & 0 \\ \hline C_1 & C_2 \end{array}\right]M\equiv\Gamma M \tag{9}$$

and M is a product of elementary row transformations on B_0 such that

$$MB_0=\left[\begin{array}{c} 0 \\ \hline B_2 \end{array}\right], \quad C_0M^{-1}=[C_1 \quad C_2]. \tag{10}$$

We note that $m\times m$ matrices B_2 and C_2 are nonsingular because of $C_0B_0=C_2B_2$ and assumption (7). It is shown in Appendix A that the meaning of T is that it decomposes the original state space of x_0 into the null space of C_0, $\mathfrak{N}(C_0)$, and the range space of B_0, $\mathfrak{R}(B_0)$. Then x_0 can be written as

$$x_0=Nz+B_0(C_0B_0)^{-1}y \tag{11}$$

where $C_0N=0$. Also

$$y=C_0x_0 \tag{12}$$

and $z=M_1x_0$ where $M_1B_0=0$. Such decompositions also appear in MacFarlane, Kouvaritakis, and Shaked [13]–[15]. The system (6) is thus transformed into

$$\dot{z}=F_{11}z+F_{12}y \tag{13}$$

$$\mu\dot{y}=\mu H_1z+(C_0B_0+\mu H_2)y. \tag{14}$$

The block triangularization [17], [18], which is simpler than Jordan transformations used in [13]–[15], is now applied to (13), (14) to exhibit the two time scale property of high-gain feedback systems. We introduce the "fast" variable

$$\eta=y+\mu Lz \tag{15}$$

where

$$L=(C_0B_0+\mu H_2)^{-1}H_1+\mu G=L_0+\mu G. \tag{16}$$

A recursive formula for calculating G and an upper bound μ_c for μ (that is, a lower bound g_c for g) are given in Appendix B. The resulting separation of slow and fast modes is now summarized in the following theorem.

Theorem 1: If the high-gain system (6) satisfies assumptions (7) and (8) and

$$0<\mu<\mu_c, \tag{17}$$

then (6) is equivalent to the block triangular system

$$\dot{z}=[F_{11}-\mu F_{12}(L_0+\mu G)]z+F_{12}\eta \tag{18}$$

$$\mu\dot{\eta}=[C_0B_0+\mu H_2+\mu^2(L_0+\mu G)F_{12}]\eta \tag{19}$$

and, hence, its eigenvalues are

$$\lambda_i^f=\frac{1}{\mu}[\lambda_i(C_0B_0)+0(\mu)] \qquad i=1,\cdots,m \tag{20}$$

$$\lambda_j^s=\lambda_j(F_{11})+0(\mu) \qquad j=1,\cdots,n-m. \tag{21}$$

Moreover, for sufficiently high gain, that is for μ sufficiently small, the fast subsystem (19) is asymptotically stable.

The proof follows from (10)–(16) and the existence of G established in Appendix B. The asymptotic stability of (19) follows from assumption (8) and from the fact that the nonzero eigenvalues of B_0C_0 are the eigenvalues of C_0B_0,

$$\lambda_i(C_0B_0)=\lambda_i(B_0C_0), \qquad \lambda_i\neq 0, \quad i=1,\cdots,m. \tag{22}$$

The two time scale property of high-gain system (6) is exemplified by the presence of $0(1/\mu)$ large eigenvalues λ_i^f and the eigenvalues λ_j^s which are $0(1)$. The fast variable η decays exponentially in the "stretched" time scale t/μ and for $t>t_s$ it is $0(e^{-\bar{\lambda}t_s/\mu})$ where $\bar{\lambda}=|\max \mathrm{Re}\lambda_i(C_0B_0)|$. Integrating (18) by parts, the slow variable can be approximated by

$$z(t)=e^{F_{11}t}z(0)+0(\mu). \tag{23}$$

Hence, the decomposition (11) of the original state space corresponds to the separation of time scales and $x_0(t)$ is approximated by

$$x_0(t)=Ne^{F_{11}t}z(0)+B_0(C_0B_0)^{-1}e^{C_0B_0t/\mu}\eta(0)+0(\mu). \tag{24}$$

which proves that, in the limit, the fast transient occurs in $\mathfrak{R}(B_0)$ and the slow motion is confined to $\mathfrak{N}(C_0)$. For $t>t_s$,

$$y(t)=\mu L_0z(t)+0(\mu^2). \tag{25}$$

If, in an output regulator problem, $y=C_0x_0$ is defined as the output of (1) to be forced to zero ("zeroing the output" [24]), then from (25), we see that $y(t)$ is reduced to an $0(\mu^2)$ quantity after $t>t_s$.

Another interpretation of Theorem 1 is that it encompasses and extends results on root locus asymptotes and transmission zeroes of multivariable systems [12]–[15]. As $g\to\infty$, $\mu\to 0$, the eigenvalues λ_i^f in (20) tend to infinity along the asymptotes defined by the directions of $\lambda_i(C_0B_0)$, which are the root locus asymptotes obtained in [15]. For large but finite values of gain g, the eigenvalues λ_i^f can now be computed from (19), or more simply approximated by neglecting the $0(\mu^2)$ terms in (19).

Wang and Davison [12] and Kouvaritakis and MacFarlane [13] have shown that the finite limits of the

eigenvalues of the high-gain feedback system (6) as gain $g \to \infty$ are the transmission zeros of the open loop system (1) with the output $y = C_0 x_0$. Therefore, the limits $\lambda_j(F_{11})$ of the eigenvalues λ_j^s are the transmission zeros. Theorem 1 and Appendix A provide the following procedure for calculation of transmission zeros. Find the matrix $M^T = [M_1^T \quad M_2^T]$ in (10) and its inverse $M^{-1} = [S_1 \quad S_2]$, where M_2^T and S_2 are $n \times m$ matrices. This is easy since M is a product of elementary row transformations. Then form $C_0 M^{-1} = [C_1 \quad C_2]$ as in (10), find C_2^{-1}, and compute the eigenvalues of

$$F_{11} = M_1 A_0 (S_1 - S_2 C_2^{-1} C_1) \tag{26}$$

which are the transmission zeros of (1), (12). The derivation of (26) is given in Appendix A where it is shown that the last matrix in (26) and the matrix N in (11) have the same range:

$$\mathcal{R}(N) = \mathcal{R}(S_1 - S_2 C_2^{-1} C_1). \tag{27}$$

Therefore, our matrix F_{11} coincides with the "zero-matrix" in [6], [13]–[15] for the specific choice of the basis for $\mathcal{N}(C_0)$. The calculation of transmission zeros from matrices of the form (26), such as in [13]–[15], [19], avoid the ill-conditioning due to the large gain factor present in [12], [20].

Pole Placement

For the remainder of the text the matrix C_0 in (2) is considered free to be chosen in the state feedback design. To stress this fact this state feedback matrix is denoted by K. We first develop a design method in which the two time scale property of high-gain systems is exploited to separately place fast and slow eigenvalues. To simplify the notation we will deal with the system

$$\dot{x}_1 = A_{11} x_1 + A_{12} x_2 \tag{28}$$

$$\dot{x}_2 = A_{21} x_1 + A_{22} x_2 + B_2 u \tag{29}$$

which is obtained from (1) using $x = [x_1 \quad x_2]^T = M x_0$ where M is as in (10). The $m \times m$ matrix B_2 is nonsingular by assumption (7) and, hence, the pair (A_{22}, B_2) is controllable for any A_{22}.

Lemma 1: If the pa·· (A_0, B_0) in (1) is controllable (stabilizable), then the pair (A_{11}, A_{12}) is controllable (stabilizable).

Proof: The controllability of (A_0, B_0) implies

$$\operatorname{rank}\begin{bmatrix} \sigma I - A_{11} & A_{12} & 0 \\ A_{21} & \sigma I - A_{22} & B_2 \end{bmatrix} = n \tag{30}$$

and, since rank $B_2 = m$,

$$\operatorname{rank}\left[\sigma I - A_{11} \quad A_{12}\right] = n - m \tag{31}$$

for all complex σ, that is, (A_{11}, A_{12}) is controllable. When (A_0, B_0) is only stabilizable, consider $u = v - B_2^{-1} A_{21} x_1$. Since (A_{22}, B_2) is controllable, rank$\left[\sigma I - A_{11} \quad A_{12}\right] < n - m$ and all the uncontrollable but stable eigenvalues of A_0 are eigenvalues of A_{11}. Hence, (A_{11}, A_{12}) is stabilizable.

Theorem 2: Let (A_0, B_0) be a controllable pair, and let the $(n-m) \times m$ matrix K_s and the $m \times m$ matrix K_f be such that

$$\lambda_j(A_{11} - A_{12} K_s) = q_j \qquad j = 1, \cdots, n-m \tag{32}$$

$$\lambda_i(B_2 K_f) = p_i, \qquad \operatorname{Re} p_i < 0, \quad i = 1, \cdots, m \tag{33}$$

where q_j and gp_i are the prescribed locations of the slow and fast eigenvalues, respectively. Then the use of the high-gain feedback

$$u = gKx = g\left[K_f K_s x_1 + K_f x_2\right] \tag{34}$$

places the eigenvalues of the system (28), (29) to $q_j + O(1/g)$ and $g[p_i + O(1/g)]$.

Proof: Substitution of (34) into (28), (29), and the transformation $z = x_1$, $y = K_f K_s x_1 + K_f x_2$ yield

$$\dot{z} = (A_{11} - A_{12} K_s) z + A_{12} K_f^{-1} y \tag{35}$$

$$\mu \dot{y} = \mu K_f [K_s A_{11} - A_{22} K_s - K_s A_{12} K_s + A_{12}] z + K_f B_2 \left[I + \mu B_2^{-1}(K_s + A_{22} K_f^{-1})\right] y. \tag{36}$$

Noting that $\lambda(K_f B_2) = \lambda(B_2 K_f)$, the proof follows from Lemma 1 and Theorem 1.

A procedure for completely separated placement of the slow and fast modes is to design K_s to place the eigenvalues of $A_{11} - A_{12} K_s$ and design K_f to place the eigenvalues of $B_2 K_f$. The n-dimensional eigenvalue placement problem is thus decomposed into lower dimensional problems. The preceding design is an improvement over the technique in [21]. The fast eigenvalues can now be placed arbitrarily in the open left half complex plane while in [21] they are restricted to lie on the negative real axis. The placement of slow eigenvalues is performed by solving a standard pole placement problem of lower dimension while in [21], it is done by solving $n-m$ linear equations $\det[\tilde{K}(q_j I - A_0)^{-1} B_0] = 0$, $j = 1, \cdots, n-m$, for the $m \times n$ elements of $\tilde{K}$.

High-Gain Regulators and Cheap Controls

High-gain feedback systems can also result from the optimization of system (1) with respect to a quadratic performance index having small penalty on u:

$$J = \frac{1}{2}\int_0^\infty \left[x_0^T Q_0 x_0 + \mu^2 u^T R u\right] dt \tag{37}$$

where $Q_0 \geq 0$ and $R > 0$ are symmetric and μ is a small scalar parameter. Such "cheap control" problems have been studied by O'Malley and Jameson [10], [22], Kwakernaak and Sivan [11], [23], Wonham [24], and others. Detailed results exist [10] for the case when

$$B_0^T Q_0 B_0 > 0. \tag{38}$$

Under this condition the resulting high-gain system satisfies our assumption (7). The analysis of the preceding sections is applicable and reveals the two time scale properties of the optimal state regulators developed in [10]. Without loss of generality, we deal again with the system (28), (29) instead of with (1).

From the fast–slow separation of the po cement design, it can be expected that a similar decomposition is feasible in the near optimum high-gain regulator design. This is done by the method of [25]. Instead of the problem (1), (37), the following two reduced order regulator problems are solved.

Problem "s": Optimize system (39) with respect to the performance index (40).

$$\dot{x}_s = A_{11}x_s + A_{12}u_s \tag{39}$$

$$J_s = \frac{1}{2}\int_0^\infty \left(x_s^TQ_{11}x_s + 2x_s^TQ_{12}u_s + u_s^TQ_{22}u_s\right)dt. \tag{40}$$

Problem "f": Optimize system (41) with respect to the performance index (42).

$$\dot{x}_f = B_2u_f \tag{41}$$

$$J_f = \frac{1}{2}\int_0^\infty \left(x_f^TQ_{22}x_f + u_f^TRu_f\right)dt. \tag{42}$$

In (39), (41), matrices A_{11}, A_{12}, and B_2 are as in (28), (29), and x_s is an $(n-m)$-vector, x_f, u_f, u_s are m-vectors, and Q_{ij} are the submatrices of $Q=(M^{-1})^TQ_0M^{-1}$ where M is as in (10).

Lemma 2: If the pair (A_0, B_0) is stabilizable and

$$(A_{11}, D) \quad \text{is detectable,} \tag{43}$$

where

$$D^TD = Q_{11} - Q_{12}Q_{22}^{-1}Q_{12}^T, \tag{44}$$

and if (38) is satisfied, then there exists a unique stabilizing solution P_s of the Riccati equation

$$0 = -P_s\left(A_{11} - A_{12}Q_{22}^{-1}Q_{12}^T\right) - \left(A_{11} - A_{12}Q_{22}^{-1}Q_{12}^T\right)^TP_s + P_sA_{12}Q_{22}^{-1}A_{12}^TP_s - D^TD \tag{45}$$

and the optimal control for Problem "s" is

$$u_s = -Q_{22}^{-1}\left(Q_{12}^T + A_{12}^TP_s\right)x_s \equiv G_sx_s. \tag{46}$$

Proof: Q_{22}^{-1} exists since $B_0^TQ_0B_0>0$ implies $B_2^TQ_{22}B_2>0$. By Lemma 1, the pair (A_{11}, A_{12}) is stabilizable and, hence, $(A_{11} - A_{12}Q_{22}^{-1}Q_{12}^T, A_{12})$ is stabilizable. This guarantees the existence and uniqueness of P_s [26].

The optimal control for Problem "f" is readily obtained as

$$u_f = -R^{-1}B_2P_fx_f \equiv G_fx_f \tag{47}$$

where, in terms of $S = B_2R^{-1}B_2^T$,

$$P_f = S^{-1/2}\left(S^{1/2}Q_{22}S^{1/2}\right)^{1/2}S^{-1/2} > 0. \tag{48}$$

A composite control u_c is now made of the controls u_s and u_f in the form of the control law (34):

$$u_c = g\left(G_fG_sx_1 + G_fx_2\right), \tag{49}$$

where the feedback gain matrices are operating on the actual state variables x_1 and x_2 of the system (28), (29) rather than on x_s and x_f. The feedback system (28), (29), (49) satisfies Theorem 2 with G_s and G_f playing the roles of K_s and K_f, respectively. Hence, J_s and J_f are the performance specifications for the slow and fast modes.

Theorem 3: Under the conditions of Lemma 2, the control u_c is near optimal in the sense that the performance J of the feedback system (28), (29), (49) is $O(\mu^2)$ close to its optimum performance.

Proof: A power series expansion of the optimal Riccati solution has been derived in [10]. Since our equations (45) and (48) coincide with (4.11) and (4.9) of [10], the leading terms K_{110} and K_{220} of the optimal expansion are identical to our P_s and P_f, respectively. From the identity, analogous to (4.10) in [10],

$$P_sA_{12} + Q_{12} = K_{120}SP_f \tag{50}$$

it follows that u_c and x of the system (28), (29), (49) are $O(\mu)$ close to the control and the state of the optimal system, and in view of [25] the performance of (1), (49) is $O(\mu^2)$ near optimal.

The results of cheap control [10] are thus recast in te of two separate problems providing a $O(\mu^2)$ near optimal solution. It is observed from Lemma 2 that the standard detectability assumption on system (1) is replaced by detectability assumptions (43) and (38) for the slow and fast subsystems. The stabilizability assumption, however, remains the same. Thus the meaning of Hypothesis (H) in [10] is to assume that the slow eigenvalues have negative real parts.

To regulate the output $y=C_0x_0$ we use $Q=C_0^TC_0$. If $\operatorname{rank} C_0 = m$, then a $O(\mu^2)$ near optimal control can be found by the preceding decomposition procedure, introducing $y_f = C_2x_f$ and $y_s = C_1x_s$ as fast and slow output variables in (42) and (40), respectively, where C_1, C_2 are as in (10). Since $Q_{11}=C_1^TC_1$, $Q_{22}=C_2^TC_2$ and C_2 is nonsingular, then $D=0$ in (44) and (43) is replaced by

$$\operatorname{Re}\lambda\left(A_{11} - A_{12}C_2^{-1}C_1\right) < 0. \tag{51}$$

The solution of (45) is $P_s=0$, that is, the slow eigenvalues are not influenced by high-gain feedback. From Theorem 1 it follows that $\lambda(A_{11} - A_{12}C_2^{-1}C_1)$ are the transmission zeros of the system (28), (29) with output $y=C_0x_0$. The meaning of the condition (51) is that they have to be in the open left half complex plane. This form is analogous to the condition on transmission zeros in [11].

A related problem is considered in [6] where a feedback gain matrix is obtained by solving a standard linear quadratic state regulator problem, but no performance index is being minimized by the high-gain feedback control.

Sliding Mode

A class of systems with discontinuous feedback control, called variable structure systems, has been developed in USSR in the last 15 years and surveyed in [8], [27], [28]. The salient feature of these systems is that the so-called sliding mode occurs on a switching surface $s(x_0)=0$. While in the sliding mode, the system remains insensitive to parameter variations and disturbances, similar to a high-gain system.

Suppose that the switching surface $s(x_0)=C_0x_0=0$ is chosen and assumption (7) is satisfied. In sliding mode, that is, when $s(x_0)=0$, system (1) with a discontinuous feedback control, componentwise,

$$u_i(x_0)=\begin{cases} u_i^+(x_0), & s_i(x_0)>0 \\ u_i^-(x_0), & s_i(x_0)<0 \end{cases} \tag{52}$$

is governed by

$$\dot{x}_0=A_0x_0+B_0u_{eq} \tag{53}$$

where u_{eq} is the "equivalent control"

$$u_{eq}=-(C_0B_0)^{-1}C_0A_0x_0 \tag{54}$$

obtained by requiring that

$$\dot{s}=C_0A_0x_0+C_0B_0u=0. \tag{55}$$

The existence and uniqueness of u_{eq} is guaranteed by assumption (7). In [9], it is proved that the feedback system (1), (54) is robust with respect to small time constants neglected in the model, nonideal switching such as relay hysteresis, in the sense that its trajectory remains close to the trajectory of the equivalent control system (53), (54). It is of practical importance to exploit the results of [9] to examine the robustness properties of high-gain systems. As a step in this direction we establish the following relationship between high-gain feedback system (1), (2) and the equivalent control system (53), (54).

Lemma 3: The motion of equivalent control system (53), (54) is identical to the motion of the slow subsystem (18) of the high-gain feedback system as $g\to\infty$.

Proof: Using the transformation $x=Mx_0$, we deal with system (28), (29) instead of (1). In terms of x the equivalent control (54) is

$$u_{eq}=-(C_2B_2)^{-1}\left[(C_1A_{11}+C_2A_{21})x_1+(C_1A_{12}+C_2A_{22})x_2\right] \tag{56}$$

where C_1, C_2 are as in (10). Noting that

$$s=C_1x_1+C_2x_2 \tag{57}$$

the equivalent control system becomes

$$\dot{x}_1=A_{11}x_1+A_{12}x_2 \tag{28}$$

$$\dot{s}=0. \tag{58}$$

Since in sliding mode $s=0$, we solve it for x_2 and substitute into (28). This yields

$$\dot{x}_1=(A_{11}-A_{12}C_2^{-1}C_1)x_1 \tag{59}$$

and $A_{11}-A_{12}C_2^{-1}C_1=F_{11}$ by Appendix A. Therefore, (59) coincides with (18) as $g\to\infty$, $\mu\to 0$.

Thus, the slow motions of high-gain systems as $g\to\infty$ are the same as sliding motions in variable structure system. It follows that the slow motions enjoy the same robustness properties of sliding motions. From our results in the previous sections and Lemma 3 it is clear that the eigenvalues of the equivalent control system (53), (54) are the transmission zeros of the system (1) with the "output" s. Furthermore, the switching surfaces can be synthesized by either placing the slow eigenvalues or solving problem "s" in the last section, a lower order state regulator problem.

Example

As a simple illustration consider a system in the form of (28), (29),

$$\dot{x}=\begin{bmatrix} 3 & 1 & 1 \\ -6 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}x+\begin{bmatrix} 0 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}u. \tag{60}$$

First, suppose that a high-gain feedback control is to be found to place the slow eigenvalue near $\lambda_1=-3$ and the fast eigenvalues near $\lambda_{2,3}=g(-1\pm j1)$. We solve the two lower order pole placement problems (32), (33) and obtain

$$K_s=\begin{bmatrix} 6 \\ 0 \end{bmatrix},\quad K_f=\begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix}. \tag{61}$$

The high-gain feedback (34) is

$$u=g\begin{bmatrix} 0 & 0 & 1 \\ -12 & -2 & 0 \end{bmatrix}x. \tag{62}$$

Second, consider the high-gain feedback system (60), (62) with (62) written as $u=gC_0x$ and apply Theorem 1. The closed loop eigenvalues are computed approximately by neglecting the μ^2 terms in (18), (19), that is,

$$\lambda_1=\lambda(F_{11}-\mu F_{12}L_0)=\lambda(-3+30\mu)=-3+30\mu \tag{63}$$

$$\begin{aligned}\lambda_{2,3}&=\frac{1}{\mu}\lambda(C_0B_0+\mu H_2)\\ &=\frac{1}{\mu}\lambda\left(\begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix}+\mu\begin{bmatrix} 3 & 0 \\ -12 & 7 \end{bmatrix}\right)\\ &=\frac{1}{\mu}\left(-1+5\mu\pm j\sqrt{1+16\mu-4\mu^2}\right).\end{aligned} \tag{64}$$

The upper bound in (17) is $\mu_c=0.0074$ corresponding to $g_c=135.6$. The exact eigenvalues are computed for comparison purposes. Denoting the error between the approximate and exact eigenvalues as ϵ_{app} and between the specified and exact eigenvalues as ϵ_{spc}, the results are summarized in Table I.

From (63), (64) it is observed that as $\mu\to 0$, $\lambda_{1,2,3}$ tend to

TABLE I

Gain g	Exact	Approximate	Specified	ϵ_{app} percent	ϵ_{spc} percent
100	-2.69324752	-2.7	-3.0	<0.05	<14
	$-95.1533625 \pm j107.955057$	$-95 \pm j107.684725$	$-100 \pm j100$		
1000	-2.96991225	-2.97	-3.0	<0.003	≈ 1
	$-995.015044 + j1007.99596$	$-995 \pm j1007.96627$	$-1000 \pm j1000$		

the specified values. For design purposes gain g does not have to be high, since with $g=100$ the accuracy of 14 percent in eigenvalue location is often acceptable.

Conclusion

Several classes of seemingly unrelated problems are shown to result in or are equivalent to high-gain feedback systems. A singular perturbation analysis reveals the two time scale properties of these systems and allows a simplified determination of eigenvalues, root locus asymptotes, and transmission zeros. A separation of slow and fast state feedback designs is also presented for both pole-placement and linear regulator problems. For simplicity of comparison the results of this paper are restricted to the class of high-gain feedback systems characterized by (7). It is recognized that this structural assumption has excluded some systems of practical interest. For example, in single input systems, (7) means that the number of zeros ν of the transfer function is $n-1$. However, classical results, such as [1], exist for $\nu<n-1$ and show that for $\nu=n-2$ the system is oscillatory and for $\nu<n-2$ it is unstable. The recent study of multivariable root locus asymptotes [15] also shows that for more general structures of B_0 and C_0 instability of the closed loop system can result from high-gain feedback.

Appendix A

Consider the transformation $\tilde{x}=Tx_0$ as two successive transformations, that is,

$$x=Mx_0, \quad \tilde{x}=\Gamma x. \tag{A1}$$

Denote

$$MA_0M^{-1}=\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \tag{A2}$$

then the system

$$\dot{x}=MA_0M^{-1}x+MB_0u \tag{A3}$$

by (A2) and (10) is (28), (29). From (9),

$$\Gamma^{-1}=\begin{bmatrix} I_{n-m} & 0 \\ -C_2^{-1}C_1 & C_2^{-1} \end{bmatrix} \tag{A4}$$

with C_1, C_2 as in (10). The system

$$\dot{\tilde{x}}=\Gamma MA_0M^{-1}\Gamma^{-1}\tilde{x}+\Gamma MB_0u \tag{A5}$$

by (A2) and (A4) and $\tilde{x}=[x \quad y]^T$ is (13), (14) with

$$F_{11}=A_{11}-A_{12}C_2^{-1}C_1 \tag{A6}$$

$$F_{12}=A_{12}C_2^{-1} \tag{A7}$$

$$H_1=C_1A_{11}+C_2A_{21}-H_2C_1 \tag{A8}$$

$$H_2=(C_1A_{12}+C_2A_{22})C_2^{-1}. \tag{A9}$$

Denote

$$M=\begin{bmatrix} M_1 \\ M_2 \end{bmatrix}, \quad M^{-1}=[S_1 \quad S_2] \tag{A10}$$

where M_1, S_1^T are $(n-m)\times n$ and M_2, S_2^T are $m\times n$ matrices, then

$$S_1M_1+S_2M_2=I. \tag{A11}$$

From (10),

$$C_1=CS_1, \quad C_2=CS_2 \tag{A12}$$

$$B_2=M_2B, \quad 0=M_1B. \tag{A13}$$

In terms of (A10), A_{11} in (A2) becomes

$$A_{11}=M_1A_0S_1. \tag{A14}$$

Substituting into (A6) yields (26). Using the expressions (A11)–(A13), $x_0=M^{-1}\Gamma^{-1}\tilde{x}$ is

$$x_0=(S_1-S_2C_2^{-1}C_1)z+B_0(C_0B_0)^{-1}y. \tag{A15}$$

From (A12)

$$C(S_1-S_2C_2^{-1}C_1)=0 \tag{A16}$$

and by defining

$$N=S_1-S_2C_2^{-1}C_1 \tag{A17}$$

(A15) is (11). By (10) and (A10), $\tilde{x}=\Gamma Mx_0$ becomes

$$\begin{bmatrix} z \\ y \end{bmatrix}=\begin{bmatrix} M_1x_0 \\ (C_1M_1+C_2M_2)x_0 \end{bmatrix}=\begin{bmatrix} M_1x_0 \\ C_0x_0 \end{bmatrix}. \tag{A18}$$

Appendix B

For the system (13), (14), by Lemma 1 in [18], if assumption (7) is satisfied and if μ satisfies

$$\mu a < \tfrac{1}{3}(b + 2\mu acd) \tag{B1}$$

where $a = \|(C_0B_0)^{-1}\|$, $b = \|F_{11}\|$, $c = \|F_{12}\|$, and $d = \|H_1\|$, then L of (16) exists. $L = L_0 + \mu G$ where

$$L_0 = (C_0B_0 + \mu H_2)^{-1}H_1. \tag{B2}$$

From (B1), μ_c in (17) is obtained,

$$\mu_c = \frac{\left(b^2 + \frac{8}{3}cd\right)^{1/2} - b}{4acd}. \tag{B3}$$

By application of the implicit function theorem to

$$(C_0B_0 + \mu H_2)G - \mu GF_{11} + \mu^2 GF_{12}L_0 + \mu^2 L_0F_{12}G + \mu^3 GF_{12}G = L_0F_{11} - \mu L_0F_{12}L_0 \tag{B4}$$

we can show that G possesses a power series at $\mu = 0$, that is,

$$G = \sum_{i=0}^{k} G_i\mu^i + 0(\mu^k). \tag{B5}$$

A recursive formula for calculating G_i is obtained by substituting (B5) into (B4) and equating coefficients of powers of μ,

$$G_0 = (C_0B_0 + \mu H_2)^{-1}L_0H_1F_{11} \tag{B6}$$

$$G_1 = (C_0B_0 + \mu H_2)^{-1}[G_0F_{11} - L_0F_{12}L_0] \tag{B7}$$

$$G_k = (C_0B_0 + \mu H_2)^{-1}\left[G_{k-1}F_{11} - G_{k-2}F_{12}L_0 - L_0F_{12}G_{k-2} - \sum_{j=0}^{k-3} G_jF_{12}G_{k-j-3}\right] \quad \text{for } k \geq 2 \text{ with } \sum_{j=0}^{-1} \equiv 0. \tag{B8}$$

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SECTION 5

DESIGN OF NONLINEAR REGULATORS

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NEAR-OPTIMAL FEEDBACK STABILIZATION OF A CLASS OF NONLINEAR SINGULARLY PERTURBED SYSTEMS*

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Abstract. A new series expansion method is developed for a class of nonlinear singularly perturbed optimal regulator problems. The resulting feedback control is near-optimal and can stabilize essentially nonlinear systems when linearized models provide no stability information. The stability domain is shown to include large initial conditions of the fast variables. The control law is implemented in two-time-scales, with the feedback from the fast state variables depending on slow state variables as parameters. The coefficients of the formal expansions of the optimal value function are obtained from equations involving only the slow variables.

1. Introduction. Compared with the rich literature on linear regulator theory, publications dealing with feedback design of nonlinear systems are a small minority. Realistic approaches to the difficult nonlinear feedback control problem usually exploit properties of special classes of systems to develop approximate methods [1], [2]. The approach in this paper exploits multiple time scale properties of a class of nonlinear singularly perturbed systems [3], [4] to achieve stabilization and near-optimality. The stabilization results obtained are essentially nonlinear in the sense that they also apply to the critical case when linearized models provide no stability information. Due to a separation of time scales, the proposed design procedure is applicable to higher order systems.

The problem considered is to optimally control the nonlinear system

$$\dot{x} = a_1(x) + A_1(x)z + B_1(x)u, \qquad x(0) = x_0, \tag{1a}$$

$$\mu\dot{z} = a_2(x) + A_2(x)z + B_2(x)u, \qquad z(0) = z_0, \tag{1b}$$

with respect to the performance index

$$J = \int_0^\infty [p(x) + s'(x)z + z'Q(x)z + u'R(x)u]\, dt, \tag{2}$$

where $\mu > 0$ is the small singular perturbation parameter, x, z are n-, m- dimensional states, respectively, u is an r-dimensional control and the prime denotes a transpose. It is assumed that there exists a domain $D \subset R^n$ containing the origin such that for all $x \in D$ and $z \in R^m$ the problem satisfies the following assumptions:

I. The functions a_1, a_2, A_1, A_2, B_1, B_2, p, s, q and R are differentiable with respect to x a sufficient number of times and a_1, a_2, p and s are all zero only at $x = 0$.

II. The matrices $Q(x)$ and $R(x)$ are positive definite, that is, $Q(x) > 0$, $R(x) > 0$. Furthermore, the scalar function $p + s'z + zQz$ of x and z is positive definite in both x and z.

III. For every fixed $x \in D$

$$\operatorname{rank}[B_2, A_2B_2, \cdots, A_2^{m-1}B_2] = m \tag{3}$$

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and hence $A_2(x)$ is assumed to be nonsingular. (If not, then using $u = \hat{u} + K(x)z$ such that $A_2 + B_2K$ is nonsingular we redefine the problem.)

Assumptions I and II establish that the origin is the desired equilibrium of (1). Assumption III and $Q(x) > 0$ simplify the derivations. Alternatively a less restrictive stabilizability-detectability condition can be used.

Finite time trajectory optimization problems for the same class of systems have been treated in [3], [4] via singularly perturbed two point boundary value problems originating from necessary optimality conditions. The resulting controls are open-loop and require boundary layer correction terms at both ends of the interval. For the infinite time regulator problem considered here the Hamilton–Jacobi–Bellman sufficiency condition is more suitable since it readily incorporates stability requirements and leads to feedback solutions. Using this condition we obtain near-optimal stabilizing controls in feedback form and avoid explicit treatment of boundary layer phenomena.

Our procedure is based on a nested power series expansion of the optimal value function in z and μ. An advantage of this procedure is that it uses lower order equations involving only the slow variable x. In applications truncated series are of interest. Stabilizing properties of various truncated designs are discussed and an explicit estimate of the stability domain is given. It is of practical importance that this domain encompasses large initial disturbances of $z(0)$. Furthermore, near-optimality of these truncated designs is established in terms of $O(\mu)$, $O(\mu^2)$, etc. A particularly useful result is that an $O(\mu)$ near optimal feedback control can be implemented without knowing the value of the small parameter μ.

The paper is organized as follows. In § 2 a reduced order problem is formulated for the slow variable x. The crucial assumption is that the properties of its solution are known. Using a truncated expansion of the optimal value function the so-called composite control is introduced in § 3. Since the leading term in the series is the optimal value function of the reduced problem, the original problem is well posed. In § 4 it is shown that the composite control guarantees a finite domain of stability for the resulting feedback system. In § 5, a formal expansion of the optimal value function is proposed and near-optimality results are discussed. An example is discussed in § 6.

2. The reduced control. In singular perturbation techniques [5], a problem for the full order system (1) where $\mu > 0$ is interpreted as a perturbation of a reduced problem

(4a) $$\dot{x} = a_1(x) + A_1(x)z + B_1(x)u, \qquad x(0) = x_0$$

(4b) $$0 = a_2(x) + A_2(x)z + B_2(x)u,$$

in which $\mu = 0$. Due to assumption III, z can be solved from (4b) and eliminated from (4a) and (2). Then the reduced problem is to optimally control the system

(5) $$\dot{x} = a_0(x) + B_0(x)u, \qquad x(0) = x_0$$

with respect to

(6) $$J_0 = \int_0^\infty [p_0(x) + 2s_0'(x)u + u'R_0(x)u]\, dt,$$

where

$$a_0 = a_1 - A_1A_2^{-1}a_2, \qquad B_0 = B_1 - A_1A_2^{-1}B_2,$$

(7) $$p_0 = p - s'A_2^{-1}a_2 + a_2'A_2'^{-1}QA_2^{-1}a_2, \qquad s_0 = B_2'A_2'^{-1}(QA_2^{-1}a_2 - \tfrac{1}{2}s),$$

$$R_0 = R + B_2'A_2'^{-1}QA_2^{-1}B_2.$$

The origin $x = 0$ is the desired equilibrium of the optimally controlled reduced system (5) for all $x \in D$, since, in view of assumption II, $a_0(0) = 0$ and

$$p_0(x) + 2s_0'(x)u + u'R_0(x)u \tag{8}$$

is positive definite in x and u.

The reduced problem (5), (6) is considerably simpler than the original problem (1), (2) because of the elimination of the fast variables and the reduction of the system order. One of the tasks of the singular perturbation analysis is to establish whether the full problem is well-posed in the sense that its solution tends to the solution of the reduced problems as $\mu \to 0$. If so, then the next task is to deduce the properties of the original problem from the properties of the reduced problem. Finally these properties are to serve as a basis for a simplified design procedure.

To formulate our basic assumption about the properties of the solution of the reduced problem we use the optimality principle

$$0 = \min_u \,[p_0(x) + 2s_0'(x)u + u'R_0(x)u + L_x(a_0(x) + B_0(x)u)], \tag{9}$$

where L is the optimal value function and L_x is its partial derivative with respect to x. This yields the minimizing control

$$u_0 = -R_0^{-1}(s_0 + \tfrac{1}{2}B_0'L_x') \tag{10}$$

whose elimination from (9) results in the Hamilton–Jacobi equation

$$0 = (p_0 - s_0'R_0^{-1}s_0) + L_x(a_0 - B_0R_0^{-1}s_0) - \tfrac{1}{4}L_xB_0R_0^{-1}B_0'L_x', \qquad L(0) = 0. \tag{11}$$

Note that, due to (8), $p_0 - s_0'R_0^{-1}s_0$ is positive definite in D. Our crucial assumption is then stated as follows.

IV. The unique positive definite solution $L(x)$ of (11) exists in D and is differentiable with respect to x a sufficient number of times. Furthermore the level surface $L = c_0 =$ constant is taken to be the boundary of the set D.

In the special case considered in [1], where the linearization of (5) at $x = 0$ is stabilizable and its states are observable in the quadratic approximation of J_0, our assumption IV is automatically satisfied for all x near the origin. It follows from assumption IV that u_0 is the unique optimal feedback control for the reduced problem and L is a Lyapunov function of the optimally controlled reduced system

$$\dot{x} = a_0 - B_0R_0^{-1}(s_0 + \tfrac{1}{2}B_0'L_x') = \bar{a}_0(x), \tag{12}$$

establishing that the origin is asymptotically stable and the set D belongs to its domain of attraction.

3. The composite control. The optimal value function $V(x, z, \mu)$ of the full problem (1), (2) satisfies the equation

$$\begin{aligned} 0 = \min_u \Big[& p + s'z + z'Qz + u'Ru + V_x(a_1 + A_1z + B_1u) \\ & + \frac{1}{\mu}V_z(a_2 + A_2z + B_2u)\Big], \end{aligned} \tag{13}$$

where V_x, V_z denote the partial derivatives of V with respect to the variables x, z, respectively. The minimizing control of (13) is

$$u = -\frac{1}{2}R^{-1}\Big(B_1'V_x' + \frac{1}{\mu}B_2'V_z'\Big). \tag{14}$$

and its substitution into (13) yields the Hamilton–Jacobi equati n

$$
(15)\qquad \begin{aligned}
0 &= p + s'z + z'Qz + V_x(a_1 + A_1 z) + \frac{1}{\mu} V_z(a_2 + A_2 z) \\
&\quad - \frac{1}{4}\left(V_x B_1 + \frac{1}{\mu} V_z B_2\right) R^{-1}\left(B_1' V_x' + \frac{1}{\mu} B_2' V_z'\right), \qquad V(0, 0, \mu) = 0.
\end{aligned}
$$

Since system (1) is linear in z and J is quadratic in z, and since $\dot{z}$ is multiplied by μ, we seek a solution of (15) in the form

$$
(16)\qquad \begin{aligned}
V(x, z, \mu) &= \bar{V}_0(x) + \mu \bar{V}_1'(x) z + \mu z' \bar{V}_2(x) z + \mu q(x, z, \mu) \\
&= \bar{V}(x, z, \mu) + \mu q(x, z, \mu), \qquad \bar{V}_0(0) = 0
\end{aligned}
$$

where

$$
(17)\qquad \partial q/\partial x = O(1), \qquad \partial q/\partial z = O(\mu).
$$

We shall investigate the expansion of q in a later section. The partial derivatives of V with respect to x, z are

$$
(18)\qquad \begin{aligned}
V_x &= \bar{V}_{0x} + O(\mu), \\
V_z &= \mu \bar{V}_1' + 2\mu z' \bar{V}_2 + 0(\mu^2).
\end{aligned}
$$

Substituting (18) into (15) and neglecting the μ-dependent terms, we obtain the equation

$$
(19)\qquad \begin{aligned}
0 &= p + \bar{V}_{0x} a_1 + \bar{V}_1' a_2 - \tfrac{1}{4}(\bar{V}_{0x} B_1 + \bar{V}_1' B_2) R^{-1} (B_1' \bar{V}_{0x}' + B_2' \bar{V}_1) \\
&\quad + [s' + 2a_2' \bar{V}_2 + \bar{V}_{0x}(A_1 - B_1 R^{-1} B_2' \bar{V}_2) + \bar{V}_1' (A_2 - B_2 R^{-1} B_2' \bar{V}_2)] z \\
&\quad + z'(Q + \bar{V}_2 A_2 + A_2' \bar{V}_2 - \bar{V}_2 B_2 R^{-1} B_2' \bar{V}_2) z.
\end{aligned}
$$

In order to satisfy (19) identically for all z, we require that

$$
(20)\qquad 0 = p + \bar{V}_{0x} a_1 + \bar{V}_1' a_2 - \tfrac{1}{4}(\bar{V}_{0x} B_1 + \bar{V}_1' B_2) R^{-1} (B_1' \bar{V}_{0x}' + B_2' \bar{V}_1), \qquad \bar{V}_0(0) = 0,
$$

$$
(21)\qquad 0 = s' + 2a_2' \bar{V}_2 + \bar{V}_{0x}(A_1 - B_1 R^{-1} B_2' \bar{V}_2) + \bar{V}_1' (A_2 - B_2 R^{-1} B_2' \bar{V}_2),
$$

$$
(22)\qquad 0 = Q + \bar{V}_2 A_2 + A_2' \bar{V}_2 - \bar{V}_2 B_2 R^{-1} B_2' \bar{V}_2.
$$

At each fixed value of x, (22) is an algebraic Riccati equation for $\bar{V}_2$. In view of (3) and $Q(x) > 0$, the unique positive definite solution $\bar{V}_2$ exists such that for all $x \in D$, the real parts of the eigenvalues of $\bar{A}_2 = A_2 - B_2 R^{-1} B_2' \bar{V}_2$, denoted by Re $(\lambda(\bar{A}_2))$, are less than a negative constant. Thus $\bar{A}_2$ is nonsingular and $\bar{V}_1$ can be expressed in terms of $\bar{V}_{0x}$ and $\bar{V}_2$ as

$$
(23)\qquad \bar{V}_1' = -[s' + 2a_2' \bar{V}_2 + \bar{V}_{0x}(A_1 - B_1 R^{-1} B_2' \bar{V}_2)] \bar{A}^{-1}.
$$

It is of crucial importance that the elimination of $\bar{V}_1$ from (21) results in an equation involving only $\bar{V}_{0x}$. For the well-posedness of the full problem it is necessary that the leading term $\bar{V}_0$ of (16) be identical to the solution L of the reduced problem.

LEMMA 1. *If assumptions* III *and* IV *are satisfied, then the unique positive definite solution* $\bar{V}_0(x)$ *of* (20)–(22) *exists in D and is identical to the solution* $L(x)$ *of the reduced problem* (5), (6).

Proof. It is shown in the appendix that eliminating $\bar{V}_1$ from (20), we obtain the Hamilton–Jacobi equation (11) with $\bar{V}_{0x}$ in place of L_x, and hence $\bar{V}_0(x) = L(x)$ with properties as in assumption IV.

By virtue of Lemma 1, $\bar{V}_0$ and $\bar{V}_2$ are solved independently from (11) and (22). This is the separation of time scales in the design of nonlinear regulators, analogous to the linear time-invariant design in [7].

Using $\bar{V}$, we derive the control

$$\begin{aligned} \bar{u} &= -\tfrac{1}{2}R^{-1}\left(B_1'\bar{V}_x' + \frac{1}{\mu}B_2'\bar{V}_z'\right) \\ &= -\tfrac{1}{2}R^{-1}[B_1'\bar{V}_{0x}' + B_2'(\bar{V}_1 + 2\bar{V}_2 z)] + O(\mu) \qquad (24) \\ &= u_c + O(\mu), \end{aligned}$$

whose main part u_c is defined as *the composite control*. Eliminating $\bar{V}_1$ from (24) using (23) and following the derivation in [7], we can write u_c as

$$\begin{aligned} u_c &= -R_0^{-1}(s_0 + \tfrac{1}{2}B_0'\bar{V}_{0x}') - R^{-1}B_2'\bar{V}_2[z + A_2^{-1}(a_2 - B_0R_0^{-1}(s_0 + \tfrac{1}{2}B_0'\bar{V}_{0x}'))] \qquad (25) \\ &= u_0 - R^{-1}B_2'\bar{V}_2(z + \bar{A}_2^{-1}\bar{a}_2), \end{aligned}$$

where

$$\bar{A}_2(x) = A_2 - B_2R^{-1}B_2'\bar{V}_2, \qquad (26a)$$

$$\bar{a}_2(x) = a_2 - \tfrac{1}{2}B_2R^{-1}(B_1'\bar{V}_{0x}' + B_2'\bar{V}_1), \qquad \bar{a}_2(0) = 0. \qquad (26b)$$

Hence the composite control u_c consists of a slow control u_0 which optimizes the reduced system (5) and a fast control $-R^{-1}B_2'\bar{V}_2(z + \bar{A}^{-1}\bar{a}_2)$ which optimizes the fast part $(z + \bar{A}^{-1}\bar{a}_2)$ of z in the sense that $\bar{V}_2$ satisfies (22). Note that when z is not penalized in (2), that is when $Q(x) = 0$, but $\mathrm{Re}\,\{\lambda(A_2)\} < 0$, then $\bar{V}_2$ is identically zero and u_c reduces to u_0 of (10). Stabilizing properties of the composite control u_c are established in the next section.

4. Stabilizing properties. System (1) controled by u_c is

$$\begin{aligned} \dot{x} &= a_1 + A_1 z + B_1 u_c = \bar{a}_1(x) + \bar{A}_1(x)z, \qquad x(0) = x_0, \\ \mu\dot{z} &= a_2 + A_2 z + B_2 u_c = \bar{a}_2(x) + \bar{A}_2(x)z, \qquad z(0) = z_0, \end{aligned} \qquad (27)$$

where

$$\begin{aligned} \bar{a}_1 &= a_1 - \tfrac{1}{2}B_1R^{-1}(B_1'\bar{V}_{0x}' + B_2'\bar{V}_1), \qquad \bar{a}_1(0) = 0, \\ \bar{A}_1 &= A_1 - B_1R^{-1}B_2'\bar{V}_2. \end{aligned} \qquad (28)$$

With the change of variables

$$\eta = z + \bar{A}_2^{-1}\bar{a}_2 \qquad (29)$$

exhibiting η as the fast part of z, system (27) becomes

$$\dot{x} = \bar{a}_0 + \bar{A}_1\eta, \qquad x(0) = x_0, \qquad (30a)$$

$$\begin{aligned} \mu\dot{\eta} &= \mu(\bar{A}_2^{-1}\bar{a}_2)_x\bar{a}_0 + [\bar{A}_2 + \mu(\bar{A}_2^{-1}\bar{a}_2)_x\bar{A}_1]\eta \\ &= \mu f(x) + [\bar{A}_2(x) + \mu F(x)]\eta, \qquad \eta(0) = z_0 + \bar{A}_2^{-1}(x_0)\bar{a}_2(x_0). \end{aligned} \qquad (30b)$$

Since the right-hand side of (30b) is an $O(\mu)$ perturbation of $\bar{A}_2(x)\eta$ and $\mathrm{Re}\,\{\lambda(\bar{A}_2)\} < 0$ in D we expect that η will rapidly decay to an $O(\mu)$ quantity. This motivates the introduction of

$$U(x, \eta; \mathscr{E}) = \bar{V}_0(x) + \mathscr{E}\eta'\bar{V}_2(x)\eta. \qquad (31)$$

as a tentative Lyapunov function for (30). Here $\mathscr{E}$ is a small positive scalar to be determined. From assumptions III and IV, $\bar{V}_0(x)$ is positive definite and $\bar{V}_2(x)>0$ in D. Hence U is positive definite for all $x \in D$ and $\eta \in R^m$. Furthermore, since $\bar{V}_0(x)=c_0>0$ for all x on the boundary of D, the surface

$$S(x,\eta;\mathscr{E})=\{x,\eta:U(x,\eta;\mathscr{E})=c_0\} \tag{32}$$

is closed in the $(n+m)$-dimensional domain $x \in D$, $\eta \in R^m$. We define S_{in} to be the domain in the interior of S.

Let D_1 be a set strictly in the interior of D, that is, the boundary of D_1 does not intersect the boundary of D, and let E be a bounded set in R^m. The presence of $\mathscr{E}$ in U extends S to encompass (x,z) for all $x \in D_1$ and for z in any prescribed set E. This crucial result is stated as follows.

LEMMA 2. *If assumption* III *and* IV *are satisfied, then there exists an* $\mathscr{E}>0$ *such that the domain* S_{in} *contains all* $x \in D_1$, $\eta \in E$.

Proof. At each point $\hat{x} \in D_1$, the projection S onto the η subspace is the ellipsoid

$$\eta' \bar{V}_2(\hat{x})\eta=(c_0-\bar{V}_0(\hat{x}))/\mathscr{E}, \tag{33}$$

implying that η extends to $O(1/\sqrt{\mathscr{E}})$. Hence for every $\hat{x}$, there exists an $\mathscr{E}(\hat{x})$ sufficiently small such that the ellipsoid (33) includes all $\eta \in E$. (Note that we must exclude the boundary of D because from (33) the projection of S at any point on the boundary of D is a single point $\eta=0$.) Hence if we choose $\mathscr{E}^*$ to be the smallest of such $\mathscr{E}(\hat{x})$, the domain S_{in} contains all $x \in D_1$, $\eta \in E$ for any $\mathscr{E} \in (0,\mathscr{E}^*]$.

By virtue of Lemma 2, the initial condition $\eta(0)$ of (30b), and hence $z(0)$ of (27), can be as far away from zero as $O(1\sqrt{\mathscr{E}})$ and still be enclosed by S. We now examine the relationship between $\mathscr{E}$ and μ.

Using (11), (22) and rearranging, we obtain the time derivative of U with respect to (30) as

$$\dot{U}=-g(x,\mathscr{E},\mu)-\frac{\mathscr{E}}{2\mu}\xi' Q(x)\xi-\frac{\mathscr{E}}{\mu}\eta' M(x,\eta,\mathscr{E},\mu)\eta \tag{34}$$

where

$$g=g_1-\frac{\mu}{2\mathscr{E}}y'Q^{-1}y, \qquad g_1=p_0-s_0'R_0^{-1}s_0+\tfrac{1}{4}\bar{V}_{0x}B_0R_0^{-1}B_0'\bar{V}_{0x}',$$

$$y=\bar{A}_1'\bar{V}_{0x}'+2\mathscr{E}\bar{V}_2 f, \qquad \xi=\eta-\frac{\mu}{\mathscr{E}}Q^{-1}y, \tag{35}$$

$$M=\frac{Q}{2}+\bar{V}_2B_2R^{-1}B_2'\bar{V}_2-\mu(\bar{V}_2\bar{F}+\bar{F}'\bar{V}_2)-\mu\hat{V}_2.$$

Since $\bar{V}_2F_2+\bar{F}'\bar{V}_2$ and $\hat{V}_2$ are bounded for all x, η in S_{in}, and since $Q(x)>0$ in D, it follows that there exists a $\mu_1^*>0$ such that $M>0$ for all x, η in S_{in} and for $\mu \in (0,\mu_1^*]$. Thus the last two terms in $\dot{U}$ are positive definite. To ensure that $g(x,\mathscr{E},\mu)$ is positive definite, we assume that the reduced problem also satisfies

V. The limit

$$\lim_{x_1\to 0}\frac{y'Q^{-1}y}{g_1}=k(\mathscr{E})<\infty \tag{36}$$

exists for all fixed $\mathscr{E}>0$.

Note that $k \geqq 0$ because $y'Q^{-1}y$ is positive semidefinite and g_1 is positive definite. The limit (36) implies that there exists a domain $\hat{D}$ about $x=0$ such that

$$y'Q^{-1}y \leqq (1+k)g_1, \tag{37}$$

that is, such that for $\mu < 2\mathscr{E}/(1+k)$, g is positive definite in $\hat{D}$; see (35). Let $\bar{k}(\mathscr{E})>0$ be the minimum value of g_1 on the boundary of $\hat{D}$. Hence in the domain

$$\hat{D}_1(x) = \{x : g_1(x) < \bar{k}\}, \tag{38}$$

g is positive definite. On the other hand, since D is bounded, there exists a $k_1(\mathscr{E})>0$ such that $y'Q^{-1}y < k_1$ for all $x \in D$, that is, such that g is positive definite when x is not in the domain

$$\tilde{D}(x) = \{x : g_1(x) < \mu k_1/2\mathscr{E}\} \tag{39}$$

about the origin. But for $\mu < 2\mathscr{E}\bar{k}/k_1$, $\tilde{D} \subset \hat{D}_1$, implying that g is positive definite in D. Thus $\dot{U}$ is negative definite for all x, η contained in S_{in}. We now conclude that U is a Lyapunov function for (30) guaranteeing that $x=0$, $\eta=0$ is asymptotically stable for all $x \in D_1$, $\eta \in E$ and for $\mu \in (0, \mu^*]$, where

$$\mu^* = \min\left(\frac{2\mathscr{E}}{1+k}, \frac{2\mathscr{E}\bar{k}}{k_1}, \mu_1^*\right). \tag{40}$$

Returning from the η variable to the z variable via $z = \eta - \bar{A}_2^{-1}\bar{a}_2$, we obtain for all $x \in D_1$, $\eta \in E$ a corresponding bounded domain E_1 for z. We summarize the above discussions on the asymptotic stabilizing property of u_c in (24) as follows.

THEOREM 1. *If assumptions* I–V *are satisfied, then there exists a* $\mu^*>0$ *such that for all* $\mu \in (0, \mu^*]$ *and for all* $x \in D_1$ *and* z *in any prescribed bounded set* E_1, *the origin* $x=0$, $z=0$ *of the feedback system* (1) *controlled by the composite control* u_c *is asymptotically stable.*

Theorem 1 can be applied in two different directions. As outlined above, for any given D_1 and E_1, we first find $\mathscr{E}^*$ such that S_{in} of (32) contains all $x \in D_1$, $z \in E_1$. Then we find μ^* from (40). This direction is suitable when μ is a parameter at the designer's disposal, such as a gain factor [9]. In the other direction, if μ represents some given physical parameters, such as time constants, we use its value to determine the smallest $\mathscr{E}$ such that $\dot{U}$ of (34) is negative definite, that is we find the largest D_1 and E_1.

As a special case of assumption V, consider that the origin $x=0$ of the reduced system (12) is exponentially stable. Then near the origin, $p_0 - s_0'R_0^{-1}s_0$, $\bar{V}_0$ grow as $|x|^2$, and $|\bar{V}_{0x}|$, $|\bar{a}_0|$ grow as $|x|$, and we can find positive constants $k_2, \cdots, k_9$ and δ such that

$$\begin{aligned} &k_2|x|^2 \leqq p_0 - s_0'R_0^{-1}s_0 \leqq k_3|x|^2, \qquad k_4|x|^2 \leqq \bar{V}_0 \leqq k_5|x|^2 \\ &k_6|x| \leqq |\bar{V}_{0x}| \leqq k_7|x|, \qquad k_8|x| \leqq |\bar{a}_0| \leqq k_9|x| \end{aligned} \tag{41}$$

for all $|x|<\delta$. It follows from (41) that there exists a fixed $k_{10}(\mathscr{E})>0$ such that

$$y'Q^{-1}y \leqq k_{10}|x|^2 \tag{42}$$

and the limit (36) is bounded by

$$\lim_{|x|\to 0} \frac{y'Q^{-1}y}{g_1} \leqq \lim_{|x|\to 0} \frac{k_{10}|x|^2}{k_2|x|^2} = \frac{k_{10}}{k_2} \tag{43}$$

satisfying assumption V.

In this case a claim stronger than Theorem 1 can be made.

COROLLARY 1. *If assumptions* I–IV *are satisfied and the origin* $x=0$ *of the reduced system is exponentially stable, then the conclusion of Theorem* 1 *holds and moreover the origin* $x=0$, $z=0$ *of* (27) *is exponentially stable.*

Proof. The first part of the corollary follows from Theorem 1. The second part follows from the linearization of (27) at the origin

$$\begin{bmatrix}\delta\dot{x}\\ \delta\dot{z}\end{bmatrix}=\begin{bmatrix}\dfrac{\partial\bar{a}_1(0)}{\partial x} & \bar{A}_1(0)\\ \dfrac{1}{\mu}\dfrac{\partial\bar{a}_2(0)}{\partial x} & \dfrac{1}{\mu}\bar{A}_2(0)\end{bmatrix}\begin{bmatrix}\delta x\\ \delta z\end{bmatrix}. \tag{44}$$

The system matrix of (44) has one group of n small eigenvalues $O(\mu)$ close to those of $(\partial\bar{a}_1/\partial x)-\bar{A}_1\bar{A}_2^{-1}(\partial\bar{a}_2/\partial x)|_{x=0}$ and another group of m large eigenvalues $O(1)$ close to those of $(1/\mu)\bar{A}_2(0)$ [8]. But $\bar{a}_1-\bar{A}_1\bar{A}_2^{-1}\bar{a}_2=\bar{a}_0$ and $(\partial\bar{a}_0/\partial x)|_{x=0}=(\partial\bar{a}_1/\partial x)-\bar{A}_1\bar{A}_2^{-1}(\partial\bar{a}_2/\partial x)|_{x=0}$ as $\bar{a}_2(0)=0$. Thus the real parts of the eigenvalues of the system matrix of (44) are all negative and $x=0$, $z=0$ is exponentially stable.

If the origin $x=0$ of the reduced system is only asymptotically stable but not exponentially stable, then in general g need not be positive definite for all $x\in D$. This situation includes the critical case when the linearized model does not provide any stability information as clarified by the example in § 6. For this situation the system is now shown to possess a weaker stability property, that is, its trajectories tend to a small sphere around the origin. Define the domain in R^n

$$\rho(x)=\{x: g(x;\bar{x},\mu)\leq 0\}, \tag{45}$$

which is contained in the domain $\bar{D}$ of (39). Due to the presence of μ in (34), $\dot{U}$ may be positive only if $x\in\rho(x)$ and $\eta=O(\mu)$. Otherwise, $\dot{U}$ is negative. If we define the surface

$$\pi(x,z)=\{x,z: x\in\rho(x;\mu),\ z=-\bar{A}_2^{-1}(x)\bar{a}_2(x)\} \tag{46}$$

about the origin in R^{m+n}, u_c defined by (24) is a stabilizing control in the following sense.

THEOREM 2. *If assumptions* I–IV *are satisfied, then there exists a* $\mu^*>0$ *such that for all* $\mu\in(0,\mu^*]$, *the feedback control* (24) *steers all* $x\in D_1$, $z\in E_1$ *of the full system* $O(\mu)$ *close to the surface* $\pi(x,z)$.

Proof. Since $U>0$ and $\dot{U}<0$ except for $x\in\rho(x)$ and $\eta=O(\mu)$, x converges to $\rho(x)$ and η decays to an $O(\mu)$ quantity. Thus in the x, z variables, (x,z) converges to an $O(\mu)$ neighborhood of the surface $\pi(x,z)$.

In the case where the fast transients of z in (1) are exponentially stable, that is, $A_2(x)$ is stable for all $x\in D$, and we are only concerned with the optimality of the reduced system (5), then the z-independent reduced control u_0 of (10) stabilizes the full system (1) with essentially the same stabilizing properties as u_c of (24). We shall not repeat the argument.

An attractive feature of the controls u_c and u_0 is that they do not require the knowledge of the actual value of μ provided that it is sufficiently small. When appropriately implemented, these controls stabilize the full system (1) and achieve optimality of the reduced system, and in the case of u_c, also optimality of the fast part of z. The above results also answer the question of well-posedness by giving the conditions under which the same optimal reduced order system is obtained when μ is set equal to zero either when system (1) is uncontrolled or when it is controlled by u_c or u_0. In contrast to many other singular perturbation results which require μ to be

sufficiently small, this section provides a method to compute an estimate of allowable values of μ given a stability domain or vice versa.

5. A formal expansion and near-optimality. The equation (16) only satisfies the Hamilton–Jacobi equation (15) to $O(\mu)$ order. We now propose to solve (15) by expanding V formally as a nested infinite power series. If this power series is convergent, then the optimal solution V of (15) exists. For x, z near the origin, it has been shown in [1] that the optimal solution exists and possesses a power series expansion when system (1) after linearization at the origin is stabilizable and the state in the quadratic approximation of J is observable. Here we are interested in a power series of V which satisfies (15) to any order of μ.

Since system (1) is linear in z and J is quadratic in z, the optimal value function can be expanded as a power series in the components of z [2]. In addition, since z is the fast variable, the z terms in the optimal value function are multiplied by appropriate powers of μ [5]. In view of these two characteristics, we seek a solution of (15) in the form

$$
\begin{aligned}
V(x, z, \mu) = {} & V_0(x, \mu) + \mu \sum_{j=1}^{m} V_{1j}(x, \mu) z_j + \mu \sum_{j=1}^{m} \sum_{k=1}^{m} V_{2jk}(x, \mu) z_j z_k \\
& + \mu^2 \sum_{j=1}^{m} \sum_{k=1}^{m} \sum_{q=1}^{m} V_{3jkq}(x, \mu) z_j z_k z_q + \cdots \\
& + \mu^{i-1} \sum_{j_1=1}^{m} \sum_{j_2=1}^{m} \cdots \sum_{j_i=1}^{m} V_{ij_1j_2\cdots j_i}(x, \mu) z_{j_1} z_{j_2} \cdots z_{j_i} + \cdots, \\
& V_0(0, \mu) = 0,
\end{aligned} \tag{47}
$$

where $V_{ij_1j_2\cdots j_i}$ is the $(j_1, j_2, \cdots, j_i)$ element of the completely symmetric generalized matrix[1] V_i of dimension m^i and z_j is the jth component of z. The summation signs in (47) and in other equations in the paper will be omitted when there is no confusion as to which indices $j_1, j_2, \cdots, j_i$ are being summed. The partial derivatives $V_x, V_{z_1}, \cdots, V_{z_m}$ expressed in terms of the vector x and the scalars $z_1, \cdots, z_m$ are

$$V_x = V_{0x} + \mu V_{1jx} z_j + \mu V_{2jkx} z_j z_k + \cdots \tag{48a}$$

$$V_{z_i} = \mu V_{1i} + 2\mu V_{2ij} z_j + 3\mu^2 V_{3ijk} z_j z_k + \cdots, \qquad i = 1, 2, \cdots, m, \tag{48b}$$

where the summation signs over j, k are omitted.

For the series (47) to satisfy (15) as an identity, we first rewrite (15) in terms of the vector x and the scalars $z_1, \cdots, z_m$,

$$
\begin{aligned}
0 = {} & p + s_i z_i + Q_{ij} z_i z_j + V_x(a_1 + A_{1i} z_i) + \frac{1}{\mu} V_{z_i}(a_{2i} + A_{2ij} z_j) \\
& - \frac{1}{4}\left(V_x B_1 + \frac{1}{\mu} V_{z_i} B_{2i} \right) R^{-1} \left(B_1' V_x' + \frac{1}{\mu} B_{2i}' V_{z_i} \right),
\end{aligned} \tag{49}
$$

where s_i, a_{2i} are the ith components of the vectors s, a_2, respectively, A_{1i} is the ith column of the matrix A_1, B_{2i} is the ith row of B_2, Q_{ij}, A_{2ij} are the (i, j) elements of Q, A_2, respectively, and the summation signs over the indices i, j are omitted. Then, upon substituting (48) into (49) and equating the coefficients of the like powers of z_i, we

[1] The $(j_1, j_2, \cdots, j_i)$ elements of V_i are identical for all permutations of the indices $j_1, j_2, \cdots, j_i$ [6].

obtain

$$0 = p + V_{0x}a_1 + V_{1i}a_{2i} - \tfrac{1}{4}(V_{0x}B_1 + V_{1i}B_{2i})R^{-1}(B_1' V_{0x}' + B_{2i}' V_{1i}), \qquad V_0(0, \mu) = 0, \tag{50a}$$

$$0 = s_i + V_{0x}A_{1i} + \mu V_{1ix}a_1 + V_{1j}A_{2ji} + 2V_{2ij}a_{2j} - \tfrac{1}{2}(V_{0x}B_1 + V_{1j}B_j) R^{-1}(\mu B_1' V_{1ix}' + 2B_{2j}' V_{2ji}), \qquad i = 1, 2, \cdots, m, \tag{50b}$$

$$\begin{aligned} 0 = {} & Q_{ij} + \mu V_{2ijx}a_1 + \mu(V_{1ix}A_{1j})_s + 2(V_{2ik}A_{2kj})_s + 3\mu V_{3ijk}a_{2k} \\ & - \tfrac{1}{2}(V_{0x}B_1 + V_{1k}B_{2k})R^{-1}(\mu B_1' V_{2ijx}' + 3\mu B_{2k}' V_{3kij}) \\ & - \tfrac{1}{4}(\mu V_{1ix}B_1 + 2V_{2ik}B_{2k})R^{-1}(\mu B_1' V_{1jx}' + 2B_{2k}' V_{2kj}), \qquad i, j = 1, 2, \cdots, m, \end{aligned} \tag{50c}$$

$$\begin{aligned} 0 = {} & \mu^2 V_{3ijkx}a_1 + \mu(V_{2ijx}A_{1k})_s + 4\mu^2 V_{4ijkq}a_{2q} + 3\mu(V_{3ijq}A_{2qk})_s \\ & - \tfrac{1}{2}(V_{0x}B_1 + V_{1q}B_{2q})R^{-1}(\mu^2 B_1' V_{3ijkx}' + 4\mu^2 B_{2q}' V_{4ijkq}) \\ & - \tfrac{1}{2}((\mu V_{1ix}B_1 + 2V_{2iq}B_{2q})R^{-1}(\mu B_1' V_{2jkx}' + 3\mu B_{2q}' V_{3qjk}))_s, \qquad i, j, k = 1, 2, \cdots, m, \end{aligned} \tag{50d[2]}$$

$$\vdots$$

where the right-hand sides of (50a), (50b), (50c), (50d), $\cdots$, are the coefficients of the z-independent terms and of the z_i, z_iz_j, $z_iz_jz_k$, $\cdots$, terms, respectively. Because of symmetry, there are $m(m+1)/2$ equations in (50c), $m(m+1)(m+2)/6$ equations in (50d) and in general, $(\prod_{k=0}^{i-1}(m+k))/i!$ equations when the coefficients of $z_{j_1}z_{j_2}\cdots z_{j_i}$, $j_1, j_2, \cdots, j_i = 1, 2, \cdots, m$, are equated.

For a simplified treatment of these equations we now exploit the presence of the small singular perturbation parameter μ. We expand each coefficient of (47) as a power series in μ:

$$V_i(x, \mu) = \sum_{j=0}^{\infty} \mu^j V_i^j(x), \qquad i = 0, 1, 2, \cdots, \tag{51}$$

where the boundary condition of V_0^j is $V_0^j(0) = 0$, $j = 0, 1, 2, \cdots$. The expressions (51) substituted into equations (50) are to satisfy them as identities in μ. Equating the coefficients of the like powers in μ, we generate sets of equations for V_i^j, $i, j = 0, 1, 2, \cdots$. The first set of equations obtained by equating the μ-independent parts in (50a), (50b), (50c), are precisely equations (20), (21), (22), respectively. Hence from the uniqueness of solutions to (20), (21), (22), conclude that

$$V_0^0 = \bar{V}_0 = L, \qquad V_1^0 = \bar{V}_1, \qquad V_2^0 = \bar{V}_2, \tag{52}$$

and $\bar{V}$ thus consists of the leading terms of V.

The second set of equations in matrix form

$$0 = V_{0x}^1 \bar{a}_1 + V_1^{1'} \bar{a}_2, \qquad V_0^1(0) = 0, \tag{53a}$$

$$0 = V_{0x}^1 \bar{A}_1 + \bar{a}_1' V_{1x}^{0'} + V_1^{1'} \bar{A}_2 + 2\bar{a}_2' V_2^1, \tag{53b}$$

$$0 = V_{2x}^0 \bar{a}_1 + \tfrac{1}{2}(V_{1x}^0 \bar{A}_1 + \bar{A}_1' V_{1x}^{0'}) + V_2^1 \bar{A}_2 + \bar{A}_2' V_2^1 + 3(V_3^0 \bar{a}_2), \tag{53c}$$

---

[2] The subscript s denotes the symmetrization operation of generalized matrices [6]. For example,

$$(V_{2ik}A_{2kj})_s = \tfrac{1}{2}(V_{2ik}A_{2kj} + V_{2jk}A_{2ki})$$

$$(V_{3ijq}A_{2qk})_s = \tfrac{1}{6}(V_{3ijq}A_{2qk} + V_{3jiq}A_{2qk} + V_{3ikq}A_{2qj} + V_{3kiq}A_{2qj} + V_{3jkq}A_{2qi} + V_{3kjq}A_{2qi}).$$

(53d) $$0 = 3(V_3^0 \bar{A}_2)_z + (V_{2x}^0 \bar{A}_1)_z,$$

obtained by equating the μ terms in (50a), (50b), (50c), (50d), respectively, involve only the unknown terms V_{0x}^1, V_1^1, V_2^1 and V_3^0. In (53) the multiplication of an $n_1 \times n_2 \times n_3$ matrix by an $n_3 \times n_4$ matrix results in an $n_1 \times n_2 \times n_4$ matrix. For convenience we suppress the last dimension of the $m \times m \times 1$ matrices $(V_{2x}^0 \bar{a}_1)$ and $(V_3^0 \bar{a}_2)$ and regard them as $m \times m$ matrices. Since $\bar{A}_2$ is stable, (53d) and (53c) can be solved sequentially for V_3^0 and V_2^1, respectively. Then V_1^1 can be solved from (53b) and its substitution into (53a) results in the partial differential equation

(54) $$0 = V_{0x}^1 \bar{a}_0 - (\bar{a}_1' V_{1x}^{0'} + 2\bar{a}_2' V_2^1)\bar{A}_2^{-1}\bar{a}_2, \qquad V_0^1(0) = 0.$$

In general, in equating the μ^i terms we obtain the $(i+1)$st set of equations involving the unknown terms V_{0x}^i, V_1^i, V_2^i, $V_3^{i-1}, \cdots, V_{i+2}^0$. The terms V_{i+1}^0, $V_i^1, \cdots, V_2^{i-1}$ are solved for sequentially and then V_0^{i-1} is to be solved from an equation similar to (41).

The main accomplishment of the nested expansions is that the first set of equations (20)–(22) can be solved independently for the first three zeroth order terms V_0^0, V_1^0, and V_2^0. Similarly, (53) and the subsequent sets of equations can be solved independently for V_0^i, $V_1^i, \cdots, V_{i+2}^0$. These equations are dependent only on x and not on z or μ. A further simplifying property is that at the first stage the equations (11), (22) for V_0^0 and V_2^0 are decoupled.

The approximation obtained by expanding V of (47), (51) to the ith set of equations is stated in the following theorem..

THEOREM 3. *Suppose that the solutions to the i-th set of equations of V exist and let V^i be the truncated series of* (47), (51) *including all the terms V_j^i up to the i-th set. Then the control*

(55) $$u_i = -\frac{1}{2}R^{-1}\left(B_1' V_x^{i\prime} + \frac{1}{\mu} B_2' V_z^{i\prime}\right)$$

is near-optimal in the sense that V^i satisfies the Hamilton–Jacobi equation (15) *to an $O(\mu^i)$ error.*

Proof. Substituting the V_j^i terms into (15) and using the first i set of equations of V, the coefficients of μ^k terms, $k < i$, in the resulting equation vanish, implying $O(\mu^i)$ near-optimality.

Thus Theorem 3 implies that u_c of (24) is an $O(\mu)$ near-optimal control because it is an $O(\mu)$ approximation of u_1 which achieves $O(\mu)$ near-optimality. In general, retaining only the μ^i terms, $k < i$, in u_i, the resulting control also is $O(\mu^i)$ near-optimal in the sense of Theorem 3.

Repeating the derivation in § 4, we can show that u_i stabilizes the full system (1) with similar stabilizing properties as u_c of (24). We first introduce the x, $\eta = z + \bar{A}_2^{-1}\bar{a}_2$ variables and consider U in (31) as a tentative Lyapunov function. The analysis is more cumbersome but results similar to Theorems 1 and 2 and Corollary 1 can be established.

6. Discussion and example. The computational advantage of the proposed procedure is that all the terms of V in (47), (51) are obtained from equations involving the slow variable x only. Moreover V_0^0 and V_2^0 are solved for independently. Explicit consideration of the initial boundary layer is avoided and it is optimally stabilized by the z variable feedback. Furthermore using the x, η variables an estimate of the domain of stability is easily obtained. Alternatively, for a stability domain to encompass a prescribed bounded set $\eta \in E \subset R^m$ a bound for μ can be determined.

Several aspects of the design procedure and the stability properties of the resulting feedback system are now illustrated by considering the optimal control problem of the second order system

$$\dot{x} = xz, \qquad \mu\dot{z} = -z + u, \tag{56}$$

with respect to the performance index

$$J = \int_0^\infty (x^4 + \tfrac{1}{2}z^2 + \tfrac{1}{2}u^2)\,dt. \tag{57}$$

Solving the reduced problem we obtain $L = V_0^0 = x^2$ and $u_0 = -x^2$. The optimally controlled reduced system (12) is $\dot{x} = -x^3$ and its unique asymptotically stable equilibrium is $x = 0$. Note that the linearization of the reduced system fails to provide any stability information at $x = 0$. Let D be the interval $[-1, 1]$, that is, $L = c_0 = 1$ at $x = \pm 1$ by assumption IV.

The pair $(A_2, B_2) = (-1, 1)$ satisfies (3) and we can solve (22) for $V_2^0 = \frac{1}{2}(\sqrt{2} - 1)$ such that $\bar{A}_2 = -\sqrt{2}$. Then the substitution of $V_0^0 = L = x^2$ and V_2^0 into (23) yields the following expressions for (24) and (16):

$$u_c = -(\sqrt{2}x^2 + (\sqrt{2} - 1)z), \tag{58}$$

$$\bar{V} = x^2 + \mu\sqrt{2}x^2 z + \mu\tfrac{1}{2}(\sqrt{2} - 1)z^2. \tag{59}$$

The resulting feedback system is

$$\dot{x} = xz, \qquad \mu\dot{z} = -\sqrt{2}x^2 - \sqrt{2}z. \tag{60}$$

This result is essentially nonlinear since the linearization of (60) at $x = 0$, $z = 0$ does not provide any stability information. After the change of variables $\eta = z + x^2$, system (60) becomes

$$\dot{x} = -x^3 + x\eta, \qquad \mu\dot{\eta} = -2\mu x^4 - (\sqrt{2} - 2\mu x^2)\eta. \tag{61}$$

Since we require $|x| \leq 1$, μ is restricted to be less than $1/\sqrt{2}$. The tentative Lyapunov function (31) is

$$U(x, \eta; \mathscr{E}) = x^2 + \tfrac{1}{2}(\sqrt{2} - 1)\mathscr{E}\eta^2. \tag{62}$$

If we require that the initial conditions of (61) be in $|x| \leq .8$, $|\eta| \leq 5$, then we must set $\mathscr{E}$ to be less than .0695 in order for the ellipse

$$S(x, \eta; \mathscr{E}) = \{x, \eta : U = x^2 + \tfrac{1}{2}(\sqrt{2} - 1)\mathscr{E}\eta^2 = 1\} \tag{63}$$

to enclose these initial conditions. Plots of S in the x, η coordinates and the x, z coordinates for $\mathscr{E} = .06$ are shown in Fig. 1. The time derivative of U with respect to (61) is

$$\dot{U} = -\left(g_1 - \frac{\mu}{\mathscr{E}}y^2\right) - \frac{\mathscr{E}}{4\mu}\xi^2 - \frac{\mathscr{E}}{\mu}M\eta^2, \tag{64}$$

where

$$\begin{aligned} g_1 &= 2x^4, \qquad & y &= 2(1 - \mathscr{E}(\sqrt{2} - 1)x^2)x^2, \\ \xi &= \eta - \frac{2\mu}{\mathscr{E}}y, \qquad & M &= \frac{7}{4} - \sqrt{2} - 2\mu(\sqrt{2} - 1)x^2. \end{aligned} \tag{65}$$

Since $\lim_{x \to 0} y^2/g_1 = 2$, assumption V is satisfied. For all x, η in the interior of S and

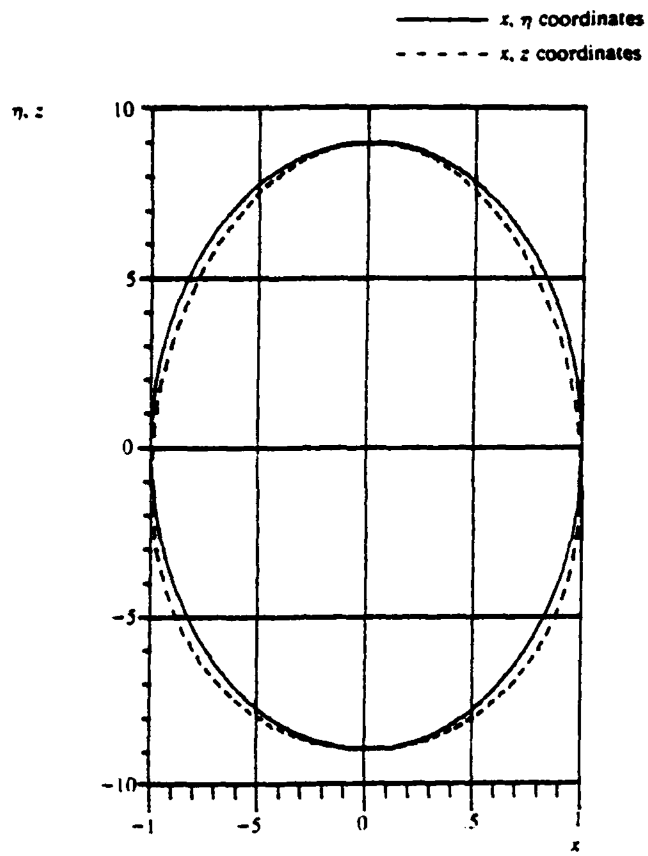

FIG. 1. *Plot of S in* (63).

$\mathscr{C} = .06$, $\dot{U}$ is negative definite for all $\mu \in (0, .03]$. Hence $x = 0$, $z = 0$ is asymptotically stable for all $|x| \leq .8$, $|z + x^2| \leq 5$ and $\mu \in (0, .03]$. Furthermore, $\bar{V}$ satisfies the Hamilton–Jacobi equation (15) with an error of $\mu 2\sqrt{2}x^2z^2$.

If we are only interested in the optimality of the reduced problem and consider the z-part as due to "system parasitics," we can apply the reduced control u_0 to (56) as $A_2 = -1$ is stable. System (56) controlled by u_0 is

$$\dot{x} = xz, \qquad \mu\dot{z} = -x^2 - z. \tag{66}$$

Transforming z to $\eta = z + x^2$, system (66) becomes

$$\dot{x} = -x^3 + x\eta, \qquad \mu\dot{z} = -2\mu x^2 - (1 - 2\mu x^2)\eta. \tag{67}$$

We use U in (62) as a Lyapunov function for (67) and the time derivative of U with respect to (67) is

$$\begin{aligned}\dot{U} = &-\left[2 - \frac{\mu}{\mathscr{C}}2(\sqrt{2}-1)(\sqrt{2}+1-\mathscr{C}x^2)^2\right]x^4 - \frac{\mathscr{C}}{\mu}\frac{\sqrt{2}-1}{2}\left[\eta - \frac{\mu}{\mathscr{C}}2(\sqrt{2}+1-\mathscr{C}x^2)x^2\right] \\ &-\frac{\mathscr{C}}{\mu}(\sqrt{2}-1)\left(\frac{1}{2} - 2\mu x^2\right)\eta^2.\end{aligned} \tag{68}$$

Thus for all x, η enclosed in S and $\mathscr{V} = .06$, $\dot{U}$ is negative definite for all $\mu \in (0, .02]$. Hence $x = 0$, $z = 0$ of (66) is asymptotically stable for all $|x| \leq .8$, $|z + x^2| \leq 5$, $\mu \in (0, .02]$.

To obtain an $O(\mu^2)$ approximation of V in the sense of Theorem 3, we solve (53) for higher order terms of V_i^j and obtain

$$u_2 = u_c - \mu 2x^2 z, \tag{69}$$

$$V^2 = \tilde{V} + \mu \frac{x^4}{\sqrt{2}} + \mu^2 x^2 z^2. \tag{70}$$

System (56) controlled by u_2 becomes

$$\dot{x} = xz, \qquad \mu \dot{z} = -\sqrt{2}x^2 - (\sqrt{2} + \mu 2x^2)z, \tag{71}$$

or, in the x, $\eta = z + x^2$ variables,

$$\dot{x} = -x^3 + x\eta, \qquad \mu\dot{\eta} = -\sqrt{2}\eta, \tag{72}$$

which is globally asymptotically stable for all $\mu > 0$. Furthermore, V^2 satisfies (15) with an error of $\mu^2(8x^4z^2 + 2x^2z^3)$.

7. Conclusions. A nested power series expansion method has been proposed for solving the optimal control problem of a class of nonlinear singularly perturbed systems. The terms in the expansion V are obtained from equations involving only the slow variable x. In addition, V_0^0 and V_2^0 are solved for independently. Explicit consideration of the initial boundary layer is avoided and it is optimized by the z variable feedback. Sufficient conditions are obtained such that feedback controls using truncated series stabilize the nonlinear systems and the stability domain can encompass large initial conditions of z. These truncated controls can achieve near-optimality of $O(\mu)$, $O(\mu^2)$, etc. In particular, an $O(\mu)$ near-optimal feed-back control can be implemented without knowing the value of the small parameter μ. The results apply to essentially nonlinear problems.

Appendix. Substituting (23) into (20) and rearranging yields

$$0 = X_1 + V_{0x}X_2 - \tfrac{1}{4}V_{0x}X_3V_{0x}',$$

where

$$X_1 = p - (s' + 2a_2'V_2)\tilde{A}_2^{-1}a_2 - (\tfrac{1}{2}s' + a_2'V_2)\tilde{A}_2^{-1}B_2R^{-1}B_2'\tilde{A}_2'^{-1}(\tfrac{1}{2}s + V_2a_2),$$

$$X_2 = \tilde{a}_2 + \tilde{B}_0R^{-1}B_2'\tilde{A}_2'^{-1}(\tfrac{1}{2}s + V_2a_2), \qquad X_3 = \tilde{B}_0R^{-1}\tilde{B}_0',$$

$$\tilde{a}_0 = a_1 - (A_1 - B_1R^{-1}B_2'V_2)\tilde{A}_2^{-1}a_2, \qquad \tilde{B}_0 = B_1 - (A_1 - B_1R^{-1}B_2'V_2)\tilde{A}_2^{-1}B_2,$$

$$\tilde{A}_2 = A_2 - B_2R^{-1}B_2V_2,$$

and the superscript 0 in V_{0x}^0 and V_2^0 has been dropped. Let $H = I + R^{-1}B_2'V_2\tilde{A}_2^{-1}B_2$. Then $H^{-1} = I - R^{-1}B_2'V_2A_2^{-1}B_2$ and $H'^{-1}RH^{-1} = R + B_2'A_2'^{-1}QA_2^{-1}B_2 = R_0$. Thus $\tilde{B}_0 = B_1H - A_1\tilde{A}_2^{-1}B_2 = B_0H$. Hence $X_3 = B_0R_0^{-1}B_0'$. Also,

$$\begin{aligned}
X_2 &= a_0 + B_0R_0^{-1}[(R + B_2'A_2'^{-1}QA_2^{-1}B_2)R^{-1}B_2'V_2\tilde{A}_2^{-1} + B_2'A_2'^{-1}V_2]a_2 \\
&\quad + \tfrac{1}{2}B_0R_0^{-1}B_2'A_2'^{-1}s \\
&= a_0 + B_0R_0^{-1}B_2'A_2'^{-1}(A_2'V_2 + QA_2^{-1}B_2R^{-1}B_2'V_2 + V_2A_2 - V_2B_2R^{-1}B_2'V_2)\tilde{A}_2^{-1} \\
&\quad + \tfrac{1}{2}B_0R_0^{-1}B_2'A_2'^{-1}s \\
&= a_0 - B_0R_0^{-1}s_0.
\end{aligned}$$

Furthermore, $\bar{A}_2^{-1}B_2R^{-1}B_2'\bar{A}_2'^{-1} = A_2^{-1}B_2HR^{-1}H'B_2'A_2'^{-1} = A_2^{-1}B_2R_0^{-1}B_2'A_2'^{-1}$ and

$$\begin{aligned}\bar{A}_2^{-1} &= A_2^{-1} + A_2^{-1}B_2R^{-1}B_2'V_2\bar{A}_2^{-1}\\ &= A_2^{-1} + A_2^{-1}B_2R_0^{-1}B_2'(V_2 + A_2'^{-1}QA_2^{-1}B_2R^{-1}B_2'V_2)\bar{A}_2^{-1}\\ &= A_2^{-1} - A_2^{-1}B_2R_0^{-1}B_2'A_2'^{-1}QA_2^{-1} - A_2^{-1}B_2R_0^{-1}B_2'A_2'^{-1}V_2.\end{aligned}$$

Thus X_1 becomes

$$\begin{aligned}X_1 = p - s'A_2^{-1}a_2 + s'A_2^{-1}B_2R_0^{-1}B_2'A_2'^{-1}QA_2^{-1} - \tfrac{1}{2}s'A_2^{-1}B_2R_0^{-1}B_2'A_2'^{-1}s\\ + a_2'V_2A_2^{-1}B_2R_0^{-1}B_2'A_2'^{-1}V_2a_2 - a_2'(V_2\bar{A}_2^{-1} + \bar{A}_2'^{-1}V_2)a_2.\end{aligned}$$

But

$$\begin{aligned}V_2\bar{A}_2^{-1} + \bar{A}_2'^{-1}V_2 &= -V_2A_2^{-1} - A_2'^{-1}V_2 + V_2A_2^{-1}B_2R_0^{-1}B_2'A_2'^{-1}QA_2^{-1}\\ &\quad + A_2'^{-1}QA_2^{-1}B_2R_0^{-1}B_2'A_2^{-1}V_2\\ &\quad + 2V_2A_2^{-1}B_2R_0^{-1}B_2'A_2'^{-1}V_2\\ &= A_2'^{-1}QA_2^{-1} - A_2'^{-1}V_2B_2R^{-1}B_2'V_2A_2^{-1}\\ &\quad + (V_2 + A_2'^{-1}Q)A_2^{-1}B_2R_0^{-1}B_2'A_2'^{-1}(V_2 + QA_2^{-1})\\ &\quad + V_2A_2^{-1}B_2R_0^{-1}B_2'A_2'^{-1}V_2 - A_2'^{-1}QA_2^{-1}B_2R_0^{-1}B_2'A_2'^{-1}QA_2^{-1},\end{aligned}$$

and

$$\begin{aligned}A_2'^{-1}V_2B_2R^{-1}B_2'V_2A_2^{-1} = [-(V_2 + A_2'^{-1}Q_2)A_2^{-1}\\ + A_2'^{-1}V_2B_2R^{-1}B_2'V_2A_2^{-1}]B_2R^{-1}B_2'V_2A_2^{-1},\end{aligned}$$

that is,

$$\begin{aligned}A_2'^{-1}V_2B_2R^{-1}B_2'V_2A_2^{-1} &= -(V_2 + A_2'^{-1}Q_2)A_2^{-1}B_2R^{-1}B_2'V_2\bar{A}_2^{-1}\\ &= (V_2 + A_2'^{-1}Q)A_2^{-1}B_2R^{-1}B_2'A_2'^{-1}(QA_2^{-1} + V_2),\end{aligned}$$

implying $X_1 = p_0 - s_0'R_0^{-1}s_0$. Hence elimination of V_1 from (20) yields the Hamilton-Jacobi equation (11) of the reduced problem.

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A TWO STAGE LYAPUNOV-BELLMAN FEEDBACK DESIGN

OF A CLASS OF NONLINEAR SYSTEMS*

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ABSTRACT

The composite control proposed in an earlier paper for a class of singularly perturbed nonlinear systems is now shown to possess properties essential for near-optimal feedback design. It asymptotically stabilizes the desired equilibrium and produces a finite cost which tends to the optimal cost for a slow problem as the singular perturbation parameter tends to zero. Thus the well-posedness of the full regulator problem is established. The stability results are also applicable to two-time scale systems which are not singularly perturbed, and the paper does not assume the knowledge of singular perturbation techniques.

1. INTRODUCTION

A conceptually appealing framework for simultaneous stabilization and optimization of feedback systems consists in requiring that the Bellman's optimal value function be in the same time a Lyapunov function. This has been elegantly achieved in Kalman's linear regulator theory as a culmination of earlier efforts by Lur'e, Krasovski, Bellman, and many others. However, in dealing with nonlinear problems, the Lyapunov-Bellman concept has serious drawbacks. One of them, the notorious "curse of dimensionality," is frustrating to practitioners. Another one, the question of existence and differentiability of the optimal value function, disturbs the analytically minded. Similar difficulties appear on the Lyapunov side because of the lack of general methods for constructing Lyapunov functions. Nevertheless, the optimum stabilization continues to be one of the fertile concepts stimulating the development of numerical and analytical methods for nonlinear regulator design [4-7]. Most analytical methods assume that the linear part of the system is dominant and design a linear regulator as a first approximation, to be subsequently corrected by series expansions [5,7]. This approach is applicable to many nonlinear systems, but it also has important limitations. First, it is not directly applicable if the linear part is not dominant, second, calculation of expansions increases the dimensionality difficulties, and, third, ill-conditioning due to fast and slow phenomena remains.

The two-time-scale approach presented in this paper avoids linearization and directly addresses the dimensionality and ill-conditioning difficulties. Its philosophy can simply be stated as follows: "Design the slow subsystem first, by assuming that the fast subsystem has already reached its steady state. Then design the fast subsystem for a set of constant values of the states of the slow subsystem. Combine the two designs by guaranteeing stability and near-optimality properties of the resulting system." The method proposed in [3] and developed here implements this design philosophy on the systems nonlinear in slow variables and linear in fast variables and control.

The class of systems considered is assumed to be in the standard singular perturbation form exhibiting explicitly a parameter μ, which can be interpreted as the order of magnitude of the ratio of the slow and fast state speeds. Although this form simplifies the definition of the subsystems, the paper does not require any familiarity with singular perturbation techniques. The slow and fast subsystems can be considered as postulates whose validity is subsequently demonstrated by the properties of the actual system controlled by the proposed composite control. Since the proofs of these properties are elementary and make use of only Bellman's principle of optimality and Lyapunov-type arguments, the paper can be read with no more than a basic background in control theory. The steps of the design procedure are presented on a simple example. The method of this paper is radically different from the finite interval trajectory optimization results of [8,9] because of the stability and boundedness requirements fundamental in infinite time problems, which require feedback solutions.

2. FULL PROBLEM

The problem considered is to optimally control the nonlinear system

$$\dot{x} = a_1(x) + A_1(x)z + B_1(x)u, \qquad x(0) = x_o \qquad (2.1a)$$

$$\mu\dot{z} = a_2(x) + A_2(x)z + B_2(x)u, \qquad z(0) = z_o \qquad (2.1b)$$

with respect to the cost function

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$$J = \int_0^\infty [p(x)+s'(x)z+z'Q(x)z+u'R(x)u]dt \tag{2.2}$$

where $\mu>0$ is the singular perturbation parameter, x, z are n-, m-dimensional states, respectively, u is an r-dimensional control and the prime denotes a transpose. Regulator problems for systems linear in the control and nonlinear in the state have been considered earlier [6]. Here the system is also linear in the fast state variable z, as is for example, the case with models of dc motors and synchronous machines [2]. We make an assumption which in addition to differentiability and positivity properties of terms in (2.1), (2.2) also guarantees that the origin is the desired equilibrium.

Assumption I: There exists a domain $D \subset R^n$, containing the origin as an interior point, such that for all $x \in D$ functions a_1, a_2, A_1, A_2, A_2^{-1}, B_1, B_2, p, s, R, and Q are differentiable with respect to x; a_1, a_2, p, and s are zero only at $x=0$; Q and R are positive definite matrices for all $x \in D$; the scalar $p+s'z+z'Qz$ is a positive definite function of its arguments x and z, that is, it is positive except for $x=0$, $z=0$ where it is zero.

An approach to the full problem (2.1), (2.2) would be to assume that a differentiable optimal value function $V(x,z,\mu)$ exists satisfying Bellman's principle of optimality

$$0 = \min_u [p+s'z+z'Qz+u'Ru+V_x(a_1+A_1z+B_1u)$$

$$+ \frac{1}{\mu} V_z(a_2+A_2z+B_2u)] \tag{2.3}$$

where V_x, V_z denote the partial derivatives of V. Since the control minimizing (2.3) is

$$u_m = -\frac{1}{2} R^{-1}(B_1'V_x' + \frac{1}{\mu} B_2'V_z') \tag{2.4}$$

the problem would consist in solving the Hamilton-Jacobi equation

$$0 = p+x'z+z'Qz+V_x(a_1+A_1z) + \frac{1}{\mu} V_z(a_2+A_2z)$$

$$-\frac{1}{4}(V_xB_1 + \frac{1}{\mu} V_zB_2)R^{-1}(B_1'V_x' + \frac{1}{\mu} B_2'V_z'),$$

$$V(0,0,\mu) = 0 \tag{2.5}$$

This would be a difficult task even for well behaved nonlinear systems. Due to the presence of $\frac{1}{\mu}$ terms in (2.5), the difficulties with singularly perturbed systems (2.1) increase. The method of this paper avoids these difficulties. In contrast we take advantage of the fact that as $\mu \to 0$ the slow and the fast phenomena in (2.1) separate. We do not deal with the problem (2.1), (2.5) directly. Instead we define two separate lower dimensional subproblems, slow and fast. The assumption about existence and differentiability of the optimal value function is then made only for the slow subproblem, while the assumption for the fast subproblem is similar to those made for linear quadratic problems. The solutions of the two subproblems are combined into a composite control whose stabilizing and near optimal properties are the main subject of the paper.

3. SLOW SUBPROBLEM

Because of the presence of μ, system (2.1) exhibits a "boundary layer," that is, a fast transient in the variable z, after whose decay both x and z vary slowly with time. Setting $\mu = 0$ the fast transient is neglected, that is,

$$\dot{x}_s = a_1(x_s) + A_1(x_s)z_s + B_1(x_s)u_s, \quad x_s(0) = x_o \tag{3.1a}$$

$$0 = a_2(x_s) + A_2(x_s)z_s + B_2(x_s)u_s, \tag{3.1b}$$

and, since A_2^{-1} is assumed to exist,

$$z_s(x_s) = -A_2^{-1}(a_2 + B_2u_s) \tag{3.2}$$

is eliminated from (3.1a) and (2.2). Then the slow subproblem is to optimally control the slow subsystem

$$\dot{x}_s = a_o(x_s) + B_o(x_s)u, \quad x_s(0) = x_o \tag{3.3}$$

with respect to

$$J_s = \int_0^\infty [p_o(x_s)+2s_o'(x_s)u_s + u_s'R_o(x_s)u_s]dt \tag{3.4}$$

where

$$a_o = a_1 - A_1A_2^{-1}a_2$$

$$B_o = B_1 - A_1A_2^{-1}B_2$$

$$p_o = p - s'A_2^{-1}a_2 + a_2'A_2'^{-1}QA_2^{-1}a_2$$

$$s_o = B_2'A_2'^{-1}(QA_2^{-1}a_2 - \frac{1}{2} s)$$

$$R_o = R + B_2'A_2'^{-1}QA_2^{-1}B_2. \tag{3.5}$$

We note that $x_s = 0$ is the desired equilibrium of the slow subsystem (3.3) for all $x_s \in D$, since, in view of Assumption I, $a_o(0) = 0$ and the integrand in (3.4) is positive definite in x_s and u_s, that is

$$p_o(x_s) + 2s_o'(x_s)u_s + u_s'R_o(x_s)u_s > 0,$$

$$x_s \neq 0,\ u_s \neq 0 \tag{3.6}$$

Our crucial Assumption II concerns the existence of the optimal value function $L(x_s)$ for the slow subproblem satisfying the optimality principle

$$0 = \min_{u_s}[p_o(x_s) + 2s_o'(x_s)u_s + u_s'R_o(x_s)u_s$$

$$+ L_x(a_o(x_s)+B_o(x_s)u_s)] \tag{3.7}$$

where L_x denotes the derivative of L with respect to its argument x_s. The elimination of the minimizing control

$$u_s = -R^{-1}(s_o + \frac{1}{2} B_o'L_x') \tag{3.8}$$

from (3.7) results in the Hamilton-Jacobi equation

$$0 = (p_o - s_o' R_o^{-1} s_o) + L_x(a_o - B_o R_o^{-1} s_o)$$

$$-\frac{1}{4} L_x B_o R_o^{-1} B_o' L_x', \qquad L(0) = 0, \tag{3.9}$$

where, due to (3.6), $p_o - s_o' R_o^{-1} s_o$ is positive definite in D.

Assumption II: For all $x_s \in D$ equation (3.9) has a unique differentiable positive definite solution $L(x_s)$ with the property that positive constants k_1, k_2, k_3, k_4 exist such that

$$k_1 L_x L_x' \le -L_x \bar{a}_o \le k_2 L_x L_x' \tag{3.10}$$

$$k_3 \bar{a}_o' \bar{a}_o \le -L_x \bar{a}_o \le k_4 \bar{a}_o' \bar{a}_o. \tag{3.11}$$

Assumption II allows $L(x_s)$ to be used as a Lyapunov function guaranteeing the asymptotic stability of $x_s = 0$ for the slow subsystem (3.3) controlled by (3.8), that is for the feedback system

$$\dot{x}_s = a_o - B_o R_o^{-1}(s_o + \frac{1}{2} B_o' L_x') = \bar{a}_o(x_s). \tag{3.12}$$

It also guarantees that D belongs to the region of attraction of $x_s = 0$. For convenience we will take a level surface $L(x_s) = c_o$ to be the boundary of D. It is pointed out that Assumption II does not guarantee the exponential stability. This would be unnecessarily restrictive and would exclude some common slow subsystems such as $\dot{x}_s = -x_s^3$.

Conditions (3.10), (3.11) characterize the slow subproblem solution L by bounding the rate $\dot{L} = L_x \bar{a}_o$ at which it decays to zero along the trajectories of (3.12). These bounds encompass a larger class of *nonlinear systems* than do some more common conditions based on exponential stability of linearized models [5,7]. When the solution L of the slow subproblem is known, conditions (3.10), (3.11) are readily verifiable. This is how they are used in our two stage design. We first solve the slow subproblem by one of the existing methods, taking advantage of the fact that its dimensionality is lower than that of the full problem. At the end of this stage L is known and (3.10), (3.11) are checked. If they are satisfied, we proceed to the second stage, that is we solve the fast subproblem.

4. FAST SUBPROBLEM

To motivate the formulation of the fast subproblem we observe that x being predominantly slow means that only an $O(\mu)$ error is made by replacing x with x_s, or vice versa. Thus, when we subtract (3.1b) from (2.1b) we obtain the system

$$\mu(\dot{z}-\dot{z}_s) = A_2(x)(z-z_s) + B_2(x)(u-u_s) - \mu\dot{z}_s \tag{4.1}$$

which can be further simplifed by neglecting the r.h.s. $O(\mu)$ term $-\mu\dot{z}_s$. Defining $z_f = z - z_s$ and $u_f = u - u_s$ the system (4.1) becomes

$$\mu\dot{z}_f = A_2(x) z_f + B_2(x) u_f, \qquad z_f(0) = z_o - z_s(0). \tag{4.2}$$

Following a similar reasoning we define

$$J_f = \int_0^\infty (z_f' Q(x) z_f + u_f' R(x) u_f)\,dt. \tag{4.3}$$

Now (4.2) and (4.3) constitute our fast subproblem for each fixed $x \in D$. It has the familiar linear quadratic form.

Assumption III: For every fixed $x \in D$

$$\mathrm{rank}[B_2, A_2 B_2, \ldots, A_2^{m-1} B_2] = m. \tag{4.4}$$

Alternatively a less demanding stabilizability assumption can be made. Recalling also that $R(x) > 0$, $Q(x) > 0$ (see Assumption I), we obtain, for each $x \in D$, the optimal solution of the fast subproblem

$$u_f(z_f, x) = -R^{-1}(x) B_2'(x) K(x) z_f \tag{4.5}$$

where K(x) is the positive definite solution of the x-dependent Riccati equation

$$0 = KA_2 + A_2' K - KB_2 R^{-1} B_2' K + Q. \tag{4.6}$$

The control (4.5) is stabilizing in the sense that the fast feedback system

$$\mu\dot{z}_f = (A_2 - B_2 R^{-1} B_2' K) z_f \triangleq \bar{A}_2(x) z_f \tag{4.7a}$$

has the property that

$$\mathrm{Re}\lambda[\bar{A}_2(x)] < 0, \qquad \forall x \in D. \tag{4.7b}$$

5. THE COMPOSITE CONTROL

Compared to the full problem (2.1) - (2.5), the subproblems are easier to solve due to the fact that the fast subproblem, although parameter dependent, is a linear regulator problem and the slow subproblem, although nonlinear, is of a lower order than the full problem. However, the controls u_s and u_f are applicable to the slow and the fast subsystems, respectively, which do not exist in reality. Our goal is to use u_s and u_f to control the actual full system (2.1). To accomplish this we now form a 'composite' control $u_c = u_s + u_f$, in which x_s is replaced by x, and z_f by $z + A_2^{-1}(a_2 + B_2 u_s(x))$. Thus the composite control is

$$u_c(x,z) = u_s(x) - R^{-1} B_2' K(z + A_2^{-1}(a_2 - B_2 u_s(x)))$$

$$= -R_o^{-1}(s_o + \frac{1}{2} B_o' L_x') - R^{-1} B_2' K(z + \bar{A}^{-1} \bar{a}_2) \tag{5.1}$$

where

$$\bar{a}_2(x) = a_2 - \frac{1}{2} B_2 R^{-1}(B_1' L_x' + B_2' V_1'), \qquad \bar{a}_2(0) = 0$$

$$V_1' = -(s' + 2a_2' K + L_x \bar{A}_1)\bar{A}_2^{-1}$$

$$\bar{A}_1 = A_1 - B_1 R^{-1} B_2' K. \tag{5.2}$$

Note that u_c is independent of μ, which simplifies the design procedure when μ is a small but unknown parameter.

For u_c to be a meaningful feedback control of the system (2.1), it must first of all be a stabilizing control. Furthermore for u_c to be a candidate for the optimization of (2.2), the full system (2.1) controlled by u_c must result in a bounded cost (2.2). As $\mu \to 0$, the full cost should approach the cost of the slow subproblem. This would imply that u_c is a near-optimal control and that the regulator problem is well-posed. The boundedness and near-optimality results in the subsequent sections are new, while the stability result is essentially the same as [3], but in a new simpler form.

6. STABILITY

The full system (2.1) controlled by the composite control (3.1) is

$$\dot{x} = a_1 + A_1 z + B_1 u_c = \bar{a}_1(x) + \bar{A}_1(x) z, \qquad x(0) = x_o$$

$$\mu\dot{z} = a_2 + A_2 z + B_2 u_c = \bar{a}_2(x) + \bar{A}_2(x) z, \qquad z(0) = z_o \tag{6.1}$$

where

$$\bar{a}_1 = a_1 - \frac{1}{2} B_1 R^{-1} (B_1' L_x' + B_2' V_1'), \qquad \bar{a}_1(0) = 0, \tag{6.2}$$

and has the following stability property.

<u>Theorem 6.1</u>: If Assumptions I - III are satisfied, there exists a $\mu^* > 0$ such that the equilibrium $x = 0$, $z = 0$ of system (6.1) is asymptotically stable for all $\mu \in (0, \mu^*]$.

<u>*Proof*</u>: Introducing

$$z_f = z + \bar{A}_2^{-1} \bar{a}_2, \quad z_f(0) = z_o + \bar{A}_2^{-1}(x_o)\bar{a}_2(x_o) = z_{fo} \tag{6.3}$$

and $F(x) = (\bar{A}_2^{-1} \bar{a}_2)_x$, we rewrite (6.1) as

$$\dot{x} = \bar{a}_o + \bar{A}_1 z_f, \tag{6.4a}$$

$$\mu\dot{z}_f = \mu F(x)\bar{a}_o + (\bar{A}_2 + \mu F(x)\bar{A}_1) z_f. \tag{6.4b}$$

Observing that (6.4a) has the form of the slow subsystem (3.12) with the additional forcing term $\bar{A}_1 z_f$ and that (6.4b) is an $O(\mu)$ perturbation of the fast subsystem (4.2) controlled by the fast control u_f (4.5), that is of (4.7a), we use the sum of the slow and the fast Lyapunov functions

$$v(x, z_f, \mu) = L(x) + \alpha\mu z_f' K(x) z_f \tag{6.5}$$

as a tentative Lyapunov function for (6.4) where α is a positive scalar to be chosen. Since $L(x) > 0$ and $K(x) > 0$ in D, v is positive definite for all $x \in D$, $z_f \in R^m$ and $\mu > 0$. The proof consists in showing that the time derivative $\dot{v}$ of v with respect to (6.4) is negative definite. After completing the squares $\dot{v}$ can be put in the form

$$\dot{v} = -g(x,\mu) - \frac{1}{2} \alpha \xi' Q(x) \xi - \alpha z_f' M(x, z_f, \mu) z_f \tag{6.6}$$

where

$$g = -L_x \bar{a}_o - y' Q^{-1} y / 2\alpha$$

$$y = \bar{A}_1' L_x' + 2\alpha\mu K F \bar{a}_o$$

$$\xi = z_f - Q^{-1} y/\alpha \tag{6.7}$$

$$M = Q/2 + K B_2 R^{-1} B_2' K - \mu(K F \bar{A}_1 + \bar{A}_1' F' K) - \mu\dot{K}$$

Using the fact that x-dependent quantities in g are bounded for $x \in D$, that is,

$$\|\bar{A}_1 Q^{-1} \bar{A}_1'\| \le k_5, \quad \|\bar{A}_1 Q^{-1} K F\| \le k_6, \quad 4\|F' K Q^{-1} K F\| \le k_7, \tag{6.8}$$

and recalling that $k_1 L_x L_x' \le -L_x \bar{a}_o$, $k_3 \bar{a}_o' \bar{a}_o \le -L_x \bar{a}_o$, see (3.10), (3.11), we obtain

$$y' Q^{-1} y \le (k_5 + 3\alpha\mu k_6) L_x L_x' + (3\alpha\mu k_6 + \alpha^2 \mu^2 k_7) \bar{a}_o' \bar{a}_o$$

$$\le -\sigma L_x \bar{a}_o \tag{6.9}$$

where

$$\sigma(\alpha\mu) = k_1^{-1}(k_5 + 3\alpha\mu k_6) + k_3^{-1}(3\alpha\mu k_6 + \alpha^2\mu^2 k_7). \tag{6.10}$$

It follows from (6.9) that

$$g \ge -L_x \bar{a}_o (1 - \sigma/2\alpha) \tag{6.11}$$

and hence, to make g positive definite, it is sufficient to choose $\alpha > \sigma/2$. A convenient choice is to take α to be the value of σ when $\alpha\mu = 1$. Since σ is a monotonically increasing function of $\alpha\mu \ge 0$, this choice implies that

$$g \ge -\frac{1}{2} L_x \bar{a}_o > 0 \quad \forall \mu \in (0, \frac{1}{\alpha}]. \qquad (6.\ldots$$

To complete the proof we need to show that M is also positive definite. Noting that the first two terms of M are positive definite we now establish that they dominate the last two terms, which are small for μ sufficiently small. Using the bounds (6.8) and

$$\|\dot{K}\| = \|K_x \dot{x}\| \le \|K_x\| \, \|\bar{a}_o + \bar{A}_1 z_f\| \tag{6.13}$$

we conclude that there exist positive constants μ_1 and k_8 such that

$$M \ge \frac{1}{4} (Q + K B_2 R^{-1} B_2' K) \qquad (6.1\ldots$$

holds for all $x \in D$, all z_f such that $\|z_f\| \le k_8$, and all $\mu \in (0, \mu_1]$. Thus for all

$$\mu \in (0, \mu^*], \qquad \mu^* = \min\left(\frac{1}{\alpha}, \mu_1\right) \tag{6.15}$$

the derivative $\dot{v}$ of v in (6.5) for system (6.1), or, equivalently, for system (6.4), is negative definite and hence the equilibrium $x = 0$, $z = 0$, is asymptotically stable.

From this proof we can readily obtain an estimate of the region of attraction of $x = 0$, $z = 0$. A well known estimate is the set of points x, z encompassed by the largest closed surface $v(x, z, \mu) = c_o$ for which $\dot{v}$ is negative definite.

To each fixed $\mu\in(0,\mu^*]$ there corresponds one such set denoted by S_μ. All S_μ sets contain all $x\in D$, but differ in the magnitudes of z, because, as it can be inferred from the above proof, the larger μ is, the smaller z_f is allowed. Thus the set corresponding to the largest value of μ, that is to μ^*, is the largest set and is denoted by S^*. Since this set is the intersection of all S_μ sets, it can serve as a common estimate for the regions of attraction for all values of $\mu\in(0,\mu^*]$. A proof of this fact consists of the calculations analogous to those leading to (6.6) through (6.15), but this time for ν with μ fixed at $\mu=\mu^*$, that is for $\nu(x,z,\mu^*)$, rather than for $\nu(x,z,\mu)$. Omitting these calculations we state the result in the form useful for our subsequent analysis.

Corollary 6.2: Under the assumptions of Theorem 6.1 there exist positive constants μ^* and c_o such that the set

$$S^*(x,z) = \{x,z: \quad \nu(x,z,\mu^*) < c_o\} \tag{6.16}$$

belongs to the region of attraction of $x=0$, $z=0$ for all $\mu\in(0,\mu^*]$, that is all trajectories of (6.1) originating in S^* at $t=0$ remain in S^* for all $t>0$ and converge to $x=0$, $z=0$, as $t\to\infty$.

7. BOUNDEDNESS OF J

Asymptotic stability of an equilibrium at the origin is not sufficient to guarantee that an integral of the type (2.2) will be finite along the trajectories asymptotically converging to this equilibrium. For example, when the control $u=-x^2-x^5$ is applied to the system $\dot{x} = x^2+u$, then the equilibrium $x=0$ of $\dot{x} = -x^5$ is asymptotically stable. However the solutions for $x(0) = x_o \neq 0$ are

$$x(t) = \text{sign}(x_o)\,(4t + (x_o)^{-4})^{-1/4}, \tag{7.1}$$

and hence the cost

$$J = \int_0^\infty (x^4 + 1/2\, u^2)dt \tag{7.2}$$

is infinite. Thus it is not sufficient that our composite control be only a stabilizing control. To qualify as a candidate for near-optimality u_c must also produce a bounded J. To show that this is the case we use the following lemma from [1], which is implicit in [4,6].

Lemma 7.1: Suppose that system (2.1) controlled by $u(x,z)$ has $x=0$, $z=0$ as its asymptotically stable equilibrium for all $x_o,z_o\in S$. Let this fact be established by a positive definite Lyapunov function $q(x,z)$, whose derivative $\dot{q}(x,z)$ is negative definite in S. If there exists a ball β centered at $x=0$, $z=0$ such that for all $x,z\in\beta$,

$$p + s'z + z'Qz + u'Ru + \dot{q} \le 0, \tag{7.3}$$

then the cost (2.2) is finite along all the trajectories which originate in S and is bounded from above by q.

Proof: Let t_β be the instant when a trajectory τ originating from $x_o,z_o\in S$ enters the ball β through x_β, z_β for the last time and stays in β thereafter. The part of the cost along τ over the finite interval $[0,t_\beta]$ is obviously finite. Denoting the remaining part of the cost over (t_β,∞) by J_β and integrating (7.3) from t_β to ∞ we obtain

$$J_\beta + [q(0,0)-q(x_\beta,z_\beta)] \le 0 \tag{7.4}$$

which in view of $q(0,0)=0$ and the fact that $q(x_\beta, z_\beta)$ is finite, proves that J_β is bounded.

To apply this lemma we substitute (5.1) and (6.3) for u_c and z, respectively into

$$J_c = \int_0^\infty (p+s'z+z'Qz+u_c'Ru_c)dt = \int_0^\infty f_c(x,z)dt \tag{7.5}$$

and rewrite the integrand as

$$f_c(x,z) = -L_x\bar{a}_o - s_1'z_f + z_f'(Q + KB_2R^{-1}B_2'K)\, z_f$$

$$= f(x,z_f) \tag{7.6}$$

where

$$s_1 = s + KB_2R^{-1}(B_1'L_x' + B_2'V_1)$$

$$+ 2(Q + KB_2R^{-1}B_2'\, K)\, \bar{A}_2^{-1}\bar{a}_2. \tag{7.7}$$

It is important to note that the dependence on z_f in (7.6) is indicated explicitly, that is, the term $L_x\bar{a}_o$ is independant of z_f. Furthermore, $f(x, -z_f)>0$ because $f(x,z_f) > 0$ for all $x\in D$ and $z_f\in R^m$, $x\neq 0$, $z_f\neq 0$.

Theorem 7.2: Under Assumptions I - III, the composite control u_c produces a cost J_c which is bounded from above by 4ν for all $\mu\in(0,\mu^*]$.

Proof: From (6.12) and (6.15) we obtain

$$f(x,z_f) + 4\dot{\nu} \le -f(x, -z_f) \le 0. \tag{7.8}$$

From Theorem 6.1 we know that 4ν is a Lyapunov function for system (6.4) and we use it as q in Lemma 7.1, which in view of (7.4) completes the proof.

8. NEAR OPTIMALITY

The question can now be posed whether u_c, being a stabilizing control which produces a bounded cost, is also near optimal in the sense that as $\mu\to 0$ the cost J_c tends to the optimal cost for $\mu=0$, that is the optimal cost $L(x)$ of the reduced problem. This question is answered by expressing J_c as

$$J_c(x,z,\mu) = L(x) + \mu V_1'(x)z + \mu z'K(x)z + \mu J_4(x,z,\mu) \tag{8.1}$$

where the first two μ-terms are suggested by the linear-quadratic form of the fast subproblem. If we prove that J_4 remains bounded as $\mu\to 0$, this will guarantee that $J_c(x,z,\mu)\to L(x)$.

Theorem 8.1: Under Assumptions I - III, the composite control produces cost (8.1) in which J_4 remains bounded as $\mu\to 0$.

Proof: Cost $J_c(x,z,\mu)$ of system (2.1) controlled by u_c satisfies partial differential equation

$$p + s'z + z'Qz + u_c'Ru_c + (J_c)_x\,(a_1 + A_1 z + B_1 u_c)$$

$$+ (J_c)_z(a_2 + A_2 z + B_2 u_c)/\mu = 0, \qquad (8.2)$$

$$J_c(0,0,\mu) = 0.$$

We have shown in [3] that the substitution of (8.1) into (8.2) and the use of (3.9), (4.6), and (5.2), reduce (8.2) to

$$J_{4x}(\bar{a}_1 + \bar{A}_1 z) + \frac{1}{\mu} J_{4z}(\bar{a}_2 + \bar{A}_2 z) =$$

$$-(V_1' z + z'Kz)_x(\bar{a}_1 + \bar{A}_1 z), \qquad J_4(0,0,\mu) = 0. \qquad (8.3)$$

This expression, and the fact following from Theorem 7.1 that μJ_4 is bounded, are used in the Appendix to complete the proof.

In addition to the near optimality of the composite control, Theorem 8.1 also shows that the full regulator problem is well posed in the sense that the same cost results from neglecting μ in the system model and then applying the control u_s to (3.3), or first applying the control u_c to (2.1) and then neglecting μ.

9. TWO STAGE DESIGN

The steps of the proposed two stage design will be presented on a simple example of the system

$$\dot{x} = -\frac{3}{4}x^3 + z \qquad (9.1a)$$

$$\mu\dot{z} = -z + u \qquad (9.1b)$$

and the cost functional

$$J = \int_0^\infty (x^6 + \frac{3}{4}z^2 + \frac{1}{4}u^2)\,dt. \qquad (9.2)$$

Step 1: The slow subproblem

$$\dot{x}_s = -\frac{3}{4}x_s^3 + u_s \qquad (9.3)$$

$$J_s = \int_0^\infty (x_s^6 + u_s^2)\,dt \qquad (9.4)$$

consists in solving the Hamilton-Jacobi equation

$$L_x = \frac{dL}{dx_s} = x_s^3, \qquad L(0) = 0 \qquad (9.5)$$

which yields

$$L = \frac{1}{4}x_s^4, \qquad u_s = -\frac{1}{2}x_s^3, \qquad \dot{x}_s = -\frac{5}{4}x_s^3. \qquad (9.6)$$

Step 2: Testing the conditions (3.10), (3.11)

$$k_1 x_s^6 \le \frac{5}{4}x_s^6 \le k_2 x_s^6, \qquad (9.7)$$

$$\frac{25}{16}k_3 x_s^6 \le \frac{5}{4}x_s^6 \le \frac{25}{16}k_4 x_s^6, \qquad (9.8)$$

we see that they are satisfied by

$$k_1 = k_2 = \frac{5}{4}, \qquad k_3 = k_4 = \frac{4}{5}. \qquad (9.9)$$

Step 3: The fast subproblem

$$\mu\dot{z}_f = -z_f + u_f \qquad (9.10)$$

$$J_f = \int_0^\infty (\frac{3}{4}z_f^2 + \frac{1}{4}u_f^2)\,dt \qquad (9.11)$$

is in this case independent of x and its solution is

$$K = \frac{1}{4}, \qquad u_f = -z_f, \qquad \mu\dot{z}_f = -2z_f. \qquad (9.12)$$

Step 4: The design is completed by forming the composite control

$$u_c = -x^3 - z \qquad (9.13)$$

and applying it to the full system (9.1). The final feedback system (6.1) is

$$\dot{x} = -\frac{3}{4}x^3 + z \qquad (9.14)$$

$$\mu\dot{z} = -x^3 - 2z. \qquad (9.15)$$

It should be noted that this system could not have been designed by methods based on linearization, since its linearized model at x=0, z=0 has a zero eigenvalue. However, Theorem 6.1 guarantees that the equilibrium x=0, z=0 is asymptotically stable for μ sufficiently small.

Step 5: With the help of Theorem 6.1 and Corollary 6.2 we can further analyze stability properties of the designed system (9.14), (9.15) which is first transformed by $z_f = z + \frac{1}{2}x^3$ into (6.4), that is into

$$\dot{x} = -\frac{5}{4}x^3 + z_f \qquad (9.16)$$

$$\mu\dot{z}_f = -\mu\frac{15}{8}x^5 - (2 - \mu\frac{3}{2}x^2)z_f. \qquad (9.17)$$

The Lyapunov function (6.5) is

$$v = \frac{1}{4}x^4 + \alpha\mu\frac{1}{4}z_f^2 \qquad (9.18)$$

and to analyze its derivative (6.6) we evaluate the bounds (6.8),

$$k_5 \ge \frac{4}{3}, \qquad k_6 \ge \frac{4}{3}\,\frac{1}{4}\,\frac{3}{2}x^2,$$

$$k_7 \ge 4\,\frac{9x^4}{4}\,\frac{1}{16}\,\frac{4}{3}. \qquad (9.19)$$

They are to be used to find an α guaranteeing that g in (6.7) is positive definite for all $x \in D$. In this example the choice of D is free, since the slow subsystem is asymptotically stable in the large. Suppose that we are interested in $x \in [-\frac{1}{2}, \frac{1}{2}]$

Then $k_6 \ge \frac{1}{8}$, $k_7 \ge \frac{3}{64}$ and α is obtained from (6.10)

as $\alpha = \sigma(1)$, that is

$$\alpha = \frac{4}{5}(\frac{4}{3} + \frac{3}{8}) + \frac{5}{4}(\frac{3}{8} + \frac{3}{64}) = \frac{881}{480}. \qquad (9.20)$$

With this α it can be easily verified that

$$g = \frac{5}{4}x^6\,(1 - \frac{8}{15\alpha}(1 - \frac{15}{16}\alpha\mu x^2)^2) > 0 \qquad (9.21)$$

$$-f_1 - (z_f' J_{2x} + L_x \bar{A}_1' \bar{A}_2^{-1} F + 2 z_f' J_3 F)\,(\bar{a}_o + \bar{A}_1 z_f)$$

$$+ z_f'(J_{3x}(\bar{a}_o + \bar{A}_1 z_f))z_f \qquad (A6)$$

is bounded from above by

$$-[(1 + c_3)/k_1 + (c_1+c_2)/k_3]L_x\bar{a}_o + (2+c_4)z_f'z_f \qquad (A7)$$

and from below by

$$-[(1 + c_3)/k_2 + (c_1+c_2)/k_4]L_x\bar{a}_o + (2+c_4)z_f'z_f \qquad (A8)$$

where

$$c_1 \geq \|H_1'H_1\|, \qquad H_1 = \bar{A}_1'\bar{A}_2^{-1}F$$

$$c_2 \geq \|H_2'H_2\|, \qquad H_2 = (\bar{V}_{1x} + 2KF)\bar{a}_o \qquad (A9)$$

$$c_3 \geq \|H_3'H_3\|, \qquad H_3 = \bar{A}_1'\bar{A}_2^{-1}F\bar{A}_1$$

$$c_4 \geq \|K + \frac{1}{2}(\bar{V}_{1x} + 2KF)\bar{A}_1 + \frac{1}{2}\bar{A}_1'(\bar{V}_{1x}' + 2F'K)\|.$$

From (6.13) we know that K, and hence c_4, remain bounded as $\mu \to 0$. Furthermore, rewriting $f(x,z_f) > 0$ in (7.6) as

$$s_1'z_f \geq -L_x\bar{a}_o + z_f'(Q + KB_2R^{-1}B_2'K)z_f, \qquad (A10)$$

and using the fact that the right hand side quantity is positive definite for all $x \in D$, $z_f \in R^m$, we obtain by substituting $\pm\bar{A}_2^{-1}(\bar{a}_o + \bar{A}_1 z_f)$ for z_f,

$$\|s_1'\bar{A}_2^{-1}(\bar{a}_o + \bar{A}_1 z_f)\| \leq -(1 + 2c_5/k_3)L_x\bar{a}_o + 2c_6 z_f'z_f \qquad (A11)$$

where

$$c_5 \geq \|N\| = \|(\bar{A}_2^{-1})'(Q+KB_2R^{-1}B_2'K)\bar{A}_2^{-1}\|$$

$$c_6 \geq \|\bar{A}_1'N\bar{A}_1\|. \qquad (A12)$$

Combining (A6) and (A11) we conclude that there exists $\gamma>0$ such that f_1 is bounded by $|\gamma\dot{v}|$, which, by Lemma 7.1, proves that

$$J_4 = \int_0^\infty f_1\,dt \qquad (A13)$$

is bounded.

Two-Time-Scale Feedback Design of a Class of Nonlinear Systems

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Abstract—For a class of nonlinear singularly perturbed systems, feedback stabilizing and near-optimal controls are designed using two separate lower order subsystems. The two-time-scale properties greatly simplify the stability analysis and the nonlinear controller design. Two electric machine examples (a dc motor and a synchronous generator) illustrate the proposed design procedure.

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I. Introduction

Until recently, singular perturbation techniques [1] have primarily focused on state feedback design of linear regulators [2]–[5] and, to a smaller extent, on open-loop trajectories of a class of nonlinear systems [6],[7]. Advantages of these techniques, such as order reduction and separation of time scales, are expected to have a more dramatic effect on feedback design of nonlinear systems. First highly encouraging results in this direction [8],[9],[15] are related to feedback design of stabilizing and near-optimal controls for nonlinear systems of the type

$$\dot{x}=f(x)+F(x)z+B_1(x)u, \qquad x(0)=x_0 \tag{1a}$$

$$\mu\dot{z}=g(x)+G(x)z+B_2(x)u, \qquad z(0)=z_0 \tag{1b}$$

where $\cdot \equiv d/dt$, $x,f\in R^n$, $z,g\in R^m$, $u\in R^r$, F, G, B_1, B_2 are matrices of appropriate dimensions, and μ is the small singular perturbation parameter. A particularly promising novelty of the results of [8],[9] is that they apply to essentially nonlinear cases of (1); that is, when the linearized system yields no stability information.

In this short paper, we clarify the results of [8],[9]. These results represent nonlinear generalization of the results in [5]. Since the details of most of the proofs are either in [8],[9] or are similar to those in [5], they are omitted here. Instead, the stress is on the meaning and applicability of the theorems. Although the analytical results here build upon those of [8],[9], for application purposes this short paper can be read without referring to [8],[9]. However, an understanding of [5] is helpful.

The outline of this short paper is as follows. We begin by a procedure for separation of slow and fast subsystems of (1). Then we justify both the modeling in the form of (1) and the proposed two-time-scale procedure by two nontrivial but common examples from power system practice: a third-order dc motor model and a fifth-order synchronous generator model. This justification is both inspiring and necessary, since the two-time-scale properties are dictated by physical laws and deduced from experience, rather than inserted into models for analytical or numerical convenience. The analytical results are then presented in three parts dealing with stability, stabilizability, and near-optimal feedback control. The short paper ends with an example demonstrating how simply a near-optimal nonlinear feedback-control law can be designed for a field-controlled dc motor.

II. Slow and Fast Subsystems

We first formulate a procedure to decompose (1) into two lower order subsystems. Because of the presence of μ, system (1) exhibits a boundary-layer phenomenon in the fast variable z. If the boundary layer decays, then the dynamics of x and z will vary slowly. Formally letting $\mu=0$, that is, neglecting the boundary layer, (1) becomes

$$\dot{\bar{x}}=f(\bar{x})+F(\bar{x})\bar{z}+B_1(\bar{x})\bar{u}, \qquad \bar{x}(0)=x_0 \tag{2a}$$

$$0=g(\bar{x})+G(\bar{x})\bar{z}+B_2(\bar{x})\bar{u} \tag{2b}$$

where the bar above the variable denotes its slow part. Assuming $G(\bar{x})$ to be nonsingular, we obtain from (2b)

$$\bar{z}=-G^{-1}(g+B_2\bar{u}) \tag{3}$$

and its elimination from (2a) yields the slow subsystem or the reduced system of (1) as

$$\begin{aligned}\dot{\bar{x}}&=(f-FG^{-1}g)+(B_1-FG^{-1}B_2)\bar{u}, \qquad \bar{x}(0)=x_0\\ &\equiv a(\bar{x})+B(\bar{x})\bar{u}.\end{aligned} \tag{4}$$

To derive the fast subsystem or the boundary-layer system, we assume that the slow variables are constant in the boundary layer; that is, $\dot{\bar{z}}=0$ and $x=\bar{x}=$ constant. Subtracting (2b) from (1b) at $t=0$, we obtain

$$\mu(\dot{z}-\dot{\bar{z}})=G(\bar{x})(z-\bar{z})+B_2(\bar{x})(u-\bar{u}). \tag{5}$$

Redefining $z_f=z-\bar{z}$ and $u_f=u-\bar{u}$, we formulate the fast subsystem of (1) as

$$\mu\dot{z}_f=G(\bar{x})z_f+B_2(\bar{x})u_f, \quad z_f(0)=z_0-\bar{z}(0) \tag{6a}$$

that is,

$$\frac{dz_f}{dt'}=G(\bar{x})z_f+B_2(\bar{x})u_f \tag{6b}$$

where $t'=t/\mu$ is the fast time scale.

Under the conditions specified later, the response of system (1) can be approximated by

$$x(t)=\bar{x}(t)+O(\mu) \tag{7a}$$

$$z(t)=\bar{z}(t)+z_f(t)+O(\mu). \tag{7b}$$

Thus the properties of system (1) can be investigated by examining the subsystems (4),(6).

III. Applications to Electrical Machine Models

That systems of the type (1) are common in application will now be shown by two well-known models of electrical machines. In these models, smallness of some time constants serves as a starting for the decomposition of the original systems into slow and fast subsystems.

In electric-drive systems, higher speeds ω are usually achieved by the so-called field weakening. In such a regime the dc motor is controlled by its field voltage V_f, while its armature voltage V_a is constant. The field, the armature, and the torque equations of the dc motor are

$$L_f\frac{di_f}{dt}=-R_fi_f+V_f \tag{8a}$$

$$L_a\frac{di_a}{dt}=-R_ai_a-c_1i_f\omega+V_a \tag{8b}$$

$$J\frac{d\omega}{dt}=-c_2\omega+c_3i_fi_a \tag{8c}$$

where i_f, R_f, L_f, i_a, R_a, L_a are the currents, resistances, and inductances of the field circuit and the armature, respectively, J is the rotor inertia and c_1, c_2, and c_3 are the back emf, the viscous damping, and the torque constants, respectively. The field, armature, and mechanical time constants are $T_f=L_f/R_f$, $T_a=L_a/R_a$, and $T_m=J/(c_2+c_1c_3i_f^{o2}/R_a)$, respectively, where i_f^o is the nominal value of i_f. Since in practice $T_f\gg T_a$, T_m, we identify i_f as the slow variable and i_a,ω as the fast variables, and set μ as the ratio of a small and a large time constant; that is, $\mu=T_a/T_f=L_aR_f/L_fR_a$.

Redefining $x=i_f-i_f^o$, $z_1=i_a-i_a^o$, $z_2=\omega-\omega^o$, $u=V_f-V_f^o$, where i_f^o, i_a^o, ω^o, V_f^o are the desired equilibrium of system (8), we obtain system (8) in the form of (1) with

$$f=\left[-\frac{x}{T_f}\right], \qquad F=0, \qquad B_1=\left[\frac{1}{L_f}\right]$$

$$g=\begin{bmatrix}\dfrac{-c_1\omega^o x}{T_fR_a}\\ \alpha c_3 i_a^o x\end{bmatrix}, \qquad G=\begin{bmatrix}-\dfrac{1}{T_f} & -\dfrac{c_1(i_f^o+x)}{T_fR_a}\\ \alpha c_3(x+i_f^o) & -\alpha c_2\end{bmatrix}, \qquad B_2=0 \tag{9}$$

where $\alpha=T_a/T_fJ$. Then the slow and fast subsystems of (9) are

$$\dot{\bar{x}}=-\frac{1}{T_f}\bar{x}+\frac{1}{L_f}\bar{u} \tag{10}$$

$$\mu\begin{bmatrix}\dot{z}_{1f}\\ \dot{z}_{2f}\end{bmatrix}=\begin{bmatrix}-\dfrac{1}{T_f} & -\dfrac{c_1(i^o+\bar{x})}{T_fR_a}\\ \alpha c_3(i_f^o+\bar{x}) & -\alpha c_2\end{bmatrix}\begin{bmatrix}z_{1f}\\ z_{2f}\end{bmatrix} \tag{11}$$

respectively.

As a second example, we consider a well-known fifth-order model of a synchronous machine [10]. Neglecting the damper windings and saturation, the equations for the direct and quadrature axis voltages v_d and v_q, and the field-flux linkage ψ_f are

$$\frac{v_d}{r_a}=-i_d-\frac{(L_dL_f-M_{df}^2)}{r_aL_f}\frac{di_d}{dt}+\frac{M_{df}}{r_aL_f}\frac{d\psi_f}{dt}+\frac{L_q\Omega i_q}{r_a} \tag{12}$$

$$\frac{v_q}{r_a} = -i_q - \frac{L_q}{r_a}\frac{di_q}{d\tau} - \frac{(L_d L_f - M_{df}^2)\Omega i_d}{r_a L_f} + \frac{M_{df}\Omega\psi_f}{r_a L_f} \tag{13}$$

$$\frac{L_f}{r_f}\frac{d\psi_f}{d\tau} = -\psi_f - M_{df} i_d + \frac{L_f v_f}{r_f} \tag{14}$$

where i_d, L_d, i_q, L_q are the currents and reactances of the d and q axes, respectively, r_a is the armature resistance, v_f, L_f, r_f are the field voltage, reactance, and resistance, respectively, M_{df} is the mutual reactance between the d axis and field circuit, Ω is the instantaneous per unit angular velocity of the rotor, and τ is the per unit time. The swing equation is

$$\frac{d\gamma}{d\tau} = \Omega - \Omega_0, \qquad \gamma = \theta - \Omega_0 \tau \tag{15}$$

$$2\omega_0 H \frac{d\Omega}{d\tau} = T_{in} - \frac{M_{df}\psi_f i_q}{L_f} \tag{16}$$

where θ is the rotor angle, Ω_0 the nominal value of Ω, ω_0 the rated frequency in radians/seconds, H the inertia constant, and T_{in} the input torque. Note that the torque $(L_d - L_q - M_{df}^2/L_f)i_d i_q$ due to saliency has been neglected in (16): The reason for this would be more apparent in a model including the field circuit on the quadratic axis of the rotor.

From experience it is known that the mechanical and field circuit transients primarily described by (14), (16) are much slower than the transients in the d and q axes (12), (13). To exhibit this two-time-scale behavior we introduce a slow time variable

$$\tau' = \frac{r_f \tau}{L_f} = \frac{\tau}{T'_{d0}} \tag{17}$$

normalized with respect to the d-axis open-circuit time constant T'_{d0}. Defining the singular perturbation parameter μ as the ratio of a small and a large time constant

$$\mu = \frac{r_f(L_d L_f - M_{df}^2)}{r_a L_f^2} \tag{18}$$

we rewrite (12), (13) in the slow time scale as

$$\frac{v_d}{r_a} = -i_d - \mu\frac{di_d}{d\tau'} + \alpha_1 \mu \frac{d\psi_f}{d\tau'} + \frac{L_q \Omega i_q}{r_a} \tag{19}$$

$$\frac{v_q}{r_a} = -i_q - \alpha_2 \mu \frac{di_q}{d\tau'} - \frac{(L_f L_d - M_{df}^2)\Omega i_d}{r_a L_f} + \frac{M_{df}\Omega\psi_f}{r_a L_f} \tag{20}$$

where

$$\alpha_1 = \frac{M_{df}}{(L_d L_f - M_{df}^2)}, \qquad \alpha_2 = \frac{L_q L_f}{(L_d L_f - M_{df}^2)}. \tag{21}$$

Similarly, we rewrite (14)–(16) in the slow time scale. The resulting system has the form (1) in which the slow variables are γ, Ω, and ψ_f, and the fast variables i_d and i_q, whose derivatives are multiplied by μ, appear linearly in these equations.

Letting $\mu = 0$, the slow subsystem of (12)–(15) is

$$\frac{d\bar\gamma}{d\tau'} = \frac{L_f}{r_f}(\bar\Omega - \Omega_0) \tag{22a}$$

$$\frac{d\bar\Omega}{d\tau'} = \frac{L_f}{2\omega_0 H r_f}\left(T_{in} - \frac{M_{df}\bar\psi_f \bar i_q}{L_f}\right) \tag{22b}$$

$$\frac{d\bar\psi_f}{d\tau'} = -\bar\psi_f - M_{df}\bar i_d + \frac{L_f v_f}{r_f} \tag{22c}$$

where the slow parts of i_d and i_q satisfy the algebraic system

$$\frac{v_d}{r_a} = -\bar i_d + \frac{L_q \bar\Omega \bar i_q}{r_a} \tag{23a}$$

$$\frac{v_q}{r_a} = -\bar i_q - \frac{(L_f L_d - M_{df}^2)\bar\Omega \bar i_d}{r_a L_f} + \frac{M_{df}\bar\Omega\bar\psi_f}{r_a L_f}. \tag{23b}$$

The fast subsystem is

$$\frac{d\tilde i_d}{d\tau''} = -\tilde i_d + \frac{L_q \bar\Omega \tilde i_q}{r_a} \tag{24a}$$

$$\frac{d\tilde i_q}{d\tau''} = -\frac{(L_f L_d - M_{df}^2)\bar\Omega \tilde i_d}{\alpha_2 r_a L_f} - \frac{\tilde i_q}{\alpha_2} \tag{24b}$$

where $\tau'' = \tau'/\mu$ and the tilde denotes the fast variables. It is crucial that in (24) the slow part $\bar\Omega$ is regarded as a constant.

The slow subsystem (22) is analogous to Kimbark's third-order model [11]. Instead of neglecting $di_d/d\tau$, $di_q/d\tau$ as it was in [11], we set $\mu = 0$ to obtain the slow subsystem. We also obtain the fast subsystem (24) governing the transient behavior of i_d and i_q. Since this order reduction is caused by a parameter perturbation from $\mu > 0$ to $\mu = 0$, we are able to use approximations of the type $\gamma \approx \bar\gamma$, $\Omega \approx \bar\Omega$, $\psi_f \approx \bar\psi_f$, $i_d \approx \bar i_d + \tilde i_d$, and $i_q \approx \bar i_q + \tilde i_q$.

IV. Stability Properties

In this section we establish some stability properties of the full system

$$\dot x = f(x) + F(x)z \tag{25a}$$

$$\mu \dot z = g(x) + G(x)z \tag{25b}$$

from an analysis of the lower order slow and fast subsystems, which are

$$\dot{\bar x} = a(\bar x) \tag{26a}$$

$$\mu \dot z_f = G(\bar x) z_f \tag{26b}$$

respectively. System (25) can be considered as the feedback-controlled system of (1). It is assumed that systems (25), (26) satisfy the following conditions for all $x, \bar x \in D$, where D is a closed set in R^n.

1) The vectors f, g and matrices F, G are bounded and differentiable with respect to x, and there exists a unique $x^* \in D$ such that $f(x^*) = 0$ and $g(x^*) = 0$.

2) The eigenvalues of G satisfy $\mathrm{Re}\{\lambda(G)\} < \sigma_1$ for a fixed $\sigma_1 < 0$. Thus G is nonsingular.

3) There exists a Lyapunov function $v_1(x)$ for (26a) of the Krasovskii's type [10], [11, p. 38]; that is,

$$v_1(\bar x) = a'(\bar x) P(\bar x) a(\bar x) \tag{27a}$$

$$\dot v_1(\bar x) = a'(\bar x) N(\bar x) a(\bar x) \tag{27b}$$

where the matrix $P(\bar x) > 0$ is differentiable with respect to $\bar x$,

$$N(\bar x) = P a_x + a_x' P + \sum_{j=1}^{n} P_{x_j} a_j < 0 \tag{28}$$

the subscript x denotes partial differentiation and x_j, a_j are the jth components of the vectors x, a, respectively. Without loss of generality, we let D be the set whose boundary is given by $v_1(\bar x) = c_0$ for a fixed $c_0 > 0$.

Condition 2) guarantees that the fast subsystem (26b) is asymptotically stable for all $\bar x \in D$ and condition 3) guarantees that x^* of the slow subsystem (26a) is asymptotically stable with D as its domain of attraction.

Theorem 1

Let D_1 be a closed set in the interior of D and E be a bounded set in R^m. If conditions 1)–3) are satisfied, then there exists a $\mu^* > 0$ such that for all $x \in D_1$, $z \in E$, the equilibrium $x = x^*$, $z = 0$ of system (25) is asymptotically stable for all $\mu \in (0, \mu^*]$.

Theorem 1 is proved in [9] and is an essentially nonlinear result. For the slow subsystem, if $\mathrm{Re}(\lambda(a_x(x^*))) < \sigma_2$ for a fixed $\sigma_2 < 0$, then we can always find the required $v_1(\bar{x})$, while the converse is not necessarily true. Consider the first-order system

$$\dot{\bar{x}} = -\bar{x}^3 \tag{29}$$

whose linearization at $\bar{x}=0$ provides no asymptotic stability information. With $P=1/6\bar{x}^2$, $N=-2/3$, a Lyapunov function for (29) of the form (27) is

$$v_1(\bar{x}) = \frac{\bar{x}^4}{6}, \quad \dot{v}_1(\bar{x}) = -\frac{2\bar{x}^6}{3} \tag{30}$$

guaranteeing that $\bar{x}=0$ is asymptotically stable. Hence Theorem 1 includes a class of essentially nonlinear systems whose linearizations at the equilibrium may fail to guarantee asymptotic stability.

If only the stability of the equilibrium is of interest, we can relax condition 3) to the following condition.

4) There exists a Lyapunov function $v_2(\bar{x})$ for (26a) guaranteeing that x^* is asymptotically stable. Furthermore, let v_2 be differentiable with respect to $\bar{x}$ and $v_2(\bar{x}) = c_0 > 0$ for all $\bar{x}$ on the boundary of D.

Theorem 2

If conditions 1), 2), and 4) are satisfied, then there exists a $\mu^* > 0$ such that for all $\mu \in (0,\mu^*]$, the states $x \in D_1$, $z \in E$ of (25) converge to a sphere centered at the equilibrium $x = x^*$, $z=0$, whose radius is $O(\mu)$.

The proof of Theorem 2 is given in [8]. In contrast to the result of Theorem 1, this theorem states that x,z can only converge to a sphere around the equilibrium. In the neighborhood of the equilibrium, the small parameter μ becomes significant and the behavior of x,z cannot be predicted by conditions 2) and 4). However, as $\mu \to 0^+$, the equilibrium is asymptotically stable which is the same result as in [14]. It is important to note that Theorem 1 includes general nonlinear slow subsystem as indicated by condition 4).

V. Stabilizing Controls

Using the results in the previous section, the design of a stabilizing feedback control for the full system can be decomposed into separate designs of feedback controls for the subsystems. The systems (1), (4), (6) are assumed to satisfy the following conditions for all $x, \bar{x} \in D$.

1′) In addition to condition 1), the matrices B_1 and B_2 are bounded and differentiable with respect to x.

2′) G is nonsingular and

$$\mathrm{rank}\left[B_2, GB_2, \cdots, G^{m-1}B_2\right] = m. \tag{31}$$

3′) There exists a vector $h(\bar{x}) \in R^r$ with $h(x^*)=0$ such that the system

$$\dot{\bar{x}} = a(\bar{x}) + B(\bar{x})h(\bar{x}) \tag{32}$$

satisfies condition 3).

4′) System (32) satisfies condition 4).

Condition 2′) guarantees the existence of a fast control $u_f(\bar{x}, z_f) = H(\bar{x})z_f$ with H an $r \times m$ matrix and $\mathrm{Re}(\lambda(G+B_2H)) < \sigma_3$ for a fixed $\sigma_3 < 0$ such that the feedback subsystem (5)

$$\dot{z}_f = \left(G(\bar{x}) + B_2(\bar{x})H(\bar{x})\right)z_f \tag{33}$$

is stable. Condition 4′) guarantees a stabilizing control $\bar{u} = h(\bar{x})$ for the slow subsystem, and condition 3′) provides a stabilizing control $\bar{u} = h(\bar{x})$ such that the slow subsystem possesses a Lyapunov function of the form (27).

Following [5], from the designs of $\bar{u}$ and u_f, we consider a composite control

$$\begin{aligned} u_c(x,z) = &\left(I + H(x)G^{-1}(x)B_2(x)\right)h(x) \\ &+ H(x)G^{-1}(x)g(x) + H(x)z \end{aligned} \tag{34}$$

for the full system (1) where I is the identity matrix of appropriate dimension. System (1) controlled by (34) becomes

$$\dot{x} = f + B_1\left(I + HG^{-1}B_2\right)h + B_1HG^{-1}g + (F + B_1H)z \tag{35a}$$

$$\mu\dot{z} = g + B_2\left(I + HG^{-1}B_2\right)h + B_2HG^{-1}g + (G + B_2H)z. \tag{35b}$$

Letting $\mu = 0$, the slow subsystem of (35) is

$$\begin{aligned} \dot{\bar{x}} = &\left(f + B_1HG^{-1}g\right) - (F + B_1H)(G + B_1H)^{-1}\left(g + B_2HG^{-1}g\right) \\ &+ \left(B_1 - (F + B_1H)(G + B_2H)^{-1}B_2\right)\left(I + HG^{-1}B_2\right)h \\ = &\, a + Bh. \end{aligned} \tag{36}$$

Hence the following theorem follows immediately from Theorems 1 and 2.

Theorem 3

If conditions 1′)–3′) are satisfied, then there exists a $\mu^* > 0$ such that for all $\mu \in (0,\mu^*]$, $x \in D_1$, $z \in E$, the equilibrium $x = x^*$, $z=0$ of system (1) controlled by (34) is asymptotically stable. If only conditions 1′), 2′), and 4′) are satisfied, then there exists a $\mu^* > 0$ such that the control (34) steers all $x \in D_1$, $z \in E$ of (1) to a sphere centered at $x = x^*$, $z=0$, whose radius is $O(\mu)$.

Thus we have designed a composite control for the full system (1) based on separate lower order designs of the slow and fast subsystems. In the special case when condition 2) is satisfied, we let $H=0$ in (34) and obtain the reduced control

$$u_r(x) = h(x). \tag{37}$$

Then the conclusions of Theorem 3 hold for system (1) controlled by (37).

VI. Near-Optimal Control

A similar decomposition method is now developed for the optimal control of the full system (1) with respect to the performance index

$$J = \int_0^{\infty}\left[p(x) + s'(x)z + z'Q(x)z + u'R(x)u\right]dt. \tag{38}$$

The problem (1), (38) satisfies the following condition for all $x \in D$:

1″) In addition to 1′, the scalar p, the vector $s \in R^m$, and the matrices Q, R are differentiable with respect to x, $p(x^*)=0$, $s(x^*)=0$, $Q(x)>0$, $R(x)>0$, and for $x \neq x^*$, $z \neq 0$

$$p + s'z + z'Qz > 0. \tag{39}$$

Thus the optimal control should steer x,z to the desired equilibrium $x = x^*$, $z=0$.

We now extract from J two performance indices, one for the slow subsystem (4) and the other for the fast subsystem (6), and formulate two separate regulator problems, denoted by s for the slow and f for the fast subsystems. From the subsystem optimal controls $\bar{u}$ and u_f, we then form a composite control u_c to be implemented to the full system (1).

Problem s

Find $\bar{u}$ to minimize

$$\bar{J} = \int_0^{\infty}\left[p(\bar{x}) + s'(\bar{x})\bar{z} + \bar{z}'Q(\bar{x})\bar{z} + \bar{u}'R(\bar{x})\bar{u}\right]dt \tag{40}$$

for the slow subsystem (4).

Using (3) to eliminate $\bar{z}$ from (40), we obtain

$$\bar{J} = \int_0^{\infty}\left[\bar{p}(\bar{x}) + 2\bar{s}'(\bar{x})\bar{u} + \bar{u}'\bar{R}(\bar{x})\bar{u}\right]dt \tag{41}$$

where

$$\begin{aligned} \bar{p} &= p - s'G^{-1}g + g'G'^{-1}QG^{-1}g \\ \bar{s} &= B_2'G'^{-1}\left(QG^{-1}g - \frac{1}{2}s\right) \\ \bar{R} &= R + B_2'G'^{-1}QG^{-1}B_2. \end{aligned} \tag{42}$$

From condition 1″), it follows that for $\bar{x}\neq x^*$, $\bar{u}\neq 0$

$$\bar{p}(\bar{x})+2\bar{s}'(\bar{x})\bar{u}+\bar{u}'\bar{R}(\bar{x})\bar{u}>0. \tag{43}$$

Applying the principle of optimality to problem s, we obtain

$$0=\min_{\bar{u}}\left[\bar{p}(\bar{x})+2\bar{s}'(\bar{x})\bar{u}+\bar{u}'\bar{R}(\bar{x})\bar{u}+v_x\big(a(\bar{x})+B(\bar{x})\bar{u}\big)\right] \tag{44}$$

where v is the optimal value function of (40) and v_x its partial derivative with respect to x. This yields the minimizing control

$$\bar{u}_0(\bar{x})=-\bar{R}^{-1}\left(\bar{s}+\frac{1}{2}B'v_x'\right) \tag{45}$$

whose elimination from (44) results in the Hamilton–Jacobi equation

$$0=\left(\bar{p}-\bar{s}'\bar{R}^{-1}\bar{s}\right)+v_x\left(a-B\bar{R}^{-1}\bar{s}\right)-\frac{1}{4}v_xB\bar{R}^{-1}B'v_x', \quad v(x^*)=0. \tag{46}$$

Now problem s is assumed to satisfy the following condition.

5) The unique solution $v(\bar{x})>0$, $\bar{x}\neq x^*$, and $v(x^*)=0$ of (46) exists and is differentiable with respect to $\bar{x}$ for all $\bar{x}\in D$. Furthermore, $v(\bar{x})=c_0>0$ is taken to be the boundary of D.

Then it follows from condition 5) that $\bar{u}_0$ is the unique optimal feedback control for problem s, and v is a Lyapunov function of the optimally controlled slow subsystem

$$\dot{\bar{x}}=a-B\bar{R}^{-1}\left(\bar{s}+\frac{1}{2}B'v_x'\right) \tag{47}$$

establishing that x^* is asymptotically stable with D as its domain of attraction.

Problem f

Find u_f to minimize

$$J_f=\int_0^\infty\left(z_f'Q(\bar{x})z_f+u_f'R(\bar{x})u_f\right)dt \tag{48}$$

for the fast subsystem (6).

Letting $\bar{x}$ be a constant vector parameter, the optimal control for (6), (48) is

$$u_f(\bar{x},z_f)=-R^{-1}(\bar{x})B_2'(\bar{x})V(\bar{x})z_f \tag{49}$$

where $V(\bar{x})$ is the positive-definite stabilizing solution of the $\bar{x}$-dependent Riccati equation

$$\begin{aligned}0=&-V(\bar{x})G(\bar{x})-G'(\bar{x})V(\bar{x})\\&+V(\bar{x})B_2(\bar{x})R^{-1}(\bar{x})B_2'(\bar{x})V(\bar{x})-Q(\bar{x}).\end{aligned} \tag{50}$$

If conditions 1″) and 2′) are satisfied, then the required stabilizing solution $V(\bar{x})$ of (50) exists.

From the solutions to problems s and f, we formulate the composite control (34) for the full problem (1), (38) as

$$\begin{aligned}u_c(x,z)=&-\left(I-R^{-1}B_2'VG^{-1}B_2\right)\bar{R}^{-1}\left(\bar{s}+\frac{1}{2}B'v_x'\right)\\&-R^{-1}B_2'VG^{-1}g-R^{-1}B_2'Vz.\end{aligned} \tag{51}$$

The stabilizing properties of u_c then follow from conditions 2′) and 5) and Theorem 3. Implementing u_c to the full system, we have the following result.

Theorem 4

If conditions 1″), 2′), and 5) are satisfied, then u_c is an $O(\mu)$ near-optimal control in the sense that

$$u_c=\tfrac{1}{2}R^{-1}\left(B_1'U_x'+\frac{1}{\mu}B_2'U_z'\right)+O(\mu) \tag{52}$$

where

$$\begin{aligned}U=v-\mu\left[\left(s'+2g'V+v_x\left(f-B_1R^{-1}B_2'V\right)\right)\left(G-B_2R^{-1}B_2'V\right)^{-1}\right]z&\\+\mu z'Vz&\end{aligned} \tag{53}$$

satisfies the Hamilton–Jacobi equation for the full problem (1), (38), to $O(\mu)$.

The proof of Theorem 4 is in [8],[15]. A series expansion of the solution to the Hamilton–Jacobi equation is proposed in [15], and its asymptotic validity is being investigated.

VII. Near-Optimal Control of DC Motor

We now consider the optimal-control problem of the dc motor model (9) with respect to

$$J=\int_0^\infty\left(px^2+\begin{bmatrix}z_1\\z_2\end{bmatrix}'Q\begin{bmatrix}z_1\\z_2\end{bmatrix}+Ru^2\right)dt \tag{54}$$

where the scalars $p>0$, $R>0$ and the matrix $Q>0$ are x-independent constants.

Since $B_2=0$, there is no control in the fast subsystem (11), and hence we only consider problem s for the slow subsystem (10). Eliminating $\bar{z}$

$$\begin{aligned}\begin{bmatrix}\bar{z}_1\\ \bar{z}_2\end{bmatrix}&=\frac{T_f}{\alpha}\left[c_2+\frac{c_1c_3\left(i_f^*+\bar{x}\right)^2}{R_a}\right]^{-1}\\&\quad\cdot\begin{bmatrix}\alpha c_2 & -\dfrac{c_1\left(i_f^*+\bar{x}\right)}{T_fR_a}\\ \alpha c_3\left(\bar{x}+i_f^*\right) & \dfrac{1}{T_f}\end{bmatrix}\begin{bmatrix}-\dfrac{c_1\omega^*}{T_fR_a}\\ \alpha c_3i_a^*\end{bmatrix}\bar{x}\\&=M(\bar{x})\bar{x}\end{aligned} \tag{55}$$

from (40), we obtain

$$\bar{J}=\int_0^\infty\left(\left[p+M'(\bar{x})QM(\bar{x})\right]\bar{x}^2+R\bar{u}^2\right)dt. \tag{56}$$

The optimal control for problem (10), (56) is

$$\bar{u}_0(\bar{x})=R_f\bar{x}\left(1-\sqrt{1+\frac{(p+M'QM)}{RR_f^2}}\right). \tag{57}$$

Although subsystem (10) is linear, $\bar{J}$ of (56) is not strictly quadratic which results in the nonlinear feedback control $\bar{u}_0$ and the nonlinear feedback slow subsystem

$$\dot{\bar{x}}=-\frac{\bar{x}}{T_f}\sqrt{1+\frac{(p+M'QM)}{RR_f^2}} \tag{58}$$

Since $u_f=0$, from (51)

$$u_c(x)=\bar{u}_0(x) \tag{59}$$

and $x=0$, $z=0$ of system (9) controlled by u_c is asymptotically stable as (58) satisfies condition 3) and $\mathrm{Re}\{G(0)\}<0$.

VIII. Conclusions

A systematic procedure is given for the order reduction and the separation of time scales in a class of nonlinear singularly perturbed systems. First, the full system is decomposed into two separate slow and fast subsystems. Second, the stability properties are investigated and feedback stabilizing and near-optimal controls for the full system (1) are designed by analyzing the lower order subsystems. Thus this procedure is applicable to large-scale systems. A dc motor and a synchronous generator are used to illustrate the modeling aspect and the design procedure.

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Asymptotic Stability of a Class of Non-linear Singularly Perturbed Systems†

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ABSTRACT: *Sufficient conditions are obtained to guarantee the asymptotic stability of a class of non-linear singularly perturbed systems. A procedure for constructing a Lyapunov function for such a class of systems is given, and a clearly defined domain of attraction of the equilibrium is obtained. A stabilizing feedback control for such systems is also proposed.*

I. Introduction

Stability properties of the non-linear system

$$\begin{aligned} \dot{x} &= \varphi\ (t, x, z, \mu) \\ \mu\dot{z} &= \psi\ (t, x, z, \mu) \end{aligned} \qquad \left(\cdot = \frac{d}{dt}\right) \tag{1}$$

where μ is a small positive parameter, x, $\varphi \in R^n$ and z, $\psi \in R^m$, have been investigated extensively (1–4, 8). In this paper we analyze a special class of system (1) in the form

$$\dot{x} = f(x) + F(x)z, \quad \mu\dot{z} = g(x) + G(x)z \tag{2}$$

which is non-linear in x and linear in z. We first consider stability properties of the reduced system

$$\begin{aligned} \dot{x} &= f(x) - F(x)G^{-1}(x)g(x) \equiv a(x) \\ z &= -G^{-1}\ (x)g(x) \end{aligned} \tag{3}$$

obtained from (2) by formally setting $\mu = 0$ and assuming $G(x)$ to be non-singular, and the boundary layer system

$$\frac{dx}{ds} = 0, \quad \frac{dz}{ds} = G(\alpha)z + g(\alpha) \tag{4}$$

where $x = \alpha =$ constant and $s = (t - t_0)/\mu$ with t_0 the initial time. Then based on the assumptions for (3), (4), we deduce stability conditions for (2).

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In Sections II and III conditions on systems (3) and (4) are formulated which guarantee the equilibrium of (2) to be asymptotically stable in a domain in R^{n+m} for μ sufficiently small. Since system (2) is simpler in structure than system (1), we relax the condition used in Refs. (1, 3) that the linearized reduced system (3) be asymptotically stable. The conditions obtained here are also easier to apply to system (2) than those given in Refs. (3, 4, 8). An example is given in Section IV and a stabilizing control for system (2) is proposed in Section V.

II. Preliminaries

System Eqs. (2)–(4) are assumed to satisfy the following conditions for all $x \in D$ where D is a closed set in R^n:

(I) The vectors f, g and matrices F, G are bounded and differentiable with respect to x and there exists a unique $x^* \in D$ such that $f(x^*) = 0$ and $g(x^*) = 0$.

(II) The real parts of the eigenvalues of G are strictly negative, that is, there exists a fixed $\sigma_1 < 0$ such that $\mathrm{Re}\{\lambda(G)\} \le \sigma_1$. Thus G is non-singular.

(III) There exists a matrix $Q(x) > 0$ satisfying the x-dependent algebraic Lyapunov function

$$Q(x)a_x(x) + a_x'(x)Q(x) = -C(x) \tag{5}$$

for some differentiable $C(x) > 0$, where the subscript x denotes partial differentiation with respect to x and the prime denotes the transpose. Let the matrices $M(x)$ and $N(x)$ be

$$M(x) = 2Q(x)a_x(x) + K(x) \tag{6a}$$

$$N(x) = Qa_x + a_x'Q + \sum_{j=1}^{n} Q_{x_j}a_j = -C + \sum_{j=1}^{n} Q_{x_j}a_j \tag{6b}$$

where K is a matrix whose jth column is $(Q_{x_j}a)$ and x_j, a_j are the jth components of the vectors x, a, respectively. It is assumed that M is bounded and $\mathrm{Re}\{\lambda(N(x))\} \le \sigma_2$ for a fixed $\sigma_2 < 0$.

Note that in condition III, Q is differentiable with respect to x, except possibly at x^* where it can be unbounded. However, $Q(x)a_x(x)$ and $Q_{x_j}(x)a(x)$, $j = 1, 2, \ldots, n$, are required to be bounded in the limit as x approaches x^*. For example consider

$$a(x) = -x^3. \tag{7}$$

Setting $C(x) = \frac{3}{2}$, we obtain $Q = 1/(4x^2)$, $M = -1$ and $N = -1$, and condition III is satisfied for all $x \in R$, although Q is unbounded at $x = 0$.

The meaning of condition III is that the reduced system (3) possesses a Lyapunov function $v(x)$ of Krasovskii's type [(5), (6, p. 38)], that is,

$$v(x) = a'(x)Q(x)a(x) > 0, \ v(x^*) = 0 \tag{8}$$

such that $v_x(x) = a'(x)M(x)$ and

$$\dot{v}(x) = a'(x)N(x)a(x) < 0, \ \dot{v}(x^*) = 0. \tag{9}$$

Thus the equilibrium x^* of (3) is asymptotically stable. Without loss of generality, D is chosen such that $v(x)=c$ for a fixed $c>0$ and all x on the boundary of D. Hence D belongs to the domain of attraction of x^*.

If Re $\{\lambda(a_x(x^*))\}\leq\sigma_3$ for a fixed $\sigma_3<0$, then the required $Q(x)$ always exists, while the converse is not true. Consider the system

$$\dot{x}=-x^3 \tag{10}$$

whose linearization at $x^*=0$ yields Re $\{\lambda(a_x(0))\}=0$ and does not guarantee that $x^*=0$ is asymptotically stable. However, a Lyapunov function (8) for system (10)

$$v=\frac{1}{4}x^4, \quad \dot{v}=-x^6 \tag{11}$$

implies that $x^*=0$ is asymptotically stable. Hence condition III guarantees the asymptotic stability of essentially nonlinear systems whose linearizations at x^* fail to provide stability information.

III. Main Result

Theorem I

Let D_1 be closed and in the interior of D, and E be a bounded set in R^m. If conditions I–III are satisfied, then there exists a $\mu^*>0$ such that for all $x\in D_1$ and $z\in E$, the equilibrium $x=x^*$, $z=0$ of system (2) is asymptotically stable for $\mu\in(0,\mu^*]$.

Proof: By condition II, there exists a matrix $P(x)>0$ satisfying the x-dependent algebraic Lyapunov equation

$$P(x)G(x)+G'(x)P(x)=-I \tag{12}$$

where I is the $n\times n$ identity matrix. Consider the function

$$L(x,z,\mu)=v+\tfrac{1}{2}\mu(z+G^{-1}g-P^{-1}(v_xFG^{-1})')'P(z+G^{-1}g-P^{-1}(v_xFG^{-1})')>0 \tag{13}$$

$$L(x^*,0,\mu)=0$$

with v as given by (8). Then there exists a μ_1^* sufficiently small such that for every $\mu\in(0,\mu_1^*]$, $L(x,z,\mu)=c$ is a closed surface S in R^{n-m} enclosing all $x\in D_1$, $z\in E$. Note that for all x, z enclosed in S, $|x|$ is $0(1)$ and $|z|$ can be at most $0(1/\sqrt{\mu})$. The time derivative of L with respect to (2) is

$$\dot{L}=v_x\dot{x}+\mu(z+G^{-1}g-Ta)'P(\dot{z}+R\dot{x})+\tfrac{1}{2}\mu(z+G^{-1}g-Ta)'U(z+G^{-1}g-Ta) \tag{14}$$

where $T(x)a(x)=P^{-1}(v_xFG^{-1})'$, $R(x)=[G^{-1}g-P^{-1}(v_xFG^{-1})']_x$ and $U(x,z)=\sum_{i=1}^{n}P_{x_i}\dot{x}_i$. Substituting (2) for $\dot{x}$ and $\dot{z}$ and rearranging, (14) becomes

$$\dot{L}=a'A_1a+\tfrac{1}{2}(z+G^{-1}g)'A_2(z+G^{-1}g)-\tfrac{1}{4}(z+G^{-1}g+\mu Ya)'(z+G^{-1}g-\mu Ya) \tag{15}$$

where

$$A_1 = N - \tfrac{1}{2}\mu(T'PR + R'PT) + \tfrac{1}{2}\mu T'UT + \tfrac{1}{4}\mu^2 Y'Y \tag{16a}$$

$$A_2 = -I + 2\mu U + 2\mu(PRF + F'R'P) \tag{16b}$$

$$Y = (PRF + U)'T - PR. \tag{16c}$$

Then there exists a $\mu_2^* > 0$ such that for $\mu \in (0, \mu_2^*]$, $A_1 < 0$ and $A_2 < 0$ since the matrices T, P, R, F are independent of μ and the elements of U are at most $0\ (1/\sqrt{\mu})$ for all x, z enclosed in S, implying that $\dot{L} < 0$ and $\dot{L}(x^*, 0, \mu) = 0$. Thus choosing $\mu^* = \min(\mu_1^*, \mu_2^*)$, L is a Lyapunov function for (2) and $x = x^*$, $z = 0$ is asymptotically stable for all $\mu \in (0, \mu^*]$ with its domain of attraction of attraction enclosed by S.

The accomplishment of Theorem I is threefold. First, condition III replaces the more restrictive condition Re $\{\lambda(a_z)\} \le \sigma_3$ proposed in Refs. (1, 3) for the general system (1). Second, Theorem I formulates a procedure to construct a Lyapunov function for (2) and provides an estimate of μ^*. Third, the domain of attraction of the equilibrium $x = x^*$, $z = 0$ is clearly defined. A similar result can be obtained for system (2) where f, g, F, and G are also functions of t.

IV. Example

Consider the second order system

$$\dot{x} = x - x^3 + z, \quad \mu\dot{z} = -x - z \tag{17}$$

whose reduced system is

$$\dot{x} = -x^3, \ z = -x \tag{18}$$

which is the same system as (10). Using (11), we construct (13) to be

$$L = \tfrac{1}{4}x^4 + \tfrac{1}{4}\mu(z + x + 2x^3)^2 \tag{19}$$

whose time derivative with respect to (17) is

$$\begin{aligned}\dot{L} = &(-1 + \mu(1 + 6x^2) + \mu^2 \tfrac{9}{4}(1 + 6x^2)^2)x^6 \\ &+ \tfrac{1}{4}(-1 + 2\mu(1 + 6x^2))(z + x)^2 - \tfrac{1}{4}(z + x + 3\mu x^3(1 + 6x^2))^2.\end{aligned} \tag{20}$$

Note that $v = x^2$ is also a Lyapunov function guaranteeing that $x = 0$ of (18) is asymptotically stable. However, if it were used in L instead of (11), $\dot{L}$ would not have been a complete square form. In the region $|x| \le \sqrt{2}$, $\mu^* = 0.01$ is sufficient to guarantee that $\dot{L} < 0$, and hence L is a Lyapunov function for (17) for $\mu \in (0, 0.01]$. The closed contours generated by $L = 1$ for $\mu = 0.01$ and $\mu = 0.005$ are shown in Fig. 1, where the domain of attraction is larger for $\mu = 0.005$ as indicated in the proof of Theorem I.

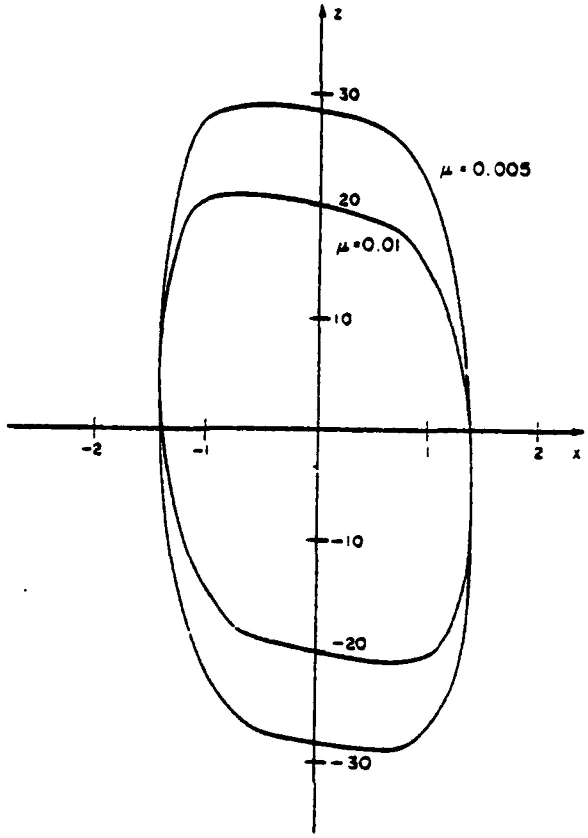

FIG. 1. Contours of $L = 1$ for $\mu = 0.01$ and $\mu = 0.005$.

V. Application to Control Systems

Theorem I is now applied to the design of a stabilizing control $u \in R^r$ for the system

$$\begin{aligned} \dot{x} &= f(x) + F(x)z + B_1(x)u \\ \mu\dot{z} &= g(x) + G(x)z + B_2(x)u. \end{aligned} \tag{21}$$

Formally letting $\mu = 0$, the reduced system of (21) is

$$\dot{x} = (f - FG^{-1}g) + (B_1 - FG^{-1}B_2)u = a + Bu. \tag{22}$$

Systems (21) and (22) are assumed to satisfy the following conditions for $x \in D$:

(I′) In addition to condition I, the matrices B_1 and B_2 are bounded and differentiable with respect to x.
(II′) G is non-singular and

$$\operatorname{rank}[B_2, GB_2, \ldots, G^{m-1}B_2] = m. \tag{23}$$

(III′) There exists a vector $h(x) \in R^r$ with $h(x^*) = 0$ such that condition III is satisfied with $a(x) + B(x)h(x)$ replacing $a(x)$.

By condition III′, there exists a stabilizing feedback control $u(x) = h(x)$ for the reduced system (22) such that the feedback controlled system

$$\dot{x} = a + Bh \tag{24}$$

possesses a Lyapunov function of the form (8) guaranteeing that $x = x^*$, $z = 0$ is asymptotically stable. Again we assume that D is the closed domain of attraction. From the reduced control, we formulate a stabilizing feedback control for (21).

Corollary 1

If conditions I′–III′ are satisfied, then there exists a $\mu^* > 0$ such that for all $x \in D_1$, $z \in E$, and $\mu \in (0, \mu^*]$, the equilibrium $x = x^*$, $z = 0$ of the system (21) controlled by

$$u(x, z) = (I + H(x)G^{-1}(x)B_2(x))h(x) + H(x)G^{-1}(x)g(x) + H(x)z \tag{25}$$

where Re $\{\lambda(G + B_2H)\} \leq \sigma_4$ for a fixed $\sigma_4 < 0$, is asymptotically stable.

Proof: Since condition II′ is satisfied, we can find $H(x)$ such that Re $\{\lambda(G + B_2H)\} \leq \sigma_4 < 0$. Thus system (21) controlled by (25) becomes

$$\begin{aligned} \dot{x} &= f + B_1\ (I + HG^{-1}B_2)h + B_1HG^{-1}g + (F + B_1H)z \\ \mu\dot{z} &= g + B_2\ (I + HG^{-1}B_2)h + B_2HG^{-1}g + (G + B_2H)z. \end{aligned} \tag{26}$$

Letting $\mu = 0$, the reduced system of (26) is

$$\begin{aligned} \dot{x} &= (f + B_1HG^{-1}g) - (F + B_1H)(G + B_2H)^{-1}(g + B_2HG^{-1}g) \\ &\quad + (B_1 - (F + B_1H)(G + B_2H)^{-1}B_2)(I + HG^{-1}B_2)h \\ &= a + Bh \end{aligned} \tag{27}$$

by using a simple algebraic manipulation. Hence from condition III′, (27) possesses a Lyapunov function of the form (8) and by Theorem I, the equilibrium $x = x^*$, $z = 0$ is asymptotically stable.

Corollary 1 outlines a low order design where the reduced system and the boundary layer system are considered separately. In addition, the parameter μ is not required to be known exactly provided that it is sufficiently small. The result here is a generalization of the design obtained for linear time-invariant singularly perturbed systems (7).

VI. Conclusion

In this paper we have presented sufficient conditions to guarantee the asymptotic stability of a class of nonlinear singularly perturbed systems. For such a class of systems, we have given a procedure for constructing a Lyapunov function, and a clearly defined domain of attraction of the equilibrium is obtained. We have also proposed a stabilizing feedback control for such systems.

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SECTION 6

TRAJECTORY OPTIMIZATION

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Optimal Open- and Closed-Loop Control of Singularly Perturbed Linear Systems

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Abstract—A simple method is given to obtain either an approximate open- or closed-loop control or an approximate solution to the linear quadratic fixed and free endpoint optimal control problems. This approximation is valid over the entire time interval, or with further computational reduction, only on the open interval. Emphasis is placed on use of control-oriented hypotheses, practical aspects of implementing the approximate controls, and interpretation of these controls. The approximating design procedure given is illustrated through examples which clarify and demonstrate it.

Introduction

SINGULAR perturbation theory can often be applied in solving linear quadratic fixed and free endpoint optimal control problems. First, however, it is necessary for the control engineer to decide if his particular system equations are of the singularly perturbed form. If small coefficients multiplying derivative terms in the system equations such as those representing capacitances, inductances, time constants, or other small parasitic parameters are present or if such coefficients within the system are separated by at least an order of magnitude, then it is highly probable that singular perturbation theory can be applied. It remains only to verify that the hypotheses of such theorems are satisfied.

The small derivative coefficients, expressible as the product of a small positive scalar μ and an appropriate constant, are responsible for creating a "stiff" system of differential equations which is difficult to solve using existing methods. An additional complication of singularly perturbed systems considered here is that both widely varying decay transients and widely varying growth transients are present as compared with only the decay transients implied by the stiffness of the system.

Use of singular perturbation theory provides the engineer with a simple means to obtain either an approximate open- or closed-loop control or an approximate solution to the original system. The approximation obtained results from the solution of a lower order system than the original. Not only is the system order reduced, but the stiffness behavior is also eliminated. If the given μ is sufficiently small, the approximate solution or control will be close to the actual. The smaller μ is, the closer the approximation. This approximation is valid over the entire time interval including both initial and terminal times, or with further computational reduction, only on the open interval. While there is no easy test to determine the bound on μ so that one knows in advance if μ is sufficiently small, the smallness criterion can be checked by other means. This can be done by using techniques such as utilizing physical intuition of the original system, testing the approximate control in the original system to determine if the predicted approximate results are obtained, or checking to determine if a Lyapunov function such as in [13] satisfies required properties for stability. Once it has been determined that μ is sufficiently small, the approximation could be utilized, for example, to save computer memory and computation time by using low order approximate equations in place of the high order original system equations. For a system required to be solved in real time on a minicomputer, the reduction in order of the system could mean the difference between feasibility and impossibility of implementation.

Although higher order approximating expansions, for which the theory developed here applies but which are not formally presented in this paper, are often given for the control or solution to such problems, they appear more of an interest from a theoretical development than in application. It is felt that problems requiring expansions greater than second order are better suited for solution by other techniques. *It is noted that increasing the number of* expansion terms in no way implies that the constant μ, the value of which is defined for a physical system, will be small enough to apply singular perturbation theory if it was not originally, even though this possibility exists.

This paper presents a thorough treatment of the singularly perturbed fixed and free endpoint optimal control problems with linear quadratic performance indices by considering both open- and closed-loop controls, requirements for the implementation of these controls, and interpretation of the form of the controls. Emphasis is placed on the development of a new theory: one that is applicable to the solution of the fixed and free endpoint problems, regardless if an open- or closed-loop control is desired, and one of great significance because interpretation of the equations and results is possible which is readily understood by control engineers. This theory uses a dichotomy transformation to separate the original optimal problem into two free endpoint problems where the latter problems, unlike the original, can be solved using initial value singular perturbation theory. The initial value singular perturbation theory is first applied to the Riccati equation

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and then to the controlled plant. The transformation can also be seen as a method to separate a two point boundary value problem into two initial value problems. The dichotomy transformation is used in the feedback form of the free endpoint problem to show that when the approximate feedback control is inserted in the higher order system, a solution close to the original solution results and is valid over the entire time interval $[t_0,T]$—the degree of closeness depending on the smallness of μ. This is neither done nor is possible using the approach given in [4],[15], which substantially extended the theory given in [9]. To make the theory most useful to control engineers, control-oriented hypotheses are given. Use of a different theory for solution of the singularly perturbed free endpoint problem is presented in [8], but the result and corresponding hypotheses are strictly mathematically oriented and applied only for the optimal open-loop control. Other papers in the general area include [3], [7], [12], [16]. To clarify and demonstrate the theory, a simple design procedure with illustrative example is given. The reader interested in applying the theory needs only to read the theorems and associated discussions since the theorems contain the design procedure.

Problem Statement

Consider a control system

$$\begin{bmatrix} \dot{x}_1 \\ \mu\dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_{11}(t,\mu) & A_{12}(t,\mu) \\ A_{21}(t,\mu) & A_{22}(t,\mu) \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1(t,\mu) \\ B_2(t,\mu) \end{bmatrix}u \quad (1)$$

where x_1, x_2, and u are n_1-, n_2-, and m-vectors, respectively, and μ is a small positive scalar. For brevity, (1) is rewritten as follows:

$$I_\mu \dot{x} = Ax + Bu, \quad I_\mu = \begin{bmatrix} I_1 & 0 \\ 0 & \mu I_2 \end{bmatrix} \quad (1)$$

where I_1 and I_2 are $n_1 \times n_1$ and $n_2 \times n_2$ identities. Also, arguments of functions are dropped when no confusion results, and a bar is used to indicate that $\mu = 0$. Thus A_{11} denotes $A_{11}(t,\mu)$, and $\bar{A}_{11} = \bar{A}_{11}(t)$ denotes $A_{11}(t,0)$.

The first problem to be studied is the minimization of

$$J = \tfrac{1}{2}\int_{t_0}^{T} [x'Qx + u'Ru]\,dt \quad (2)$$

with $x = x(t,\mu)$ specified at both t_0 and T as

$$x_1(t_0,\mu) = x_1^0, \qquad x_1(T,\mu) = x_1^T \quad (3)$$

$$x_2(t_0,\mu) = x_2^0, \qquad x_2(T,\mu) = x_2^T. \quad (4)$$

In the second problem, $x(T,\mu)$ is free, and a terminal cost is added to (2).

Given a $\mu^* > 0$, the following hypothesis is made:

Hypothesis 1: For all $t\in[t_0,T]$ and $\mu\in[0,\mu^*]$, the elements of A, B, Q, and R are twice continuously differentiable functions of t and μ, R is symmetric positive definite, and Q is symmetric positive semidefinite.

In the standard necessary optimality conditions, the adjoint variable λ is replaced by $I_\mu\lambda$, and $BR^{-1}B'$ is denoted by S. It is then seen that $x(t,\mu)$, $\lambda(t,\mu)$ must satisfy

$$\begin{bmatrix} I_\mu\dot{x} \\ I_\mu\dot{\lambda} \end{bmatrix} = \begin{bmatrix} A & -S \\ -Q & -A' \end{bmatrix}\begin{bmatrix} x \\ \lambda \end{bmatrix} \quad (5)$$

and boundary conditions (3) and (4). Similarly, the control u is given by

$$u = -R^{-1}B'\lambda. \quad (6)$$

When μ is set equal to zero in (5), its order reduces from $2(n_1 + n_2)$ to $2n_1$. Thus (3)–(5) constitute a "singularly perturbed" two point boundary value problem. The *reduced problem* is

$$\begin{bmatrix} \dot{\bar{x}}_1 \\ \dot{\bar{\lambda}}_1 \end{bmatrix} = \left\{\begin{bmatrix} \bar{A}_{11} & -\bar{S}_{11} \\ -\bar{Q}_{11} & -\bar{A}_{11}' \end{bmatrix} - \begin{bmatrix} \bar{A}_{12} & -\bar{S}_{12} \\ -\bar{Q}_{12} & -\bar{A}_{21}' \end{bmatrix} \times \begin{bmatrix} \bar{A}_{22} & -\bar{S}_{22} \\ -\bar{Q}_{22} & -\bar{A}_{22}' \end{bmatrix}^{-1}\begin{bmatrix} \bar{A}_{21} & -\bar{S}_{12}' \\ -\bar{Q}_{12}' & -\bar{A}_{12}' \end{bmatrix}\right\}\begin{bmatrix} \bar{x}_1 \\ \bar{\lambda}_1 \end{bmatrix} \quad (7)$$

subject to (3) if the indicated inverse exists. The remaining variables $\bar{x}_2$ and $\bar{\lambda}_2$ are algebraically related to $\bar{x}_1$ and $\bar{\lambda}_1$ by

$$\begin{bmatrix} \bar{x}_2 \\ \bar{\lambda}_2 \end{bmatrix} = -\begin{bmatrix} \bar{A}_{22} & -\bar{S}_{22} \\ -\bar{Q}_{22} & -\bar{A}_{22}' \end{bmatrix}^{-1}\begin{bmatrix} \bar{A}_{21} & -\bar{S}_{12}' \\ -\bar{Q}_{12}' & -\bar{A}_{12}' \end{bmatrix}\begin{bmatrix} \bar{x}_1 \\ \bar{\lambda}_1 \end{bmatrix} \quad (8)$$

and, in general, $\bar{x}_2$ does not satisfy (4). The corresponding reduced control is then

$$\bar{u} = -\bar{R}^{-1}\bar{B}'\bar{\lambda}. \quad (9)$$

Equations (7)–(9) could also have been developed from the reduced optimal problem defined by (1)–(3) where μ is first set equal to zero.

Of particular interest is when the reduced solution $\bar{x}(t)$, $\bar{\lambda}(t)$ can approximate the actual solution $x(t,\mu)$, $\lambda(t,\mu)$ on an open interval of $[t_0,T]$ for μ sufficiently small. Since $\bar{x}_2(t)$ can violate (4), large discrepancies or "boundary layers" are anticipated at the ends of the interval. *Layer correction terms* must thus be added to the reduced solution for the approximation to be valid over the entire interval. Since the boundary layers result from the fast transients of $x(t,\mu)$, $\lambda(t,\mu)$, it is important that the correction terms be obtained separately from $\bar{x}(t)$, $\bar{\lambda}(t)$ in a "stretched" time scale. The crucial hypothesis of the time scale separation method used here is that system (1) be *boundary layer controllable* and *boundary layer observable*.[1]

Hypothesis 2: For all $t\in[t_0,T]$,

$$\text{rank } [\bar{B}_2,\bar{A}_{22}\bar{B}_2,\cdots,(\bar{A}_{22})^{n_2-1}\bar{B}_2] = n_2$$

$$\text{rank } [\bar{C}_2',\bar{A}_{22}'\bar{C}_2',\cdots,(\bar{A}_{22}')^{n_2-1}\bar{C}_2'] = n_2$$

where $\bar{C}_2$ satisfies $\bar{C}_2'\bar{C}_2 = \bar{Q}_{22}$.

Here Hypothesis 2 guarantees the existence of the inverse in (7) and (8) and also will allow the systems from which the boundary layer terms are obtained to be stabilized.

[1] A less restrictive stabilizability condition can be assumed but adds unwanted complexity to the subsequent proofs. It is readily apparent from the proof of the subsequent theorem that the theorem applies to the problem $J = \frac{1}{2}\int_0^T u^2\,dt$, $\mu\dot{x}_2 = x_2 + u$, and x_2^0, x_2^T specified. Here Hypothesis 2 is not satisfied. See [5] for a further discussion on stabilizability.

Fundamental Result

The time scale separation method is based on a theorem which defines an $0(\mu)$ approximation[2] of the exact solution $x(t,\mu)$, $\lambda(t,\mu)$. The formulation of the theorem is done in such a way to provide the steps of a design procedure to be discussed in the next section. The rigorous proof is postponed until a later section.

Theorem:

1) Assume that Hypotheses 1 and 2 hold and that $\bar{x}(t)$, $\bar{\lambda}(t)$ exists and is the uniquely defined reduced solution satisfying (7) with boundary condition (3).

2) Let $\bar{P}_{22}$ and $\bar{N}_{22}$ be the positive and negative definite symmetric roots, respectively, of the algebraic equation

$$K_{22}\bar{A}_{22} + \bar{A}_{22}'K_{22} - K_{22}\bar{S}_{22}K_{22} + \bar{Q}_{22} = 0 \quad (10)$$

for all $t \in [t_0,T]$.

3) Let $\mathcal{L}(\tau)$ and $\mathcal{R}(\sigma)$ be the solutions of the two mutually independent time invariant initial value problems

$$\frac{d\mathcal{L}(\tau)}{d\tau} = [\bar{A}_{22}(t_0) - \bar{S}_{22}(t_0)\bar{P}_{22}(t_0)]\mathcal{L}(\tau) \quad (11)$$

$$\mathcal{L}(0) = x_2^0 - \bar{x}_2(t_0) \quad (12)$$

and

$$\frac{d\mathcal{R}(\sigma)}{d\sigma} = [\bar{A}_{22}(T) - \bar{S}_{22}(T)\bar{N}_{22}(T)]\mathcal{R}(\sigma) \quad (13)$$

$$\mathcal{R}(0) = x_2^T - \bar{x}_2(T). \quad (14)$$

4) Let $\mathcal{U}_L(\tau)$ and $\mathcal{U}_R(\sigma)$ be the "left" and "right" boundary terms defined by

$$\mathcal{U}_L(\tau) = -\bar{R}^{-1}(t_0)\bar{B}_2'(t_0)\bar{P}_{22}(t_0)\mathcal{L}(\tau) \quad (15)$$

$$\mathcal{U}_R(\sigma) = -\bar{R}^{-1}(T)\bar{B}_2'(T)\bar{N}_{22}(T)\mathcal{R}(\sigma). \quad (16)$$

Then there exists $\mu^* > 0$ such that for all $t \in [t_0,T]$, $\mu \in (0,\mu^*]$

$$x_1(t,\mu) = \bar{x}_1(t) + 0(\mu) \quad (17)$$

$$x_2(t,\mu) = \bar{x}_2(t) + \mathcal{L}(\tau) + \mathcal{R}(\sigma) + 0(\mu) \quad (18)$$

$$\lambda_1(t,\mu) = \bar{\lambda}_1(t) + 0(\mu) \quad (19)$$

$$\lambda_2(t,\mu) = \bar{\lambda}_2(t) + \bar{P}_{22}(t)\mathcal{L}(\tau) + \bar{N}_{22}(t)\mathcal{R}(\sigma) + 0(\mu) \quad (20)$$

and

$$u(t,\mu) = \bar{u}(t) + \mathcal{U}_L(\tau) + \mathcal{U}_R(\sigma) + 0(\mu) \quad (21)$$

where $\tau = (t - t_0)/\mu$ and $\sigma = (t - T)/\mu$ which define the "stretched" time scales.

Remark: Hypotheses 1 and 2 guarantee that $\bar{P}_{22}$ and $\bar{N}_{22}$ defined in 2) exist and are unique and that $(\bar{A}_{22} - \bar{S}_{22}\bar{P}_{22})$ and $-(\bar{A}_{22} - \bar{S}_{22}\bar{N}_{22})$ are stable matrices[3] for each $t \in [t_0,T]$. Hence the norm of $\mathcal{L}(\tau)$ is bounded by an exponentially decaying function of τ. In t-scale, the decay is very rapid since its time constant is $0(\mu)$. The same is true about the decay of $\mathcal{R}(\sigma)$ in $(T - t)$-scale. Therefore, (12) and (14) are in agreement with (18). Equation (10) possesses a unique symmetric positive definite solution $\bar{P}_{22}$, and $(\bar{A}_{22} - \bar{S}_{22}\bar{P}_{22})$ is a stable matrix for each $\bar{t} \in [t_0,T]$ as is evident from the solution to problem (22) where t_0 is replaced by $\bar{t}$ in (22). A solution to problem (22) is guaranteed to exist as a consequence of Hypotheses 1 and 2 [1, p. 785]. A similar result holds for the negative definite case using [14]. Also, within an $0(\mu)$ approximation, $\bar{P}_{22}(t)\mathcal{L}(\tau)$ can be replaced by $\bar{P}_{22}(t_0)\mathcal{L}(\tau)$ and $\bar{N}_{22}(t)\mathcal{R}(\sigma)$ by $\bar{N}_{22}(T)\mathcal{R}(\sigma)$; that is, a simplified form of (20) is

$$\lambda_2(t,\mu) = \bar{\lambda}_2(t) + \bar{P}_{22}(t_0)\mathcal{L}(\tau) + \bar{N}_{22}(T)\mathcal{R}(\sigma) + 0(\mu). \quad (20')$$

Similar simplifications based on a rapid decay of $\mathcal{L}(\tau)$ and $\mathcal{R}(\sigma)$ are made throughout the text. Also it is noted that assumption 1) of the theorem is sufficient for guaranteeing the existence of a solution to the problem (3)–(5) for $\mu \in (0,\mu^*]$, which in turn implies its optimality.

[2] A scalar, vector, or a matrix function $f(t,\mu)$ on an interval $[t_0,T]$ is $0(\mu)$ if there exist positive constants α and μ^* such that the norm of f satisfies $|f| \leq \alpha\mu$ for all $\mu \epsilon [0,\mu^*]$.

[3] A stable matrix is one in which the real parts of all its eigenvalues are less than a fixed negative number.

Discussion

In applications of the theorem, the two major jobs are first, to solve the reduced problem and second, to obtain the layer correction terms. The time scales for these two operations can be selected to be independent. For the reduced problem, a standard two point boundary value technique is used. The advantage over the original problem is that the order is lower, and the fast phenomena due to μ are eliminated.

The x_2 boundary layer correction terms $\mathcal{L}$ and $\mathcal{R}$ are interpreted as the solutions of two time invariant regulator problems in stretched time scale. The $\mathcal{L}$ problem is

$$J = \int_0^\infty (\mathcal{L}'\bar{Q}_{22}(t_0)\mathcal{L} + \mathcal{U}'\bar{R}(t_0)\mathcal{U})\, d\tau$$

$$\frac{d\mathcal{L}}{d\tau} = \bar{A}_{22}(t_0)\mathcal{L} + \bar{B}_2(t_0)\mathcal{U}, \qquad \mathcal{L}(0) = x_2^0 - \bar{x}_2(t_0), \quad (22)$$

and the $\mathcal{R}$ problem is

$$J = \int_{-\infty}^0 (\mathcal{R}'\bar{Q}_{22}(T)\mathcal{R} + \mathcal{U}'\bar{R}(T)\mathcal{U})\, d\sigma$$

$$\frac{d\mathcal{R}}{d\sigma} = \bar{A}_{22}(T)\mathcal{R} + \bar{B}_2(T)\mathcal{U}, \qquad \mathcal{R}(0) = x_2T - \bar{x}_2(T) \quad (23)$$

where the minimizing controls are (15) and (16), respectively. Hence, the original optimal control problem (1)–(4) has been divided into three optimal control problems for x_2: $\mathcal{U}_L$, (15), forces x_2 to rapidly approach $\bar{x}_2$ in the initial boundary layer; $\bar{u}$, (9), results in x_2 being close to $\bar{x}_2$ except at the ends of the interval; and $\mathcal{U}_R$, (16), forces x_2 to rapidly separate from $\bar{x}_2$ to attain its terminal condition in the final boundary layer. Only $\bar{u}$ is needed for the x_1 variable. Control $\mathcal{U}_L$ stabilizes the left boundary layer even if unstable

modes are present in $\bar{A}_{22}(t_0)$; i.e., eigenvalues with positive real parts. Likewise, a similar statement holds for $\mathfrak{U}_R$.

In actual application, the desired results will not always be obtained if the approximate control given by (21) with $0(\mu)$ equal to zero is inserted into the original system (1), as also observed by Dr. P. Sannuti. Undesirable results occur when unstable modes are present in A_{22}. It is well known that the stability of a system is not affected by an open-loop control. A perturbed initial condition in x_2 from, for example, a disturbance or noise will result in an error. This error increases with time and gets worse as μ gets smaller: e.g., the solution corresponding to a perturbation in the initial condition of $\mu\dot{z} = z$, $z(t_0) = 0$ is $\delta z = e^{t/\mu}\delta z^0$. Now assume such disturbances and noise can be disregarded. Contrary to expectation, it frequently is the case that the closer the approximation, the further one is from the optimal response. Possibilities existing for the solution of (1) due to the approximate open-loop controller are: closeness to the optimal solution everywhere, closeness except in the boundary layer, or divergence from it as μ becomes smaller. These problems can be circumvented by using a combination open- and closed-loop controller of the form $u = Mx_2 + v$ where M is chosen such that the resulting coefficient matrix $(A_{22} + B_2M)$ of x_2 in the $\mu\dot{x}_2$ equation of (1) be stable. Hypothesis 2 guarantees the existence of such an M. Having determined M and the approximate x_2 and u solutions given by (18) and (21), respectively, v can then be found from the combination controller equation. Requiring also that the boundary layer solution (22) be optimal for each fixed $\bar{t} \in [t_0, T]$, where t_0 has been replaced by $\bar{t}$, requires M be equal to $(-\bar{R}^{-1}\bar{B}_2'\bar{P}_{22})$. Hence, the combination open- and closed-loop controller is given by

$$u = -\bar{R}^{-1}\bar{B}_2'\bar{P}_{22}x_2 + v. \tag{24}$$

Details of the time scale separation design procedure and the development of the approximate control in the case of unstable modes are illustrated through example. The first example illustrates steps (1)–(20) of the procedure. The system, the performance index, and the boundary conditions are

$$\dot{x}_1 = x_2$$
$$\mu\dot{x}_2 = tx_2 + u \qquad \text{(E1)}^4$$

$$J = \tfrac{1}{2}\int_1^2 [x_1^2 + (9 - t^2)x_2^2 + u^2]\, dt \qquad \text{(E2)}$$

$$x_1(1,\mu) = x_1^0, \qquad x_1(2,\mu) = x_1^T \qquad \text{(E3)}$$

$$x_2(1,\mu) = x_2^0, \qquad x_2(2,\mu) = x_2^T. \qquad \text{(E4)}$$

Since $A_{22} = t$, $B_2 = 1$, and $Q_{22} = 9 - t^2$, Hypothesis 2 holds for $0 < t < 3$. The exact optimal solution must satisfy

$$\dot{x}_1 = x_2$$
$$\mu\dot{x}_2 = tx_2 - \lambda_2$$
$$\dot{\lambda}_1 = -x_1$$
$$\mu\dot{\lambda}_2 = -(9 - t^2)x_2 - \lambda_1 - t\lambda_2 \qquad \text{(E5)}$$

subject to (E3), (E4). When μ is set equal to zero, the reduced problem is

$$\dot{\bar{x}}_1 = -\tfrac{1}{9}\bar{\lambda}_1$$
$$\dot{\bar{\lambda}}_1 = -\bar{x}_1 \qquad \text{(E7)}$$

subject to (E3). Its solution $\bar{x}_1(t)$, $\bar{\lambda}_1(t)$ is easily found using the eigenvalues $\bar{s}_1 = \frac{1}{3}$ and $\bar{s}_2 = -\frac{1}{3}$ of the system matrix in (E7). To complete the first part of the procedure, $\bar{x}_2$ and $\bar{\lambda}_2$ are evaluated from

$$\bar{x}_2 = -\frac{1}{9}\bar{\lambda}_1, \qquad \bar{\lambda}_2 = -\frac{t}{9}\bar{\lambda}_1. \qquad \text{(E8)}$$

In the second part, the Riccati equation is

$$2K_{22}t - K_{22}^2 + (9 - t^2) = 0 \qquad \text{(E10)}$$

and its roots are

$$\bar{P}_{22}(t) = t + 3, \qquad \bar{N}_{22}(t) = t - 3.$$

The use of $\bar{P}_{22}(1) = 4$ in (11) yields

$$\frac{d\mathfrak{L}}{d\tau} = -3\mathfrak{L}; \qquad \text{(E11)}$$

hence,

$$\mathfrak{L} = [x_2^0 - \bar{x}_2(1)]e^{-3(t-1)/\mu}.$$

Similarly, when $\bar{N}_{22}(2) = -1$ is substituted in (13),

$$\frac{d\mathfrak{R}}{d\sigma} = 3\mathfrak{R} \qquad \text{(E13)}$$
$$\mathfrak{R} = [x_2^T - \bar{x}_2(2)]e^{3(t-2)/\mu}.$$

The approximation (17)–(20) can now be formed. Consider only its x part:

$$x_1(t,\mu) = c_1e^{-t/3} + c_2e^{t/3} + 0(\mu) \qquad \text{(E17)}$$
$$x_2(t,\mu) = c_3e^{-t/3} + c_4e^{t/3} + c_5e^{-3(t-1)/\mu} + c_6e^{3(t-2)/\mu} + 0(\mu) \qquad \text{(E18)}$$

where $c_1,\cdots,c_6$ are known constants. Note that (E18) has four modes whose time constants are ± 3, $\pm\mu/3$. The layer correction modes are $9/\mu$ times faster than those of the reduced solution, and for $\mu = 0.01$, the ratio of the time constants is 900. Does the exact solution possess this "two-time-scale" property? Fortunately, the explicit solution of the time varying system (E5) is readily available for comparison. Its x_2 solution is

$$x_2(t,\mu) = \alpha_1e^{s_1t} + \alpha_2e^{s_2t} + \alpha_3e^{s_3t} + \alpha_4e^{s_4t}$$

where constants $\alpha_1,\cdots,\alpha_4$ may depend on μ, and $s_1,\cdots,s_4$

[4] For convenience, equation numbers are kept the same as in the main text, and "E" refers to "example."

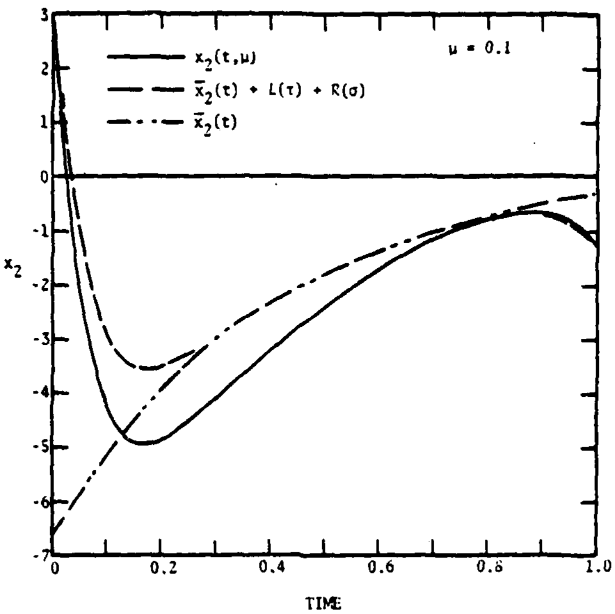

Fig. 1. Fixed endpoint problem.

are the roots of the characteristic equation

$$\mu^2 s^4 - (\mu + 9)s^2 + 1 = 0.$$

It can easily be shown that

$$s_1, s_2 = \pm \tfrac{1}{3} + 0(\mu); \qquad s_3, s_4 = \pm \frac{3}{\mu} + 0(\mu).$$

Hence, the time constants of the modes in (E18) indeed represent an $0(\mu)$ approximation of the time constants of the exact modes.

The second example

$$\dot{x}_1 = 1.5x_2$$

$$\mu\dot{x}_2 = -1.5x_1 + 0.5x_2 - u \tag{25}$$

$$J = \int_0^1 (2x_1^2 + x_2^2 + \tfrac{1}{2}u^2)\, dt \tag{26}$$

$$x_1^0 = 4, \qquad x_2^0 = 3, \qquad x_1^T = 0.5, \qquad x_2^T = -1.3 \tag{27}$$

is used to demonstrate the closeness and composition of the approximate x_2 as defined in (18) with $0(\mu)$ set to zero as well as to illustrate the results using both an open- and closed-loop approximating control in (25). In Fig. 1, $\mu = 0.1$, and the way in which the approximate solution is formed from both the reduced solution and the layer corrections is apparent. Without layer correction $\mathcal{L}(\tau)$, the error at $t = 0$ would be as large as 9.6, while with $\mathcal{L}(\tau)$, the maximum error is 1.5 and occurs at $t = 0.2$. For $\mu = 0.01$, the approximate and the exact solutions shown in Fig. 2 are virtually identical, and their behavior at $t = 0$ and $t = 1$ is a pictorial justification for the term "boundary layer." The dotted curve in Fig. 3 illustrates the divergence characteristic which results when the control given by (21) with $0(\mu)$ equal to zero is inserted in original plant (25) which possesses an unstable mode: i.e., the eigenvalue of $A_{22} = 0.5$. The divergence increases as μ decreases! By selecting the combination open- and closed-loop controller

$$u = 2x_2 + v \tag{28}$$

defined by (24), plant (25) no longer possesses an unstable mode in A_{22} upon substitution of this control there. The theorem can now be applied in a straightforward manner to find the open-loop optimal control v in the form of (21). The dot-dash-dot curve in Fig. 3 shows the resulting x_2 solution. Observe how close the approximation is.

Proof of Theorem

In the proof, the two point boundary value problem for (5) is transformed into two initial value problems. The transformation involves positive and negative definite solutions of a Riccati equation whose properties are now reviewed. Let

$$\lambda = K^\mu x, \qquad K^\mu = \begin{bmatrix} K_{11} & \mu K_{12} \\ K_{12}' & K_{22} \end{bmatrix} \tag{29}$$

and note that, in view of (5), K^μ satisfies the singularly perturbed Riccati equation

$$I_\mu \dot{K}^\mu = -K^{\mu\prime}A - A'K^\mu + K^{\mu\prime}SK^\mu - Q. \tag{30}$$

It is well known that Hypothesis 1 guarantees the existence and uniqueness of a solution to (30) for $t \in [t_0, T]$ and $\mu > 0$ if $K^\mu(T,\mu) = \Pi^\mu$ where $I_\mu \Pi^\mu$ is symmetric positive semidefinite for all $\mu \in [0, \mu^*]$, and Π^μ is defined by (29) where Π_{ij} replaces K_{ij}. Furthermore, $I_\mu K^\mu(t,\mu)$ is symmetric positive definite except possibly at $t = T$ where it can be semidefinite if $I_\mu \Pi^\mu$ is. For $\mu = 0$, the $(n_1 + n_2) \times (n_1 + n_2)$ differential equation (30) reduces to one $n_1 \times n_1$ differential equation and two algebraic equations

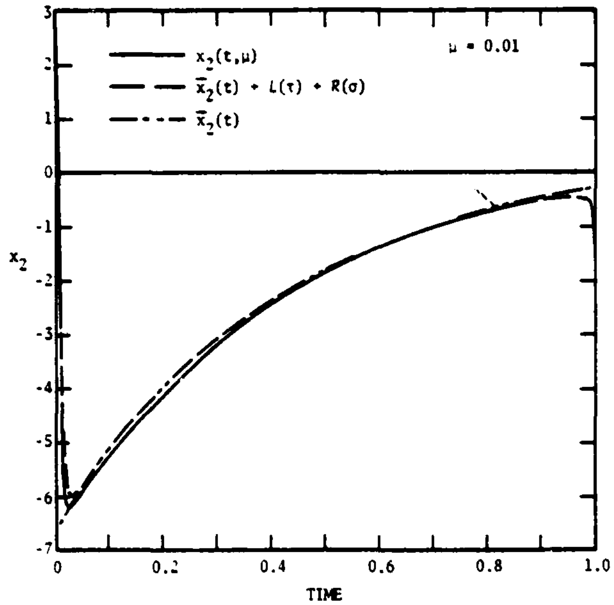

Fig. 2. Fixed endpoint problem.

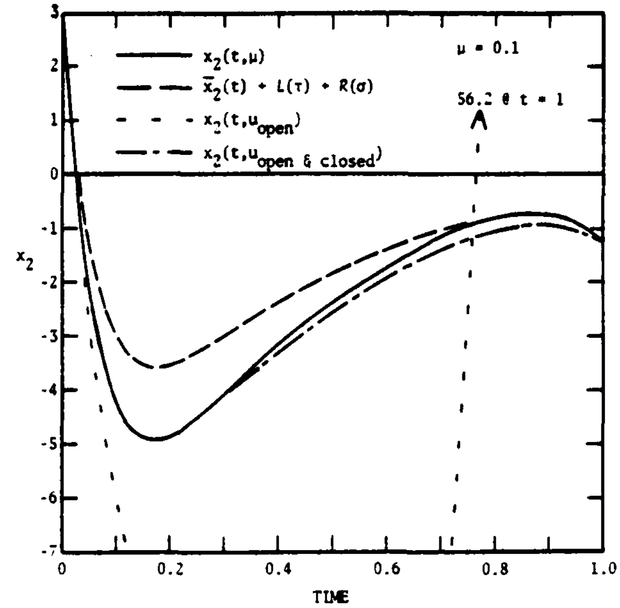

Fig. 3. Fixed endpoint problem.

$$\dot{K}_{11} = -K_{11}(\bar{A}_{11} - \bar{S}_{12}K_{12}') - (\bar{A}_{11} - \bar{S}_{12}K_{12}')K_{11} - K_{12}\bar{A}_{21} - \bar{A}_{21}'K_{12}' + K_{11}\bar{S}_{11}K_{11} + K_{12}\bar{S}_{22}K_{12}' - \bar{Q}_{11} \quad (31)$$

$$0 = -K_{12}(\bar{A}_{22} - \bar{S}_{22}K_{22}) - K_{11}\bar{A}_{12} + K_{11}\bar{S}_{12}K_{22} - \bar{A}_{21}'K_{22} - \bar{Q}_{12} \quad (32)$$

$$0 = -K_{22}\bar{A}_{22} - \bar{A}_{22}'K_{22} + K_{22}\bar{S}_{22}K_{22} - \bar{Q}_{22}. \quad (33)$$

Notice that (33) is identical to (10) and independent of (31) and (32). In [4] it was shown that Hypotheses 1 and 2 guarantee the existence of a unique $\bar{P}_{ij}$ solution to (31)–(33) subject both to $\bar{P}_{11}(T) = \Lambda_{11}$, where Λ_{11} is an arbitrary $n_1 \times n_1$ symmetric positive semidefinite matrix, and to $\bar{P}_{22}$ being the unique symmetric positive definite solution of (33). For this case, $\bar{P}_{11}(t)$ is a symmetric positive definite solution except possibly at $t = T$ where it can be semidefinite if Λ_{11} is. Similarly, with the additional use of [14], it also follows that there exists a unique $\bar{N}_{ij}$ solution to (31)–(33) upon selection of the unique symmetric negative definite solution $\bar{N}_{22}$ of (33) and subject to $\bar{N}_{11}(t_0) = \Gamma_{11}$, where Γ_{11} is an arbitrary $n_1 \times n_1$ symmetric negative semidefinite matrix. Here $\bar{N}_{11}(t)$ is the unique symmetric negative definite solution except possibly at $t = t_0$ where it can be semidefinite if Γ_{11} is. Let

$$P^\mu = \begin{bmatrix} P_{11} & \mu P_{12} \\ P_{12}' & P_{22} \end{bmatrix}, \qquad N^\mu = \begin{bmatrix} N_{11} & \mu N_{12} \\ N_{12}' & N_{22} \end{bmatrix} \quad (34)$$

be the two solutions of (30) subject to

$$P^\mu(T,\mu) = \begin{bmatrix} \Lambda_{11} & \mu\bar{P}_{12}(T) \\ \bar{P}_{12}'(T) & \bar{P}_{22}(T) \end{bmatrix} \quad (35)$$

$$N^\mu(t_0,\mu) = \begin{bmatrix} \Gamma_{11} & \mu\bar{N}_{12}(t_0) \\ \bar{N}_{12}'(t_0) & \bar{N}_{22}(t_0) \end{bmatrix}. \quad (36)$$

These Riccati solutions exist since $I_\mu P^\mu(T,\mu)$ and $-I_\mu N^\mu(t_0,\mu)$ are symmetric positive semidefinite matrices for μ sufficiently small. Hence, P_{ii} and $-N_{ii}$, $i = 1,2$ are symmetric positive semidefinite. Then there exists $\mu^* > 0$ such that the reduced Riccati solutions $\overline{P^\mu}$ and $\overline{N^\mu}$ approximate the exact Riccati solutions over the entire interval; that is,

$$\begin{aligned} P_{ij}(t,\mu) &= \bar{P}_{ij}(t) + 0(\mu) \\ N_{ij}(t,\mu) &= \bar{N}_{ij}(t) + 0(\mu) \end{aligned} \qquad i,j = 1,2 \quad (37)$$

for all $t \in [t_0,T]$, $\mu \in [0,\mu^*]$. This is a special case of [6],[10] when boundary layers are "erased" due to the matching of end conditions as accomplished by (35) and (36).

Matrices P^μ and N^μ are now used to introduce the transformation

$$x = \ell + r \quad (38)$$

$$\lambda = P^\mu(t,\mu)\ell + N^\mu(t,\mu)r. \quad (39)$$

The nonsingularity of the coefficient matrix of the transformation for $t \in [t_0,T]$, $\mu \in [0,\mu^*]$ is now shown by proving the determinant of this matrix is nonsingular. Rewriting this transformation in the form $[x_1,\lambda_1,x_2,\lambda_2]' = TR[\ell_1,r_1,\ell_2,r_2]'$, the determinant of TR can be written as

$$\det\begin{bmatrix} I_1 & I_1 \\ P_{11} & N_{11} \end{bmatrix} \cdot \det\left\{\begin{bmatrix} I_2 & I_2 \\ P_{22} & N_{22} \end{bmatrix} - \mu\begin{bmatrix} 0 & 0 \\ P_{12}' & N_{12}' \end{bmatrix}\begin{bmatrix} I_1 & I_1 \\ P_{11} & N_{11} \end{bmatrix}^{-1}\begin{bmatrix} 0 & 0 \\ P_{12} & N_{12} \end{bmatrix}\right\} \quad (40)$$

upon application of the expression in [2, p. 46] to find the determinant of a block partitioned matrix. The first determinant is nonzero since it is equal to the determinant of $(N_{11} - P_{11})$, a symmetric negative definite matrix. Semidefiniteness is not possible since P_{ii}, $i = 1,2$, can be singular only at $t = T$ and N_{ii}, $i = 1,2$ only at $t = t_0$. Thus from (37), the determinant of TR becomes

$$\det(N_{11} - P_{11}) \cdot \det(N_{22} - P_{22}) + 0(\mu) \quad (41)$$

for μ sufficiently small which thus is clearly nonzero. Substitution of (38) and (39) into (5) "dichotomizes" this two point boundary value problem into an initial value problem starting at $t = t_0$,

$$I_\mu \dot{\ell} = (A - SP^\mu)\ell, \qquad \ell(t_0,\mu) = \ell^0(\mu) \quad (42)$$

and an initial value problem in reverse time starting at $t = T$,

$$I_\mu \dot{r} = (A - SN^\mu)r, \qquad r(T,\mu) = r^T(\mu) \quad (43)$$

providing there exist $\ell^0(\mu)$ and $r^T(\mu)$ uniquely expressible in terms of x^0 and x^T. If these boundary conditions do exist, (42) and (43) can be analyzed as singularly perturbed initial value problems. Note that (42) corresponds to the solution of a free endpoint optimal problem in forward time while (43) corresponds to the solution of a free endpoint optimal problem in reverse time. Since there are no boundary layer jumps in P^μ and N^μ, the right hand sides of these equations are continuous in μ. Furthermore, since their associated boundary layer systems (11) and (13) are asymptotically stable uniformly in parameters t_0 and T, in view of a theorem on singular perturbations of initial value problems,[5] there exists $\mu^* > 0$ such that for all $t \in [t_0,T]$, $\mu \in (0,\mu^*]$ the solutions of (42) and (43) are approximated by

$$\ell(t,\mu) = \begin{bmatrix} \bar{\ell}_1(t) \\ \bar{\ell}_2(t) + \mathcal{L}(\tau) \end{bmatrix} + 0(\mu) \quad (44)$$

$$r(t,\mu) = \begin{bmatrix} \bar{r}_1(t) \\ \bar{r}_2(t) + \mathcal{R}(\sigma) \end{bmatrix} + 0(\mu) \quad (45)$$

where $\bar{\ell}(t)$, $\bar{r}(t)$ are the reduced solutions of (42) and (43) formed by setting $\mu = 0$ in them and given by

$$\begin{bmatrix} \dot{\bar{\ell}}_1 \\ 0 \end{bmatrix} = (\bar{A} - \overline{SP^\mu})\ell, \qquad \bar{\ell}_1(t_0) = \ell_1^0(0) \quad (46)$$

$$\begin{bmatrix} \dot{\bar{r}}_1 \\ 0 \end{bmatrix} = (\bar{A} - \overline{SN^\mu})r, \qquad \bar{r}_1(T) = r_1^T(0) \quad (47)$$

and $\mathcal{L}(\tau)$, $\mathcal{R}(\sigma)$ are the layer terms of (42) and (43) given by (11), (13) with initial conditions $\ell_2^0(0) - \bar{\ell}_2(t_0)$, $r_2^T(0) - \bar{r}_2(T)$, respectively, and, as will be shown, expressible in terms of x variables as in (12) and (14). Furthermore, it will subsequently be shown that

$$\bar{x} = \bar{\ell} + \bar{r} \quad (48)$$

$$\bar{\lambda} = \overline{P^\mu}\bar{\ell} + \overline{N^\mu}\bar{r}. \quad (49)$$

Thus the substitution of (44) and (45) into transformation (38) and (39), in view of (37), (48), and (49), yields theorem equations (17)–(20). Theorem equation (21) then follows after the substitution of (19) and (20) into (6) and the use of identities (9), (15), and (16).

[5] This basic theorem is well documented in the literature. Readers unfamiliar with singular perturbations may consult [6],[10], and [11].

It will now be proven that the existence of the reduced solution $\bar{x}$, $\bar{\lambda}$ is sufficient for proving the existence of a solution $x(t,\mu)$, $\lambda(t,\mu)$ for $\mu \in (0,\mu^*]$. This is equivalent to showing there exist $\ell^0(\mu)$ and $r^T(\mu)$ uniquely expressible in terms of x^0, x^T. Expressing $\ell(t,\mu)$ and $r(t,\mu)$, using fundamental matrices, by

$$\ell(t,\mu) = \Phi(t,t_0,\mu)\ell^0(\mu), \qquad r(t,\mu) = \Psi(t,T,\mu)r^T(\mu) \quad (50)$$

results in the initial value singularly perturbed matrix differential equations

$$I_\mu \dot{\Phi} = (A - SP^\mu)\Phi, \qquad \Phi(t_0,t_0,\mu) = I \quad (51)$$

$$I_\mu \dot{\Psi} = (A - SN^\mu)\Psi, \qquad \Psi(T,T,\mu) = I. \quad (52)$$

Since the associated differential equations with each column of Φ, Ψ are identical in form with (42) and (43), respectively, the latter of which were stated as satisfying an initial value singular perturbation theorem, the following relation holds:

$$\begin{bmatrix} \phi_{1j}(t,t_0,\mu) & \psi_{1j}(t,T,\mu) \\ \phi_{2j}(t,t_0,\mu) & \psi_{2j}(t,T,\mu) \end{bmatrix} = \begin{bmatrix} \bar{\phi}_{1j} & \bar{\psi}_{1j} \\ \bar{\phi}_{2j} & \bar{\psi}_{2j} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ \phi_{2j}(\tau) & \psi_{2j}(\sigma) \end{bmatrix} + 0(\mu) \quad (53)$$

for $j = 1,2$, and $t \in [t_0,T]$, $\mu \in (0,\mu^*]$. Here $\phi_{2j}(\tau) \to 0$ as $\tau \to \infty$ and $\psi_{2j}(\sigma) \to 0$ as $\sigma \to -\infty$. Reduced solutions of (51), (52) are $[\bar{\phi}_{i2} \ \bar{\psi}_{i2}] = [0 \ 0]$ for $i = 1,2$. Relating the boundary conditions using (3), (4), (38), (50), and (53) yields

$$\begin{bmatrix} x^0 \\ x^T \end{bmatrix} = \left\{ \begin{bmatrix} I_1 & 0 & \bar{\psi}_{11}(t_0) & 0 \\ 0 & I_2 & \bar{\psi}_{21}(t_0) & 0 \\ \bar{\phi}_{11}(T) & 0 & I_1 & 0 \\ \bar{\phi}_{21}(T) & 0 & 0 & I_2 \end{bmatrix} + 0(\mu) \right\} \begin{bmatrix} \ell^0(\mu) \\ r^T(\mu) \end{bmatrix}. \quad (54)$$

It suffices to now prove that the coefficient matrix relating the boundary conditions is nonsingular. Clearly this is true if the coefficient matrix is nonsingular neglecting $0(\mu)$. Its determinant is the determinant of $[I - \bar{\phi}_{11}(T)\bar{\psi}_{11}(t_0)]$ and also results from the transformation relating x_1^0, x_1^T to $\ell_1^0(0)$, $r_1^T(0)$. This last transformation is surely nonsingular for recall equations (48), (49) hold as a consequence of assumption 1) of the Theorem which assumes the existence of the reduced $\bar{x}$, $\bar{\lambda}$ solution and the fact that transformation (38), (39) relating x, λ to ℓ, r is nonsingular at $\mu = 0$. The proof is now complete and the initial conditions $\ell_2^0(0) - \bar{\ell}_2(t_0)$, $r_2^T(0) - \bar{r}_2(T)$ are seen to be equivalent to (12), (14), respectively, using (48), reduced form of (50), and (54) with $0(\mu)$ set to zero. Note that the $\mathcal{L}(0)$ and $\mathcal{R}(0)$ boundary conditions given by (12) and (14) are directly obtainable from (18) with $0(\mu)$ set to zero. Consider, for example, x_2 at $t = t_0$. Upon substitution of (44), (45) into (38) and the subsequent use of

(48) yields

$$x_2^0 = \bar{x}_2(t_0) + \mathcal{L}(0) + \mathcal{R}((t_0 - T)/\mu) + 0(\mu). \quad (55)$$

Thus (12) is satisfied upon noting $\mathcal{R}((t_0 - T)/\mu)$ approaches zero as μ does. A similar argument holds for finding $\mathcal{R}(0)$. In fact, $\mathcal{L}(0)$ and $\mathcal{R}(0)$ are uniquely determined by this method.

Extension

The conditions of the theorem are now modified to apply to the terminal cost problem; that is, the problem of optimum control of (1) with the performance index

$$J = \tfrac{1}{2}x'I_\mu\Pi^\mu x|_{t=T} + \tfrac{1}{2}\int_{t_0}^{T} (x'Qx + u'Ru)\, dt \quad (56)$$

where $I_\mu\Pi^\mu$ is symmetric positive semidefinite for all $\mu \in [0,\mu^*]$ and

$$I_\mu\Pi^\mu = \begin{bmatrix} \Pi_{11} & \mu\Pi_{12} \\ \mu\Pi_{12}' & \mu\Pi_{22} \end{bmatrix}. \quad (57)$$

In this problem, x is free at $t = T$, and the boundary conditions for (5) are

$$x(t_0,\mu) = x^0, \qquad \lambda(T,\mu) = \Pi^\mu x(T,\mu). \quad (58)$$

The only change to be made in the theorem is a different initial condition for $\mathcal{R}(\sigma)$.

Corollary 1: Let Hypotheses 1, 2, and assumptions 2)–4) of the theorem be satisfied except for (14) which is replaced by

$$\mathcal{R}(0) = [\Pi_{22} - \bar{N}_{22}(T)]^{-1}[\bar{\lambda}_2(T) - \Pi_{12}'\bar{x}_1(T) - \Pi_{22}\bar{x}_2(T)]. \quad (59)$$

Then the theorem applies to the solution $x(t,\mu)$, $\lambda(t,\mu)$ of the two point boundary value problem (5) subject to (58).

Remarks: Assumption 1) of the theorem is more stringent than required here. Hypotheses 1 and 2 are sufficient to guarantee the existence and uniqueness of the reduced solution $\bar{x}(t)$, $\bar{\lambda}(t)$ as shown in [4, Lemma 3].

It readily follows upon replacing all of the variables by their $0(\mu)$ approximations that the performance index J, given by (2) for the fixed endpoint problem and by (56) for the free endpoint problem, can be expressed by $J = \bar{J} + 0(\mu)$. Here $\bar{J}$ is defined as J where all variables have been replaced by their reduced solutions. The absence of boundary layer terms in J is a consequence of the negligible area associated with such terms under the performance integral for small μ.

The proof of this corollary follows the proof of the theorem. As before, (48) and (49) can be shown to hold. This used in conjunction with (37)–(39), (44), and (45) yields (17)–(20). Using fundamental matrices, it follows as in the theorem that the coefficient matrix relating the $\ell^0(\mu)$, $r^T(\mu)$ to the x^0 boundary condition is nonsingular for μ sufficiently small. Here it will only be shown that $\mathcal{R}(0)$ as defined by (59) satisfies (38) and (39). At $t = T$, they reduce to

$$x_1(T,\mu) = \bar{x}_1(T) + 0(\mu) \quad (60)$$

$$x_2(T,\mu) = \bar{x}_2(T) + \mathcal{R}(0) + 0(\mu) \quad (61)$$

$$\lambda_2(T,\mu) = \bar{\lambda}_2(T) + \bar{N}_{22}(T)\mathcal{R}(0) + 0(\mu) \quad (62)$$

and must satisfy (58). The λ_2 component of (58) is

$$\lambda_2(T,\mu) = \Pi_{21}'x_1(T,\mu) + \Pi_{22}x_2(T,\mu). \quad (63)$$

Thus $\mathcal{R}(0)$, defined by (59), satisfies (60)–(63) when μ equals zero. This corollary will be illustrated in the next example given.

Closed-Loop Control

Up to this point, emphasis has been on open- (at least partially) loop controls. Considered now are approximate forms of the optimal feedback control of the free endpoint problem just treated. The response of the optimally controlled system is given by

$$I_\mu\dot{x} = (A - SK^\mu)x, \qquad x(t_0,\mu) = x^0 \quad (64)$$

where $K^\mu = K^\mu(t,\mu)$ is the solution of (30) with the end condition

$$K^\mu(T,\mu) = \Pi^\mu. \quad (65)$$

It was shown in [15], that if the optimal control is replaced by the reduced feedback control

$$u = -R^{-1}B'\overline{K^\mu}x \quad (66)$$

where $\overline{K^\mu}$ is the solution of (31)–(33) with $\bar{K}_{11}(T,\mu) = \Pi_{11}$ and $\bar{K}_{22}$ is the unique symmetric positive definite solution of (33), the plant response would be close to that of the optimal one of the open interval (t_0,T) providing μ was *sufficiently small. It was further shown there that* by appending the right layer correction term $\mathcal{K}(\sigma)$ of K^μ to $\overline{K^\mu}$ in (66) where $\mathcal{K}_{11}(\sigma) = 0$ and $\mathcal{K}_{12}(\sigma)$, $\mathcal{K}_{22}(\sigma)$ satisfy

$$\frac{d\mathcal{K}_{12}}{d\sigma} = -\mathcal{K}_{12}[\bar{A}_{22}(T) - \bar{S}_{22}(T)\bar{K}_{22}(T)] - [\bar{A}_{21}'(T) - \bar{K}_{11}(T)\bar{S}_{12}(T) - \bar{K}_{12}(T)\bar{S}_{22}(T)]\mathcal{K}_{22} + \mathcal{K}_{12}\bar{S}_{22}(T)\mathcal{K}_{22} \quad (67)$$

$$\frac{d\mathcal{K}_{22}}{d\sigma} = -\mathcal{K}_{22}[\bar{A}_{22}(T) - \bar{S}_{22}(T)\bar{K}_{22}(T)] - [\bar{A}_{22} - \bar{S}_{22}(T)\bar{K}_{22}(T)]'\mathcal{K}_{22} + \mathcal{K}_{22}\bar{S}_{22}(T)\mathcal{K}_{22} \quad (68)$$

and initial conditions

$$\mathcal{K}_{12}(0) = \Pi_{12} - \bar{K}_{12}(T), \qquad \mathcal{K}_{22}(0) = \Pi_{22} - K_{22}(T) \quad (69)$$

the feedback control

$$u = -R^{-1}B'(\overline{K^\mu} + \mathcal{K}(\sigma))x \quad (70)$$

would yield a plant response close to the optimal one on the partially closed interval $[t_0,T)$ but *did not* predict the

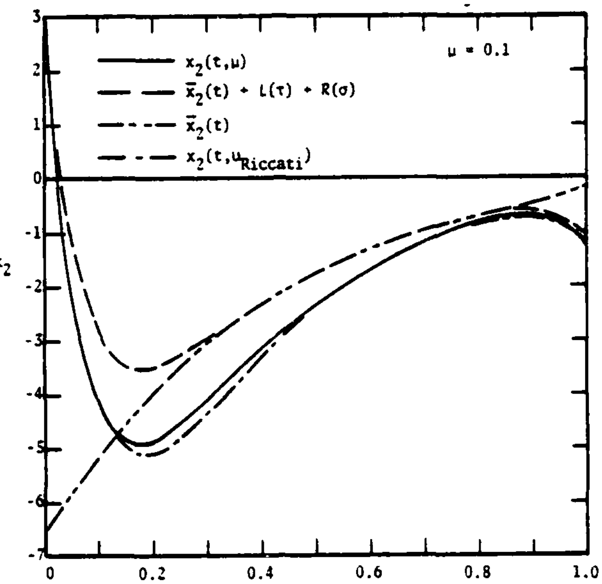

Fig. 4. Free endpoint problem.

behavior of the plant at the terminal interval. Here

$$K_{ij}(t,\mu) = \bar{K}_{ij}(t) + K_{ij}(\sigma) + 0(\mu) \qquad (71)$$

for all $t \in [t_0,T]$, $\mu \in (0,\mu^*]$, and $\mathcal{K}_{ij}(\sigma) \to 0$ as $\sigma \to -\infty$. Equations (67)–(69) can easily be derived from (30) and (58) by replacing $K_{ij}(t,\mu)$ in them by $[\bar{K}_{ij}(t) + \mathcal{K}_{ij}(\sigma)]$ for $i = 1,2$, where $\mathcal{K}_{11}(\sigma)$ is understood to be zero, and then setting μ equal to zero. These same equations are related to the $\mathcal{L}_1$, $\mathcal{L}_2$ boundary layer equations in [4],[15] by the relation

$$\mathcal{L}_i(\tau) = \bar{K}_{i2}(t) + \mathcal{K}_{i2}(\sigma), \qquad i = 1,2 \qquad (72)$$

where there it was shown the $\mathcal{L}_i$ solutions exist. This then implies the existence of the $\mathcal{K}_{i2}(\sigma)$ solutions. His technique of analyzing plant (1) controlled by (70) using initial value singular perturbation theory could not be applied when including the terminal time since the right hand side of (1) was not continuous at $\mu = 0$ as required by such theory. Use of the dichotomy transformation avoids this problem, and hence, Corollary 2 proves the response of (1) to control (70) is close to that using the optimal feedback control. The smaller μ is, the better the approximation. Before proving Corollary 2, a numerical example is given to illustrate this.

Example 3 uses the same plant used in example 2 but with the performance index

$$J = \tfrac{1}{2}x_1^2(T) + \int_0^1 (2x_1^2 + x_2^2 + \tfrac{1}{2}u^2)\, dt \qquad (73)$$

and initial conditions

$$x_1^0 = 4, \qquad x_2^0 = 3. \qquad (74)$$

See Fig. 4 for a comparison of the response of (25) using the optimal feedback control with the response using the approximating control (70). Although not shown, the two solutions are nearly identical for $\mu = 0.01$. For this example, the approximating feedback gains $\hat{K}_{ij}$ given by (71) with $0(\mu)$ equal to zero are

$$\hat{K}_{11}(t,\mu) = \frac{9 - 4e^{5(t-1)}}{3 + 2e^{5(t-1)}} \qquad (75)$$

$$\hat{K}_{12}(t,\mu) = \frac{3 - 8e^{5(t-1)}}{3 + 2e^{5(t-1)}} + \frac{5e^{1.5\sigma} - 2e^{3\sigma}}{1 + 2e^{3\sigma}} \qquad (76)$$

$$\hat{K}_{22}(t,\mu) = 2 - \frac{6e^{3\sigma}}{1 + 2e^{3\sigma}}. \qquad (77)$$

Fig. 4 also illustrates the closeness of the approximation for x_2 using Corollary 1.

Corollary 2: Let Hypotheses 1 and 2 be satisfied. Then the solution of

$$I_\mu \dot{x} = [A - S(\bar{K}^\mu + \mathcal{K}(\sigma))]x, \qquad x(t_0,\mu) = x^0 \quad (78)$$

is within $0(\mu)$ of the optimal solution (64) for all $t \in [t_0,T]$, $\mu \in (0,\mu^*]$. Furthermore, if the definitions (11) and (12) of $\mathcal{L}(\tau)$ in assumption 3) of the theorem are retained but the definitions (13) and (14) of $\mathcal{R}(\sigma)$ are replaced by

$$\mathcal{R}(\sigma) = [\bar{N}_{22}(T) - \bar{K}_{22}(T) - \mathcal{K}_{22}(\sigma)]^{-1} \times [\mathcal{K}_{12}'(\sigma)\bar{x}_1(T) + \mathcal{K}_{22}(\sigma)\bar{x}_2(T)] \quad (79)$$

and assumptions 2) and 4) are added, then the theorem applies to the solution $x(t,\mu)$ of the free endpoint problem (56), (58).

To prove this corollary, start with

$$r = (N^\mu - K^\mu)^{-1}(K^\mu - P^\mu)\ell \qquad (80)$$

which follows from (29), (38), and (39). The optimal solution to the free endpoint problem (56), (58) is expressed in x, ℓ variables, after substitution of (80) into (38) and re-

writing the ℓ differential equation (42), by

$$x = [I + (N^{\mu} - K^{\mu})^{-1}(K^{\mu} - P^{\mu})]\ell \tag{81}$$

$$I_{\mu}\dot{\ell} = (A - SP^{\mu})\ell, \qquad \ell(t_0,\mu) = \ell^0(\mu). \tag{82}$$

The existence of the optimal solution, and hence (81), (82), is guaranteed to exist by Hypotheses 1 and 2 for all $\mu \in (0,\mu^*]$. The proof now consists of showing that the solution of (81), (82) is within $0(\mu)$ of the solution (81), (82) where K^{μ} is replaced by its $0(\mu)$ approximation $[\overline{K^{\mu}} + \mathcal{K}(\sigma)]$. It will first be shown that the K^{μ} approximation changes the coefficient matrix in (81) by only an $0(\mu)$ amount and second that it changes $\ell^0(\mu)$ by only a similar amount. Knowing this and upon expanding the coefficient matrix and expanding ℓ by (44), the optimal x solution can be expressed as equal to the approximate solution plus $0(\mu)$.

From the properties previously mentioned pertaining to singularly perturbed Riccati systems, it is known that the determinant of $(N^{\mu} - K^{\mu})$ is nonsingular, approaches a nonzero value as μ approaches zero, and is continuous in μ for $\mu \in (0,\mu^*]$. Thus the K^{μ} approximation changes the coefficient matrix in (81) by only an $0(\mu)$ amount. Now choose Λ_{11} in (35) equal to Π_{11} so that $\overline{P^{\mu}} = \overline{K^{\mu}}$. Next, in (80), substitute (37) for P^{μ} and N^{μ}, (71) for K^{μ}, and (44) for ℓ. This yields

$$\mathfrak{r}_1 = 0(\mu) \tag{83}$$

$$\mathfrak{r}_2 = [\bar{N}_{22} - \bar{K}_{22} - \mathcal{K}_{22}]^{-1}[\mathcal{K}_{12}'\bar{\ell}_1 + \mathcal{K}_{22}(\bar{\ell}_2 + \mathcal{L})] + 0(\mu). \tag{84}$$

From (84), it is seen that $\mathfrak{r}_2$ at $t = t_0$ is $0(\mu)$; hence, $\ell^0(\mu) = x^0 + 0(\mu)$ using (38) which shows $\ell^0(\mu)$ is changed only by $0(\mu)$ from the K^{μ} approximation. From the homogeneous $\mathfrak{r}$ system (43), it follows that $\bar{\mathfrak{r}}(t) = 0$ which implies $\bar{x}(t) = \bar{\ell}(t)$. This and (38) results in $\mathcal{L}(0)$ as given by (12). Also, (79) then follows from (84) upon recognizing $\mathcal{L} \to 0$ as $\tau \to \infty$. Since the approximation of this corollary is within $0(\mu)$ of the optimal solution, $\mathcal{R}(\sigma)$ defined by (79) is within $0(\mu)$ of $\mathcal{R}(\sigma)$ defined by (13), (59). See [12] for an example illustrating this. Note that the reduced solution $\bar{x}$ can easily be found by solving only an initial value problem $\bar{\ell}$.

Conclusions

The control engineer, through the use of singular perturbation theory, has been provided with a simple means to obtain either an approximate open- or closed-loop control or an approximate solution to his original system, where the closeness of the approximation is determined by the smallness of μ. Requirements for implementation of these controls and interpretations of the control form have also been presented. The basis of the new theory developed has been using a dichotomy transformation to separate a two point boundary value problem into two initial value problems. Throughout this paper, the theory has been presented with control-oriented hypotheses and interpretations. A simple design procedure and corresponding example have provided clarification of this theory. Numerical examples have illustrated the corollaries.

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AD-A123 960 SINGULAR PERTURBATIONS AND TIME SCALES IN MODELING AND CONTROL OF DYNAMIC SYSTEMS(U) ILLINOIS UNIV AT URBANA DECISION AND CONTROL LAB P V KOKOTOVIC ET AL. NOV 80 3/4

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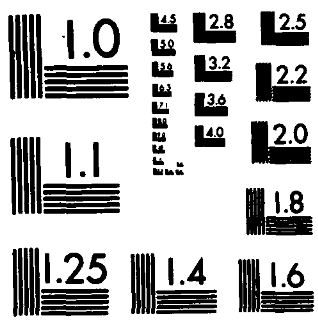

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A Class of Singularly Perturbed, Nonlinear, Fixed-Endpoint Control Problems[1,2]

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Abstract. Singular perturbation techniques are applied to a class of nonlinear, fixed-endpoint control problems to decompose the full-order problem into three lower-order problems, namely, the reduced problem and the left and right boundary-layer problems. The boundary-layer problems are linear-quadratic and, contrary to previous singular perturbation works, the reduced problem has a simple formulation. The solutions of these lower-order problems are combined to yield an approximate solution to the full nonlinear problem. Based on the properties of the lower-order problems, the full problem is shown to possess an asymptotic series solution.

Key Words. Optimal control problems, singularly perturbed nonlinear systems, time scale decomposition, asymptotic expansion, fixed-endpoint problems.

1. Introduction

High dimensionality and nonlinearity are among the factors complicating the solution of optimal control problems. For systems having slow and fast parts, the full optimal control problem can be decomposed into séparate lower-order problems, whose solutions are combined to yield a near-optimal control for the full problem. For linear–quadratic problems, this was demonstrated in Ref. 1. Here, we consider the fixed-endpoint problem of

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minimizing the performance index

$$J=\int_0^T [V_1(x,t,\mu)+V_2'(x,t,\mu)z+z'V_3(x,t,\mu)z+u'R(x,t,\mu)u]\,dt$$

$$=\int_0^T [V(x,z,t,\mu)+u'R(x,t,\mu)u]\,dt, \tag{1}$$

subject to the class of nonlinear systems

$$dx/dt=a_1(x,t,\mu)+A_1(x,t,\mu)z+B_1(x,t,\mu)u, \tag{2-1}$$

$$x(0,\mu)=x_0(\mu), \qquad x(T,\mu)=x_T(\mu), \tag{2-2}$$

$$\mu(dz/dt)=a_2(x,t,\mu)+A_2(x,t,\mu)z+B_2(x,t,\mu)u, \tag{2-3}$$

$$z(0,\mu)=z_0(\mu), \qquad z(T,\mu)=z_T(\mu), \tag{2-4}$$

where the states are $x\in R^n$ and $z\in R^m$, the control is $u\in R^r$, μ is a small parameter, and the primes denote transposition. We assume that the matrices V_3 and R are positive definite in x and that V is a positive definite function of x and z. Furthermore, we assume that a_i, A_i, B_i, x_0, x_T, z_0, z_T, V, R have asymptotic power series expansions as $\mu\to 0$ with infinitely differentiable coefficients. In addition, A_2 is assumed to be nonsingular. The Hamiltonian for problem (1)–(2) is

$$H(x,z,u,p,q,t,\mu)=V+u'Ru+p'(a_1+A_1z+B_1u)+q'(a_2+A_2z+B_2u), \tag{3}$$

where the costate variables p and μq satisfy the equations

$$dp/dt=-\nabla_x H(x,z,u,p,q,t,\mu), \tag{4-1}$$

$$\mu(dq/dt)=-\nabla_z H(x,z,u,p,q,t,\mu). \tag{4-2}$$

The control which solves

$$\partial H/\partial u=2Ru+B_1'p+B_2'q=0 \tag{5}$$

is the minimizing control

$$u=-\tfrac{1}{2}R^{-1}(B_1'p+B_2'q). \tag{6}$$

The substitution of the control (6) into systems (2) and (4) yields a nonlinear, two-point boundary-valued (TPBV) problem

$$dx/dt=a_1+A_1z-\tfrac{1}{2}B_1R^{-1}(B_1'p+B_2'q)=g_1(x,p,z,q,t,\mu), \tag{7-1}$$

$$dp/dt=-\nabla_x H(x,z,-\tfrac{1}{2}R^{-1}(B_1'p+B_2'q),p,q,t,\mu)$$
$$=g_2(x,p,z,q,t,\mu), \tag{7-2}$$

$$\mu(dz/dt) = a_2 + A_2 z - \tfrac{1}{2}B_2 R^{-1}(B_1' p + B_2' q) = g_3(x, p, z, q, t, \mu), \quad (7\text{-}3)$$

$$\mu(dq/dt) = -V_2' - 2V_3 z - A_1' p - A_2' q = g_4(x, p, z, q, t, \mu), \quad (7\text{-}4)$$

where the boundary conditions are

$$x(0, \mu) = x_0(\mu), \qquad x(T, \mu) = x_T(\mu), \quad (7\text{-}5)$$

$$z(0, \mu) = z_0(\mu), \qquad z(T, \mu) = z_T(\mu). \quad (7\text{-}6)$$

Optimal control problems of the type (1)–(2) with free endpoints are treated in Refs. 2–4, while more general free-endpoint problems are considered in Refs. 5–7. The fixed-endpoint problem has been treated in Ref. 8, where the performance index (1) is in quadratic form and the system (2) is linear. For the nonlinear fixed-endpoint problem, our approach is to decompose the full problem (1)–(2) into three separate lower-order problems, an n-dimensional, nonlinear *reduced problem* and two m-dimensional, linear–quadratic *boundary-layer problems*. Thus, the technique in Ref. 8 for linear–quadratic problems is now extended to nonlinear problems. A further result is that our formulation of the reduced problem is particularly simple. Then, similarly to the results of Refs. 2, 3, 4, 7, a solution x, z, p, q, u to the full TPBV problem (7) is shown to possess an asymptotic series expansion in μ and is approximated to $O(\mu)$ by combining the solutions of the reduced problem and the boundary-layer problems. For practical implementation, we propose a partially closed-loop control to achieve stability of the fast variable z.

The organization of the paper is as follows. First, we formulate the lower-order problems in Section 2 and propose an $O(\mu)$ approximate design (Theorem 3.1). For readers who are interested only in applying the theory, it is sufficient to read Sections 2 and 3 and the example in Section 4 illustrating the design procedure. The series expansions of x, z, p, q, u are dealt with in Section 6 and the Appendix.

2. Lower-Order Problems

Due to the presence of μ, the system (2) possesses a two-time-scale property, that is, the variable x varies slowly, while the variable z has a rapidly varying part. Letting $\mu = 0$ in system (2), which is equivalent to neglecting the fast part in z, we obtain

$$d\bar{x}/dt = \bar{a}_1 + \bar{A}_1 \bar{z} + \bar{B}_1 \bar{u}, \qquad \bar{x}(0) = x_0(0), \qquad \bar{x}(T) = x_T(0), \quad (8\text{-}1)$$

$$0 = \bar{a}_2 + \bar{A}_2 \bar{z} + \bar{B}_2 \bar{u}. \quad (8\text{-}2)$$

Here and in the following, an overbar indicates that $\mu = 0$. Assuming that $\bar{A}_2$ is nonsingular, the slow part $\bar{z}$ of z is solved from Eq. (8-2) as

$$\bar{z} = -\bar{A}_2^{-1}(\bar{a}_2 + \bar{B}_2\bar{u}). \tag{9}$$

Eliminating $\bar{z}$ from Eq. (8-1) and the performance index (1) evaluated at $\mu = 0$, we define the reduced problem as follows.

Reduced Problem. The reduced problem is to minimize the performance index

$$\bar{J} = \int_0^T [L_1(\bar{x}, t) + 2L_2'(\bar{x}, t)\bar{u} + \bar{u}'L_3(\bar{x}, t)\bar{u}]\, dt, \tag{10}$$

subject to

$$d\bar{x}/dt = \bar{a}(\bar{x}, t) + \bar{B}(\bar{x}, t)\bar{u}, \qquad \bar{x}(0) = x_0(0), \qquad \bar{x}(T) = x_T(0), \tag{11}$$

where

$$\begin{aligned} L_1 &= \bar{V}_1 - \bar{V}_2'\bar{A}_2^{-1}\bar{a}_2 + \bar{a}_2'\bar{A}_2'^{-1}\bar{V}_3\bar{A}_2^{-1}\bar{a}_2, \\ L_2 &= \bar{B}_2'\bar{A}_2'^{-1}(\bar{V}_3\bar{A}_2^{-1}\bar{a}_2 - \tfrac{1}{2}\bar{V}_2), \\ L_3 &= \bar{R} + \bar{B}_2'\bar{A}_2'^{-1}\bar{V}_3\bar{A}_2^{-1}\bar{B}_2, \\ \bar{a} &= \bar{a}_1 - \bar{A}_1\bar{A}_2^{-1}\bar{a}_2, \\ \bar{B} &= \bar{B}_1 - \bar{A}_1\bar{A}_2^{-1}\bar{B}_2. \end{aligned} \tag{12}$$

The Hamiltonian of the reduced problem is

$$\bar{H}(\bar{x}, \bar{p}, \bar{u}, t) = L_1 + 2L_2'\bar{u} + \bar{u}'L_3\bar{u} + \bar{p}'(\bar{a} + \bar{B}\bar{u}), \tag{13}$$

where the costate variable $\bar{p}$ satisfies

$$d\bar{p}/dt = -\nabla_{\bar{x}}\bar{H}(\bar{x}, \bar{p}, \bar{u}, t). \tag{14}$$

The control which solves

$$\partial\bar{H}/\partial\bar{u} = 2L_2 + 2L_3\bar{u} + \bar{B}'\bar{p} = 0 \tag{15}$$

is the minimizing control

$$\bar{u} = -\tfrac{1}{2}L_3^{-1}(2L_2 + \bar{B}'\bar{p}). \tag{16}$$

Substitution of the control (16) into the systems (11) and (14) results in the reduced TPBV problem

$$d\bar{x}/dt = \bar{a} - \bar{B}L_3^{-1}L_2 - \tfrac{1}{2}\bar{B}L_3^{-1}\bar{B}'\bar{p} = f_1(\bar{x}, \bar{p}, t), \qquad \bar{x}(0) = x_0(0),$$
$$\bar{x}(T) = x_T(0), \tag{17-1}$$
$$d\bar{p}/dt = -\nabla_{\bar{x}}\bar{H}(\bar{x}, \bar{p}, -\tfrac{1}{2}L_3^{-1}(2L_2 + \bar{B}'\bar{p}), t) = f_2(\bar{x}, \bar{p}, t). \tag{17-2}$$

The following hypothesis is crucial.

Hypothesis (H1). The reduced TPBV problem (17) has a unique solution $\bar{x}^*(t)$, $\bar{p}^*(t)$, $\bar{u}^*(t)$ for all $t \in [0, T]$.

Linearizing the system (17) along $\bar{x}^*, \bar{p}^*$, we obtain the variational equation as follows:

$$(d/dt)\begin{bmatrix}\delta\bar{x}\\ \delta\bar{p}\end{bmatrix} = \begin{bmatrix} f_{1\bar{x}} & f_{1\bar{p}} \\ f_{\bar{x}\bar{x}} & f_{2\bar{p}}\end{bmatrix}\begin{bmatrix}\delta\bar{x}\\ \delta\bar{p}\end{bmatrix} = \begin{bmatrix} C_1 & -C_2 \\ -C_3 & -C_1'\end{bmatrix}\begin{bmatrix}\delta\bar{x}\\ \delta\bar{p}\end{bmatrix}, \tag{18}$$

where

$$C_2 = \tfrac{1}{2}\bar{B}L_3^{-1}\bar{B}'.$$

The system (18) is assumed to satisfy the following hypothesis.

Hypothesis (H2). C_3 is positive semidefinite along $\bar{x}^*, \bar{p}^*$.

Hypothesis (H2) rules out the occurrence of conjugate points (Ref. 9) and guarantees that $\bar{x}^*, \bar{p}^*$ yield a local minimum. This hypothesis is also crucial for finding higher-order terms in the asymptotic series expansions for the solution of the TPBV problem (6)–(7).

Since $\bar{z}^*$ of Eq. (9) in general does not satisfy the end conditions (2-2), we assume that the variable z contains an initial (left) boundary layer $\lambda(\tau)$ and an end (right) boundary layer $\rho(\sigma)$ such that

$$z(t) = \bar{z}^*(t) + \lambda(\tau) + \rho(\sigma), \tag{19}$$

where

$$\tau = t/\mu \qquad \text{and} \qquad \sigma = (t-T)\mu$$

are the *stretched* time scales. Substituting Eq. (19) into the system (2-2) and equating the layer terms, we obtain the boundary layer systems as

$$d\lambda(\tau)/d\tau = \bar{A}_2(0)\lambda(\tau) + \bar{B}_2(0)u_\lambda(\tau), \qquad \lambda(0) = z_0(0) - \bar{z}^*(0), \tag{20}$$

$$d\rho(\sigma)/d\sigma = \bar{A}_2(T)\rho(\sigma) + \bar{B}_2(T)u_\rho(\sigma), \qquad \rho(0) = z_T(0) - \bar{z}^*(T), \tag{21}$$

where u is also decomposed into

$$u(t) = \bar{u}^*(t) + u_\lambda(\tau) + u_\rho(\sigma). \tag{22}$$

Substituting Eqs. (19) and (22) into the performance index (1) and retaining only the quadratic terms in λ, p, u_λ, u_ρ, we define the boundary layer problems as follows.

Boundary-Layer Problems. The left boundary-layer problem (LBLP) is to minimize

$$J_\lambda = \int_0^\infty [\lambda'\bar{V}_3(0)\lambda + u_\lambda'\bar{R}(0)u_\lambda]\, d\tau, \tag{23}$$

subject to the system (20). The right boundary-layer problem (RBLP) is to minimize

$$J_\rho = \int_{-\infty}^{0} [\rho' \bar{V}_3(T)\rho + u_\rho' \bar{R}(T) u_\rho]\, d\rho, \tag{24}$$

subject to the system (21).

We now make the following hypothesis.

Hypothesis (H3). For all $t \in [0, T]$ and along the trajectory $\bar{x}^*$,

$$\operatorname{rank}[\bar{B}_2, \bar{A}_2\bar{B}_2, \ldots, \bar{A}_2^{m-1}\bar{B}_2] = m,$$
$$\operatorname{rank}[\bar{H}_2', \bar{A}_2'\bar{H}_2', \ldots, \bar{A}_2'^{m-1}\bar{H}_2'] = m,$$

where $\bar{H}_2$ satisfies $\bar{H}_2'\bar{H}_2 = \bar{V}_3$.

Hypothesis (H3) is equivalent to assuming that the pair $[\bar{A}_2(t), \bar{B}_2(t)]$ is completely controllable and the pair $[\bar{A}_2(t), \bar{H}_2(t)]$ is completely observable for all $t \in [0, T]$. Under Hypothesis (H3), the solutions to LBLP and RBLP exist and are given by

$$u_\lambda(\tau) = -\bar{R}^{-1}(0)\bar{B}_2'(0)K_\lambda(0)\lambda(\tau), \tag{25}$$
$$u_\rho(\sigma) = -\bar{R}^{-1}(T)\bar{B}_2'(T)K_\rho(T)\rho(\sigma), \tag{26}$$

where $K_\lambda(0)$ is the positive-definite solution and $K_\rho(T)$ is the negative-definite solution of the Riccati equation

$$K\bar{A}_2 + \bar{A}_2K - K\bar{B}_2\bar{R}^{-1}\bar{B}_2'K + \bar{V}_3 = 0 \tag{27}$$

a $t = 0$ and $t = T$, respectively. Due to the presence of boundary layer terms in u, Eqs. (6) and (22) show that there are also boundary layers in the costate variable q, which is explored in the following section.

3. Main Theorem

The decomposition of the full problem (1)–(2) into the reduced problem and two boundary-layer problems is justified in the following theorem.

Theorem 3.1. If Hypotheses (H1)–(H3) hold, then there exists a $\mu^* > 0$ such that, for all $\mu \in (0, \mu^*]$, an asymptotic series solution x^*, z^*, p^*, q^*, u^* to the TPBV problem (6)–(7) exists and is approximated to $O(\mu)$ by

the solutions of the reduced problem and the boundary-layer problems as follows:

$$x^*(t, \mu) = \bar{x}^*(t) + O(\mu), \tag{28-1}$$

$$z^*(t, \mu) = \bar{z}^*(t) + \lambda(\tau) + \rho(\sigma) + O(\mu), \tag{28-2}$$

$$p^*(t, \mu) = \bar{p}^*(t) + O(\mu), \tag{28-3}$$

$$q^*(t, \mu) = -\bar{A}_2'^{-1}(\bar{V}_2 + 2\bar{V}_3\bar{z}^* + \bar{A}_1'\bar{p}^*) + 2K_\lambda(0)\lambda(\tau)$$

$$+ 2K_\rho(T)\rho(\sigma) + O(\mu), \tag{28-4}$$

$$u^*(t, \mu) = \bar{u}^*(t) + u_\lambda(\tau) + u_\rho(\sigma) + O(\mu). \tag{28-5}$$

The meaning of this theorem is that we can obtain an $O(\mu)$ approximate solution to the full TPBV problem (7) by solving for the reduced problem, the LBLP, and the RBLP. The reduced problem is of lower order and does not involve the small parameter μ. Thus, we can use large stepsizes in the numerical computation of the reduced problem. The LBLP and the RBLP are linear–quadratic problems, and their computation requires a small fraction of the time required to compute nonlinear problems of the same dimension. Thus, solving for the lower-dimension problems results in considerable savings of computation time.

In the actual implementation of the control (22) for the physical system (2), undesirable behavior will occur if A_2 is unstable. Since the control (22) is open loop, it does not affect the stability of the system (2). Hence, if A_2 is unstable, the higher-order μ terms in the approximation (28) which are not compensated will grow as $O(\exp(1/\mu))$. Furthermore, the error increases as μ decreases. An example illustrating this behavior in linear, time-varying systems is given in Ref. 8. This problem can be avoided by using a partially closed-loop control

$$u = M(x(t), t)z + v, \tag{29}$$

such that $A_2 + B_2M$ is stable along the trajectory $\bar{x}^*$. Hypothesis (H3) guarantees that such an M exists and can be computed as

$$M = -\bar{R}^{-1}\bar{B}_2'K(t), \tag{30}$$

where $K(t)$ is the positive-definite solution to Eq. (27) for $t \in [0, T]$. Then, the open-loop control v is computed as

$$v = u - Mz. \tag{31}$$

4. Example

We consider the optimal control of the system

$$\begin{aligned} \dot{x} &= xz, \qquad x(0) = 1/\sqrt{2}, \qquad x(1) = 0.5, \\ \mu\dot{z} &= -z + \mu, \qquad z(0) = 0, \qquad z(1) = 0, \end{aligned} \tag{32}$$

with respect to the performance function

$$J = \int_0^1 (x^4 + \tfrac{1}{2}z^2 + \tfrac{1}{2}u^2)\, dt. \tag{33}$$

The Hamiltonian for problem (32)–(33) is

$$H = x^4 + \tfrac{1}{2}z^2 + \tfrac{1}{2}u^2 + p(xz) + q(-z + u), \tag{34}$$

and the minimizing control is ·

$$u = -q, \tag{35}$$

as

$$\partial H/\partial u = u + q = 0.$$

Thus, the TPBV problem is

$$\begin{aligned} \dot{x} &= xz, \\ \mu\dot{z} &= -z - q, \\ \dot{p} &= -\partial H/\partial x = -4x^3 - pz, \\ \mu\dot{q} &= -\partial H/\partial z = -px - z + q. \end{aligned} \tag{36}$$

We now apply the decomposition procedure to obtain $O(\mu)$ approximations of x, z, p, q, u.

The reduced problem corresponding to the full problem (32)–(33) is to optimally control the reduced system

$$\dot{\bar{x}} = \bar{x}\bar{u}, \qquad \bar{x}(0) = 1/\sqrt{2}, \qquad \bar{x}(1) = 0.5, \tag{37}$$

with respect to

$$\bar{J} = \int_0^\infty (\bar{x}^4 + \bar{u}^2)\, dt. \tag{38}$$

The Hamiltonian for this reduced problem is

$$\bar{H} = \bar{x}^4 + \bar{u}^2 + \bar{p}\bar{x}\bar{u}, \tag{39}$$

and the reduced control $\bar{u}$ satisfies

$$\partial \bar{H}/\partial \bar{u} = 2\bar{u} + \bar{p}\bar{x} = 0, \tag{40}$$

that is,

$$\bar{u} = -\bar{p}\bar{x}/2. \tag{41}$$

The reduced TPBV problem becomes

$$\begin{aligned} \dot{\bar{x}} &= -\bar{x}^2\bar{p}/2, \qquad \bar{x}(0) = 1/\sqrt{2}, \qquad \bar{x}(1) = 0.5, \\ \dot{\bar{p}} &= -\partial\bar{H}/\partial\bar{x} = -4\bar{x}^3 - \bar{p}\bar{u} = -4\bar{x}^3 + \bar{x}\bar{p}^2/2. \end{aligned} \tag{42}$$

The unique solution to system (42) is found analytically to be

$$\begin{aligned} \bar{x}^*(t) &= 1/\sqrt{[2(t+1)]}, \\ \bar{p}^*(t) &= 2\bar{x}^*(t) = \sqrt{[2/(t+1)]}, \\ \bar{u}^*(t) &= -\bar{x}^{*2}(t) = -1/[2(t+1)], \\ \bar{z}^*(t) &= \bar{u}^*(t). \end{aligned} \tag{43}$$

Linearization of system (42) along $\bar{x}^*$, $\bar{p}^*$ reveals that

$$C_3 = 5/(t+1);$$

hence, Hypothesis (H2) is satisfied.

The LBLP is to optimally control

$$d\lambda/d\tau = -\lambda + u_\lambda, \qquad \lambda(0) = 0-(-1/2) = 1/2, \qquad \tau = t/\mu, \tag{44}$$

with respect to

$$J_\lambda = \tfrac{1}{2}\int_0^\infty (\lambda^2 + u_\lambda^2)\, d\tau. \tag{45}$$

The optimal control is

$$u_\lambda(\tau) = -k_\lambda\lambda(\tau) = -(\sqrt{2}-1)\lambda(\tau), \tag{46}$$

where k_λ is the positive-definite solution to the Riccati equation

$$0 = 2k + k^2 - 1. \tag{47}$$

Thus, the optimally controlled LBL system is

$$d\lambda/d\tau = -\sqrt{2}\lambda, \tag{48}$$

yielding

$$\lambda(\tau) = \lambda(0)\exp(-\sqrt{2}\tau), \qquad 0 \le \tau < \infty. \tag{49}$$

Similarly, the RBLP is to optimally control

$$d\rho/d\sigma = -\rho + u_\rho, \qquad \rho(0) = 0-(-1/4) = 1/4, \qquad \sigma = (t-1)/\mu, \tag{50}$$

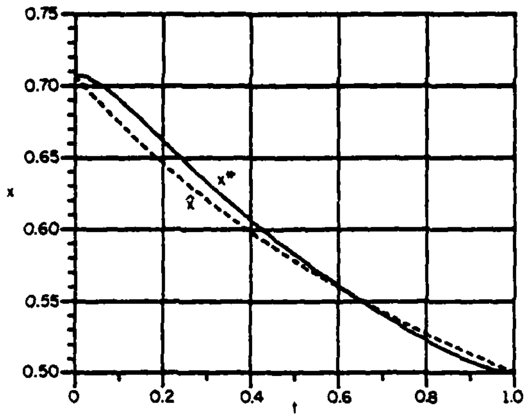

Fig. 1. Plots of x^* and $\hat{x}$.

with respect to

$$J_\rho = \tfrac{1}{2}\int_{-\infty}^{0} (\rho^2 + u_\rho^2)\, d\sigma. \tag{51}$$

The optimal control is

$$u_\rho = -k_\rho \rho(\tau) = (1+\sqrt{2})\rho(\tau), \tag{52}$$

where k_ρ is the negative-definite solution to the Riccati equation (47), resulting in the feedback system

$$d\rho/d\sigma = \sqrt{2}\rho, \tag{53}$$

such that

$$\rho(\sigma) = \rho(0)\exp(\sqrt{2}\,\sigma), \qquad -\infty < \sigma \le 0. \tag{54}$$

Thus, an $O(\mu)$ approximation to the solution of the full-order problem (36) is

$$\begin{aligned}
\hat{x}(t) &= 1/\sqrt{[2(t+1)]},\\
\hat{z}(t) &= -1/[2(t+1)] + [\exp(-\sqrt{2}\,t/\mu)]/2 + (\exp[\sqrt{2}\,(t-1)/\mu])/4,\\
\hat{p}(t) &= \sqrt{[2/(t+1)]},\\
\hat{q}(t) &= 1/[2(t+1)] + (\sqrt{2}-1)[\exp(-\sqrt{2}t/\mu)]/2\\
&\quad -(1+\sqrt{2})(\exp[\sqrt{2}(t-1)/\mu])/4,\\
\hat{u}(t) &= -\hat{q}(t).
\end{aligned} \tag{55}$$

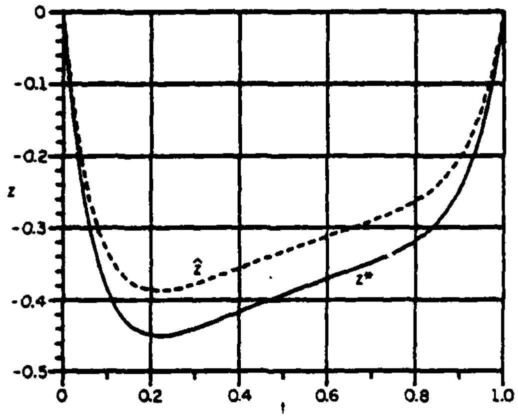

Fig. 2. Plots of z^* and $\hat{z}$.

For $\mu = 0.1$, the trajectories are shown as dashed lines in Figs. 1–5. The Newton–Raphson algorithm in Ref. 10 is modified for fixed-endpoint problems by using the dichotomy transformation in Ref. 11 and is used to compute an optimal solution of the TPBV problem (36). Using the approximate trajectory (55) as the initial guess, the computation converges in four iterations, and the optimal trajectories x^*, z^*, p^*, q^*, u^* are shown as solid lines in Figs. 1–5. Note the presence of boundary layers in z^*, q^*, u^*. The closeness of the trajectory (55) to the optimal trajectory is obvious.

5. Asymptotic Expansions

We now proceed to obtain asymptotic series expansions of x, z, p, q, u in μ. Then, Theorem 3.1 follows from the fact that approximation (28) consists of the leading terms in the expansions.

Lemma 5.1. Under Hypotheses (H1)–(H3), the TPBV problem (7) possesses a solution of the form

$$\begin{aligned}
x(t,\mu) &= X^N(t,\mu) + \mu m_1^N(\tau,\mu) + \mu n_1^N(\sigma,\mu) + \mu^N x^N(t,\mu),\\
p(t,\mu) &= P^N(t,\mu) + \mu m_2^N(\tau,\mu) + \mu n_2^N(\sigma,\mu) + \mu^N p^N(t,\mu),\\
z(t,\mu) &= z^N(t,\mu) + m_3^N(\tau,\mu) + n_3^N(\sigma,\mu) + \mu^N z^N(t,\mu),\\
q(t,\mu) &= Q^N(t,\mu) + m_4^N(\tau,\mu) + n_4^N(\sigma,\mu) + \mu^N q^N(t,\mu),
\end{aligned} \tag{56}$$

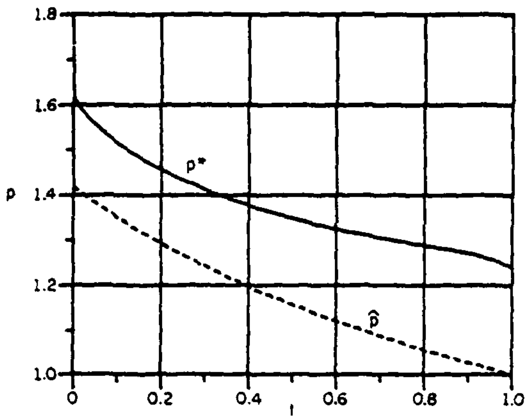

Fig. 3. Plots of p^* and $\hat{p}$.

for all $N \geq 0$, where

$$X^N = \sum_{i=0}^{N} \mu^i X_i(t, \mu), \qquad P^N = \sum_{i=0}^{N} \mu^i P_i(t, \mu),$$
$$Z^N = \sum_{i=0}^{N} \mu^i Z_i(t, \mu), \qquad Q^N = \sum_{i=0}^{N} \mu^i Q_i(t, \mu), \tag{57}$$

constitute the outer expansion,

$$m_k^N = \sum_{i=0}^{N-1} \mu^i m_{ki}(\tau, \mu), \qquad k = 1, 2, \qquad m_k^N = \sum_{i=0}^{N} \mu^i m_{ki}(\tau, \mu), \qquad k = 3, 4,$$
$$n_k^N = \sum_{i=0}^{N-1} \mu^i n_{ki}(\sigma, \mu), \qquad k = 1, 2, \qquad n_k^N = \sum_{i=0}^{N} \mu^i n_{ki}(\sigma, \mu), \qquad k = 3, 4, \tag{58}$$

are the left and the right boundary-layer expansions, respectively, and x^N, p^N, z^N, q^N are $O(\mu)$ uniformly in the interval $t \in [0, T]$ for μ sufficiently small.

Proof. Outer Expansion. The outer expansion is obtained by substituting (X^N, P^N, Z^N, Q^N) into the TPBV problem (7) and equating the coefficients of like power of μ^i, $0 \leq i \leq N$. At $i = 0$, (X_0, P_0, Z_0, Q_0) is the

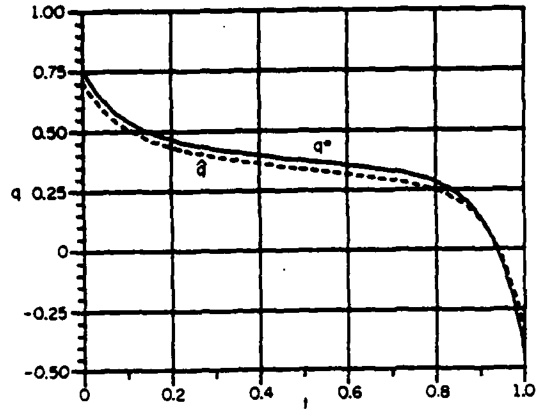

Fig. 4. Plots of q^* and $\hat{q}$.

solution to

$$dX_0/dt = \bar{a}_1 + \bar{A}_1 Z_0 - \tfrac{1}{2}\bar{S}_1 P_0 - \tfrac{1}{2}\bar{S}Q_0, \qquad X_0(0) = x_0(0), \qquad X_0(T) = x_T(0), \tag{59-1}$$

$$dP_0/dt = -\nabla_x H(X_0, Z_0, -\tfrac{1}{2}\bar{R}^{-1}(\bar{B}_1' P_0 + \bar{B}_2' Q_0), P_0, Q_0, t, 0), \tag{59-2}$$

$$0 = \bar{a}_2 + \bar{A}_2 Z_0 - \tfrac{1}{2}\bar{S}' P_0 - \tfrac{1}{2}\bar{S}_2 Q_0, \tag{59-3}$$

$$0 = -\bar{V}_2 - 2\bar{V}_3 Z_0 - \bar{A}_1' P_0 - \bar{A}_2' Q_0, \tag{59-4}$$

where $\bar{a}_1, \bar{a}_2, \bar{A}_1, \bar{A}_2, \bar{B}_1, \bar{B}_2$,

$$\bar{S}_1 = \bar{B}_1 \bar{R}^{-1} \bar{B}_1', \qquad \bar{S}_2 = \bar{B}_2 \bar{R}^{-1} \bar{B}_2', \qquad \bar{S} = \bar{B}_1 \bar{R}^{-1} \bar{B}_2',$$

$\bar{V}_1, \bar{V}_2, \bar{V}_3$ are evaluated at (X_0, P_0, Z_0, Q_0) and $\mu = 0$. In general, for $1 \le i \le N$, (X_i, P_i, Z_i, Q_i) is the solution to

$$dX_i/dt = g_{1x}(t)X_i - \tfrac{1}{2}\bar{S}_1(t)P_i + \bar{A}_1(t)Z_i - \tfrac{1}{2}\bar{S}(t)Q_i + \alpha_{1i}(t), \tag{60-1}$$

$$dP_i/dt = g_{2x}(t)X_i + g_{2p}(t)P_i + g_{2z}(t)Z_i + g_{2q}(t)Q_i + \alpha_{2i}(t), \tag{60-2}$$

$$0 = g_{3x}(t)X_i - \tfrac{1}{2}\bar{S}'(t)P_i + \bar{A}_2(t)Z_i - \tfrac{1}{2}\bar{S}_2(t)Q_i + \alpha_{3i}(t), \tag{60-3}$$

$$0 = g_{4x}(t)X_i - \bar{A}_1'(t)P_i - 2\bar{V}_3(t)Z_i - \bar{A}_2'(t)Q_i + \alpha_{4i}(t), \tag{60-4}$$

where the matrix coefficients of (X_i, P_i, Z_i, Q_i) are evaluated at (X_0, P_0, Z_0, Q_0) and $\mu = 0$, and $\alpha_{ki}(t)$, $k = 1, 2, 3, 4$, are known functions of

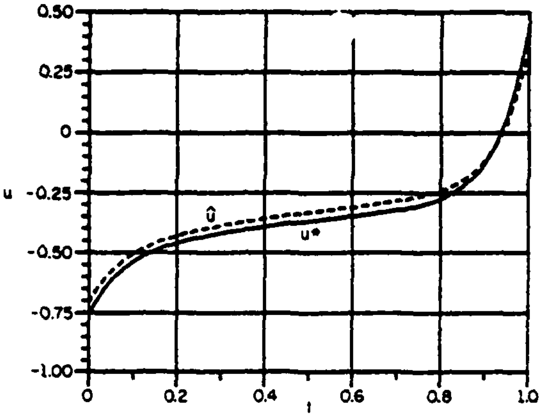

Fig. 5. Plots of u^* and $\hat{u}$.

(X_r, P_r, Z_r, Q_r) for $r \le i-1$. From Hypothesis (H3), it follows that

$$\begin{bmatrix} \bar{A}_2(t) & -\frac{1}{2}\bar{S}_2(t) \\ -2\bar{V}_3(t) & -\bar{A}_2'(t) \end{bmatrix} \tag{61}$$

is nonsingular (Ref. 8). Hence, Z_i, Q_i can be solved uniquely from Eqs. (60-3) and (60-4) and eliminated from the equations for dX_i/dt, dP_i/dt. For $i=0$, it is shown in the Appendix that the elimination of Z_0, Q_0 from Eqs. (59-1) and (59-2) yields the reduced TPBV problem (17), with X_0, P_0 replacing $\bar{x}$, $\bar{p}$. Hence, from Hypothesis (H1), the uniqueness of solution gives

$$X_0 = \bar{x}^*, \qquad P_0 = \bar{p}^*, \tag{62}$$

and Z_0, Q_0 are given in Eq. (92) of the Appendix.

It then follows immediately that, for $1 \le i \le N$, the elimination of Z_i, Q_i from Eqs. (60-1) and (60-2) yields

$$\begin{aligned} dX_i/dt &= C_1(t)X_i - C_2(t)P_i + \hat{\alpha}_{1i}(t), \\ dP_i/dt &= -C_3(t)X_i - C_1'(t)P_i + \hat{\alpha}_{2i}(t), \end{aligned} \tag{63}$$

where C_1, C_2, C_3 are given in Eq. (18). The boundary conditions for the system (63) are

$$X_i(0) = a^i \quad \text{and} \quad X_i(T) = b^i,$$

where a^i, b^i will be specified later. Since C_3 is positive semidefinite and L_3 is positive definite, the solution $K_p(t)$ to

$$\dot{K} = -KC_1 - C_1'K + KC_2K - C_3, \tag{64}$$

with end condition

$$K(T) = \pi_1, \tag{65}$$

is positive definite in $t \in [0, T)$ for any π_1 positive semidefinite, and the solution $K_n(t)$ to Eq. (64), with end condition

$$K(0) = \pi_2, \tag{66}$$

is negative definite in $t \in (0, T]$ for any π_2 negative semidefinite (Ref. 11). Introducing the dichotomy transformation (Ref. 11)

$$X_i = y_i + w_i, \qquad P_i = K_p y_i + K_n w_i, \tag{67}$$

and substituting into Eq. (63), we obtain the equations for y_i, w_i as follows:

$$dy_i/dt = (C_1 - C_2 K_p) y_i + \alpha_{yi}; \tag{68-1}$$

$$dw_i/dt = (C_1 - C_2 K_n) w_i + \alpha_{wi}; \tag{68-2}$$

hence,

$$\begin{bmatrix} \alpha_{yi} \\ \alpha_{wi} \end{bmatrix} = \begin{bmatrix} I & I \\ K_p & K_n \end{bmatrix}^{-1} \begin{bmatrix} \hat{\alpha}_{1i} \\ \hat{\alpha}_{2i} \end{bmatrix}, \tag{69}$$

and I is the $n \times n$ identity matrix. Let Φ_1, Φ_2 be the state transition matrices of the system (68-1) and (68-2), respectively. Then, the solution to Eq. (68) is

$$\begin{aligned} y_i(t) &= \Phi_1(t, 0) y_i(0) + \int_0^t \Phi_1(t, s) \alpha_{yi}(s)\, ds, \\ w_i(t) &= \Phi_2(t, T) w_i(T) + \int_T^t \Phi_2(t, s) \alpha_{wi}(s)\, ds. \end{aligned} \tag{70}$$

To determine $y_i(0)$, $w_i(T)$, we evaluate the expression (70) at $t = 0$ and $t = T$; and, due to Hypotheses (H1) and (H2), the end conditions are determined uniquely (Ref. 8) as

$$\begin{bmatrix} y_i(0) \\ w_i(T) \end{bmatrix} = \begin{bmatrix} I & \Phi_2(0, T) \\ \Phi_1(T, 0) & I \end{bmatrix}^{-1} \begin{bmatrix} a^i - \int_T^0 \Phi_2(0, s) \alpha_{wi}(s)\, ds \\ b^i - \int_0^T \Phi_1(T, s) \alpha_{yi}(s)\, ds \end{bmatrix}. \tag{71}$$

Thus, given $a^i b^i$, the unique solutions y_i, w_i to Eq. (68), and hence the unique solutions X_i, P_i to Eq. (63), can be computed.

Proof. Boundary-Layer Expansions. Since $Z(0, \mu)$ does not in general satisfy the initial condition $z_0(\mu)$, it is necessary to account for this

boundary layer by $m_k^N(\tau, \mu)$, $k = 1, 2, 3, 4$, which satisfy at $t = 0$ the equations

$$[dm_1^N/d\tau] = [dx/dt - dX^N/dt]_{t=\mu\tau}, \qquad [dm_2^N/d\tau] = [dp/dt - dP^N/dt]_{t=\mu\tau},$$
$$[dm_3^N/d\tau] = [dz/dt - dz^N/d\tau]_{t=\mu\tau}, \qquad [dm_4^N/d\tau] = [dq/dt - dQ^N/dt]_{t=\mu\tau} \tag{72}$$

to $O(\mu^N)$. Hence, we have

$$\begin{aligned} dm_k^N/d\tau &= \hat{g}_k(m_1^N, m_2^N, m_3^N, m_4^N, \tau, \mu) \\ &= g_k(X^N(\mu\tau) + \mu m_1^N(\tau), P^N(\mu\tau) + \mu m_2^N(\tau), \\ &\quad Z^N(\mu\tau) + m_3^N(\tau), Q^N(\mu\tau) + m_4^N(\tau), \mu\tau, \mu) \\ &\quad - g_k(X^N(\mu\tau), P^N(\mu\tau), Z^N(\mu\tau), Q^N(\mu\tau), \mu\tau, \mu), \quad k = 1, 2, 3, 4. \end{aligned} \tag{73}$$

Substituting Eq. (72) into Eq. (73) and equating the coefficients of like power of μ^i, $0 \le i \le N$, we obtain a system of equations for the left boundary layer. At $i = 0$, the zeroth-order terms of m_k^N satisfy

$$\begin{aligned} dm_{10}/d\tau &= \bar{A}_1(0)m_{30} - \tfrac{1}{2}\bar{S}(0)m_{40}, \\ dm_{20}/d\tau &= g_2(X_0(0), P_0(0), Z_0(0) + m_{30}(\tau), Q_0(0) + m_{40}(\tau), 0, 0) \\ &\quad - g_2(X_0(0), P_0(0), Z_0(0), Q_0(0), 0, 0), \\ dm_{30}/d\tau &= \bar{A}_2(0)m_{30} - \tfrac{1}{2}\bar{S}_2(0)m_{40}, \\ dm_{40}/d\tau &= -2\bar{V}_3(0)m_{30} - \bar{A}_2'(0)m_{40}, \end{aligned} \tag{74}$$

where the initial condition for m_{30} is

$$m_{30}(0) = z_0(0) - Z_0(0).$$

Furthermore, $m_{i0}(\tau)$ tends to zero as τ tends to infinity. Letting

$$m_{40}(\tau) = 2K_\lambda(0)m_{30}(\tau), \tag{75}$$

where $K_\lambda(0)$ is the positive-definite solution of Eq. (27) at $t = 0$, we obtain

$$dm_{30}/d\tau = [\bar{A}_2 - \bar{S}_2K_\lambda]_{t=0}m_{30}, \qquad m_{30}(0) = z_0(0) - Z_0(0). \tag{76}$$

Thus, $\bar{A}_2 - \bar{S}_2K_\lambda$ is stable; and $m_{30}(\tau)$, and hence $m_{40}(\tau)$, decay exponentially to zero. Now, comparing the system (76) to the system (20) controlled by u_λ, we obtain

$$m_{30}(\tau) = \lambda(\tau). \tag{77}$$

Furthermore, $m_{10}(\tau)$, $m_{20}(\tau)$ are given by

$$\begin{aligned} m_{10}(\tau) &= [(\bar{A}_1 - \bar{S}K_\lambda)(\bar{A}_2 - \bar{S}_2K_\lambda)^{-1}]_{t=0}m_{30}(\tau), \\ m_{20}(\tau) &= \int_\infty^\tau [dm_{20}(s)/d\tau]\,ds. \end{aligned} \tag{78}$$

Then, the initial condition a^1 of X_1 in the system (63) is given by

$$\begin{aligned} a^1 = X_1(0) &= [(\partial/\partial\mu)x_0(\mu)]_{\mu=0} - m_{10}(0) \\ &= [(\partial/\partial\mu)x_0(\mu)]_{\mu=0} - [(\bar{A}_1 - \bar{S}K_\lambda)(\bar{A}_2 - \bar{S}_2K_\lambda)^{-1}]_{t=0}(z_0(0) - Z_0(0)). \end{aligned} \tag{79}$$

In general $m_{ki}(\tau)$ satisfies the equations

$$dm_{1i}/d\tau = \bar{A}_1(0)m_{3i} - \tfrac{1}{2}\bar{S}(0)m_{4i} + M_{1i}(\tau), \tag{80-1}$$

$$\begin{aligned} dm_{2i}/d\tau = {} & g_{2z}(X_0(0), P_0(0), Z_0(0) + m_{30}(\tau), Q_0(0) \\ & + m_{40}(\tau), 0, 0)m_{3i} + g_{2q}(X_0(0), P_0(0), Z_0(0) \\ & + m_{30}(\tau), Q_0(0) + m_{40}(\tau), 0, 0)m_{4i} + M_{2i}(\tau), \end{aligned} \tag{80-2}$$

$$dm_{3i}/d\tau = \bar{A}_2(0)m_{3i} - \tfrac{1}{2}\bar{S}_2(0)m_{4i} + M_{3i}(\tau), \tag{80-3}$$

$$dm_{4i}/d\tau = -2\bar{V}_3(0)m_{3i} - \bar{A}_2'(0)m_{4i} + M_{4i}(\tau), \tag{80-4}$$

where the exponentially decaying terms $M_{ki}(\tau)$, $k = 1, 2, 3, 4$, are known successively. We solve for the systems (80-3) and (80-4) by letting

$$m_{4i}(\tau) = 2K_\lambda(0)m_{3i}(\tau) + \beta_i(\tau), \tag{81}$$

where β_i satisfies the linear system

$$d\beta_i/d\tau = -[\bar{A}_2 - \bar{S}_2K_\lambda]'_{t=0}\beta_i - 2K_\lambda(0)M_{3i}(\tau) + M_{4i}(\tau). \tag{82}$$

In Ref. 4, it is shown that there exists a unique exponentially decaying solution β_i to system (82). Then, $m_{3i}(\tau)$ is given by

$$dm_{3i}/d\tau = [\bar{A}_2 - \bar{S}_2K_\lambda]_{t=0}m_{3i} - \tfrac{1}{2}\bar{S}_2(0)\beta_i(\tau) + M_{3i}(\tau), \tag{83}$$

with initial condition

$$m_{3i}(0) = [\partial^i z_0(\mu)/\partial\mu^i]_{\mu=0} - Z_i(0).$$

Hence, the solutions $m_{ki}(\tau)$, $k = 1, 2, 3, 4$, to the system (80) are exponentially decaying. The initial condition a^i of X_i in system (63) is given by

$$a^i = X_i(0) = [\partial^i x_0(\mu)/\partial\mu^i]_{\mu=0} - m_{1i}(0). \tag{84}$$

In a similar manner, the right boundary layer correction terms $n_k^N(\sigma, \mu)$ satisfy the equations

$$
\begin{aligned}
dn_k/d\sigma &= \bar{g}_k(n_1, n_2, n_3, n_4, \sigma, \mu) \\
&= g_k(X(\mu\sigma) + \mu n_1(\sigma), P(\mu\sigma) + \mu n_2(\sigma), Z(\mu\sigma) + n_3(\sigma), \\
&\quad Q(\mu\sigma) + n_4(\sigma), \mu\sigma, \mu) \\
&\quad - g_k(X(\mu\sigma), P(\mu\sigma), Z(\mu\sigma), Q(\mu\sigma), \mu\sigma, \mu), \qquad k = 1, 2, 3, 4.
\end{aligned} \tag{85}
$$

We shall not give the procedure of solving n_{ki}, which is exactly the same as that of solving for m_{ki}, except that we are now solving in inverse time, that is, from $\sigma = 0$ to $\sigma = -\infty$. We only examine the equations for n_{30}, n_{40}, which are

$$
\begin{aligned}
&dn_{30}/d\sigma = \bar{A}_2(T)n_{30} - \tfrac{1}{2}\bar{S}_2(T)n_{40}, \qquad n_{30}(T) = z_T(0) - Z_0(T), \\
&dn_{40}/d\sigma = -2\bar{V}_3(T)n_{30} - \bar{A}_2'(T)n_{40}.
\end{aligned} \tag{86}
$$

Let

$$n_{40}(\sigma) = 2K_p(T)n_{30}(\sigma), \tag{87}$$

where $K_p(T)$ is the negative-definite solution to Eq. (27) at $t = T$. Since $-[\bar{A}_2 - \bar{S}_2K_p]_{t=T}$ is stable,

$$dn_{30}/d\sigma = [\bar{A}_2 - \bar{S}_2K_p]_{t=T}n_{30} \tag{88}$$

is stable in negative time and decays to zero as $\sigma \to -\infty$. Comparing the system (88) to the system (21) controlled by u_p, we obtain

$$n_{30}(\sigma) = \rho(\sigma). \tag{89}$$

The end conditions for $n_{3i}(T)$, $X_i(T)$ are

$$
\begin{aligned}
n_{3i}(T) &= [\partial^i z_T(\mu)/\partial\mu^i]_{\mu=0} - Z_i(T), \\
X_i(T) &= [\partial^i x_T(\mu)/\partial\mu^i]_{\mu=0} - n_{1i}(T) = b^i.
\end{aligned} \tag{90}
$$

It remains to be shown that x^N, p^N, z^N, q^N are $O(\mu)$. However, this asymptotic property has been shown in detail in Ref. 3 for the free-endpoint problem, and can be translated for the fixed-endpoint problem without major changes. Thus, we shall omit this part of the proof.

From Lemma 5.1, we obtain the approximation (28) of Theorem 3.1 by observing that Eqs. (28-1) and (28-3) follow from Eq. (62); Eq. (28-2) follows from Eqs. (92), (77), (89); Eq. (28-4) follows from Eqs. (92), (75), (87); and finally Eq. (28-5) follows from Eqs. (91) and (6).

6. Conclusions

A singularly perturbed, nonlinear, fixed-endpoint problem is decomposed into three lower-order problems, namely, the reduced problem and the left and the right boundary-layer problems. Two special features are that the reduced problem does not involve the singular perturbation

parameter and the boundary-layer problems are linear–quadratic. Combining the solutions to these lower-order problems, we obtain an $O(\mu)$ approximation (28) of a solution to the full TPBV problem corresponding to problem (1)–(2). Based on the properties of the lower-order problems, we obtain an asymptotic expansion for this solution of the full TPBV problem. An example illustrates the design procedure and the computation of a locally optimal solution using a Newton–Raphson algorithm and the $O(\mu)$ approximate solution as the initial guess. Finally, it is emphasized that, if the fast variable z in system (2) is unstable, the partially closed-loop control (29) which stabilizes the fast variable should be applied to the system (2).

7. Appendix: Equivalence of $\bar{x}, \bar{p}$ and X_0, P_0

Let

$$U_0 = -\tfrac{1}{2}\bar{R}^{-1}(\bar{B}_1' P_0 + \bar{B}_2' Q_0). \tag{91}$$

From Eq. (59-2), we have

$$\begin{aligned} Z_0 &= -\bar{A}_2^{-1}(\bar{a}_2 + \bar{B}_2 U_0) \\ &= -(\bar{A}_2 + \bar{S}_2 \bar{A}_2'^{-1} \bar{V}_3)^{-1}(\bar{a}_2 + \tfrac{1}{2}\bar{S}_2 \bar{A}_2'^{-1} \bar{V}_2 - \tfrac{1}{2}\bar{B}_2 \bar{R}^{-1} \bar{B}' P_0), \end{aligned} \tag{92-1}$$

$$\begin{aligned} Q_0 &= -\bar{A}_2'^{-1}(\bar{V}_2 + 2\bar{V}_3 Z_0 + \bar{A}_1' P_0) \\ &= -(\bar{A}_2 + \bar{S}_2 \bar{A}_2'^{-1} \bar{V}_3)'^{-1}[(\bar{V}_2 - 2\bar{V}_3 \bar{A}_2^{-1} \bar{a}_2) + (\bar{A}_1' + \bar{V}_3 \bar{A}_2^{-1} \bar{S}') P_0]. \end{aligned} \tag{29-2}$$

Therefore,

$$\begin{aligned} U_0 &= -\tfrac{1}{2}\bar{R}^{-1}[\bar{B}_1' - \bar{B}_2'(\bar{A}_2 + \bar{S}_2 \bar{A}_2'^{-1} \bar{V}_3)'^{-1}(\bar{A}_1' + \bar{V}_3 \bar{A}_2^{-1} \bar{S}')]P_0 \\ &\quad - \tfrac{1}{2}\bar{R}^{-1}\bar{B}_2'(\bar{A}_2 + \bar{S}_2 \bar{A}_2'^{-1} \bar{V}_3)'^{-1}(\bar{V}_2 - 2\bar{V}_3 \bar{A}_2^{-1} \bar{a}_2) \\ &= -\tfrac{1}{2}L_3^{-1}[\bar{B}_1' + \bar{B}_2' \bar{A}_2'^{-1} \bar{V}_3 \bar{A}_2^{-1} \bar{S}' - \bar{B}_2' \bar{A}_2'^{-1}(\bar{A}_2' + \bar{V}_3 \bar{A}_2^{-1} \bar{S}_2) \\ &\qquad \cdot (\bar{A}_2 + \bar{S}_2 \bar{A}_2'^{-1} \bar{V}_3)'^{-1}(\bar{A}_1' + \bar{V}_3 \bar{A}_2^{-1} \bar{S})]P_0 - \tfrac{1}{2}L_3^{-1}\bar{B}_2' \bar{A}_2'^{-1} \\ &\qquad \cdot (\bar{A}_2' + \bar{V}_3 \bar{A}_2^{-1} \bar{S}_2)(\bar{A}_2 + \bar{S}_2 \bar{A}_2'^{-1} \bar{V}_3)'^{-1}(\bar{V}_2 - 2\bar{V}_3 \bar{A}_2^{-1} \bar{a}_2) \\ &= -\tfrac{1}{2}L_3^{-1}[\bar{B}_1' + \bar{B}_2 \bar{A}_2'^{-1} \bar{V}_3 \bar{A}_2^{-1} \bar{S} - \bar{B}_2' \bar{A}_2'^{-1}(\bar{A}_1' + \bar{V}_3 \bar{A}_2^{-1} \bar{S})]P_0 \\ &\quad - \tfrac{1}{2}L_3^{-1}\bar{B}_2' \bar{A}_2'^{-1}(\bar{V}_2 - 2\bar{V}_3 \bar{A}_2^{-1} \bar{a}_2) \\ &= -L_3^{-1}(L_2 + \tfrac{1}{2}\bar{B}' P_0). \end{aligned} \tag{93}$$

Substituting Eq. (93) into Eq. (59-1) yields

$$dX_0/dt = (\bar{a}_1 - \bar{A}_1\bar{A}_2^{-1}\bar{a}_2) + (\bar{B}_1 - \bar{A}_1\bar{A}_2^{-1}\bar{B}_2)U_0 = \bar{a} + \bar{B}U_0, \tag{94}$$

$$\begin{aligned}
dP_0/dt = & -[\bar{V}_{1x} + Z_0'\bar{V}_{2x} + Z_0'\bar{V}_{3x}Z_0 + U_0'\bar{R}_xU_0 \\
& + P_0'(\bar{a}_{1x} + \bar{A}_{1x}Z_0 + \bar{B}_{1x}U_0) + Q_0'(\bar{a}_{2x} + \bar{A}_{2x}Z_0 + \bar{B}_{2x}U_0)]' \\
= & -[\bar{V}_{1x} - \bar{a}_2'\bar{A}_2'^{-1}\bar{V}_{2x} + \bar{a}_2'\bar{A}_2'^{-1}\bar{V}_{3x}\bar{A}_2^{-1}\bar{a}_2 \\
& - (\bar{V}_2' - 2\bar{a}_2'\bar{A}_2'^{-1})\bar{A}_2^{-1}(\bar{a}_2 - \bar{A}_{2x}\bar{A}_2^{-1}\bar{a}_2) \\
& + P_0'(\bar{a}_{1x} + \bar{A}_{1x}\bar{A}_2^{-1}(-\bar{a}_2 - \bar{B}_2U_0) + \bar{B}_{1x}U_0 \\
& - \bar{A}_1\bar{A}_2^{-1}(\bar{a}_{2x} - \bar{A}_{2x}\bar{A}_2^{-1}(\bar{a}_2 + \bar{B}_2U_0 \\
& + \bar{B}_{2x}U_0))) + U_0'(-\bar{B}_2'\bar{A}_2^{-1}\bar{V}_{2x} + 2\bar{B}_2'\bar{A}_2'^{-1}\bar{V}_{3x}\bar{A}_2^{-1}\bar{a}_2 \\
& + 2\bar{B}_2'\bar{A}_2'^{-1}\bar{V}_3\bar{A}_2^{-1}(\bar{a}_{2x} + \bar{A}_{2x}\bar{a}_2) - \bar{B}_{2x}'\bar{A}_2'^{-1}(\bar{V}_2 - 2\bar{V}_3\bar{A}_2^{-1}\bar{a}_2) \\
& - \bar{B}_2'\bar{A}_2'^{-1}\bar{A}_{2x}'\bar{A}_2'^{-1}(\bar{V}_2 - 2\bar{V}_3\bar{A}_2^{-1}\bar{a}_2)) + U_0'(\bar{R}_x + \bar{B}_2'\bar{A}_2^{-1}\bar{V}_{3x}\bar{A}_2^{-1}\bar{B}_2 \\
& + 2\bar{B}_2'\bar{A}_2'^{-1}\bar{V}_3\bar{A}_2'^{-1}(-\bar{A}_{2x}\bar{A}_2^{-1}\bar{B}_2 + \bar{B}_{2x}))U_0]' \\
= & -[L_{1x} + 2U_0'L_{2x} + U_0'L_{3x}U_0 + P_0'(\bar{a}_x + \bar{B}_xU_0)]' \\
= & -\nabla_x\bar{H}(X_0, P_0, U_0, t).
\end{aligned} \tag{95}$$

In Eq. (95), the partial derivative of an $n_1 \times n_2$ matrix G with respect to x results in an $n_1 \times n_2 \times n$ matrix G_x and its premultiplication and postmultiplication by an n_1-vector h_1 and an n_2-vector h_2 is defined as

$$h_1'G_xh_2 = \sum_{i=1}^{n_1}\sum_{j=1}^{n_2} h_{1i}h_{2j}G_{ijx}, \tag{96}$$

where h_{1i} is the ith component of h_1, h_{2j} is the jth component of h_2, and G_{ij} is the (i, j)th element of G. Thus, the equations for X_0, P_0, U_0 are identical to the equations for $\bar{x}$, $\bar{p}$, $\bar{u}$. Hence, from the uniqueness Hypothesis (H1),

$$X_0 = \bar{x}^*, \qquad P_0 = \bar{p}^*, \qquad U_0 = \bar{u}^*. \tag{97}$$

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SECTION 7

LINEAR STOCHASTIC CONTROL

Stochastic Control of Linear Singularly Perturbed Systems

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Abstract—This paper applies singular perturbation theory to the stochastic control for the Linear-Quadratic-Gaussian (L-Q-G) problem for systems with fast and slow modes. The limiting behavior of the optimal control and the performance index is investigated. It is shown that the optimal control can be approximated by a near optimal control which is obtained as a combination of a slow control and a fast control computed in separate time scales.

I. Introduction

The singular perturbation approach [1] to the linear regulator and filtering has decomposed the problem into two lower order problems treated separately in different time scales for fast and slow subsystems [2]–[9]. As a continuation, this paper presents the results for the stochastic control of linear singularly perturbed systems with quadratic performance index and Gaussian noise. Some of the results of the filtering problem [9] and the regulator problem [3] carry over to the combined Linear-Quadratic-Gaussian (L-Q-G) control problem, but there are important differences. The major difference from the deterministic control problem is the presence of the white noise which can result in an ill-defined quadratic performance index in the limiting procedure. Similarly, the stochastic control problem differs from the estimation problem in that in the estimation problem one may be required to minimize the error variances of the fast and slow states (which have different orders of magnitudes); these variances appear separately and not in a single sum. In the control problem, the quadratic functional to be minimized may contain a sum of terms involving such covariances and one of these terms may dominate. Therefore, the singularly perturbed stochastic control problem considered here contains new aspects not present in the earlier results.

The system to be controlled is assumed to have been transformed by a nonsingular transformation [10] into the block diagonal form

$$\dot{x} = A_o x + B_o u + G_o w \tag{1}^1$$

$$\mu \dot{z} = A_2 z + B_2 u + G_2 w \tag{2}^1$$

with observations

$$y = C_o x + C_2 z + v \tag{3}$$

where x, z, y and u are n-, m-, q- and r-dimensional vectors, and w and v are uncorrelated white-Gaussian noise vectors with covariances

$$\begin{aligned} E\{w(t)w'(\tau)\} &= Q(t)\delta(t-\tau) \\ E\{v(t)v'(\tau)\} &= R(t)\delta(t-\tau). \end{aligned} \tag{4}$$

The parameter $\mu > 0$ represents small time constants and similar physical quantities making (2) the fast subsystem since as $\mu \to 0$, then $\dot{z} \to \pm\infty$. There is no loss in generality in considering systems of the form (1), (2) which have been used in the filtering [9] and the time-optimal [6] problems to simplify the derivations. For A_2 nonsingular, every singularly perturbed linear system can be transformed into the form (1), (2) and, therefore, all the results and derivations hold. In applications, however, there is no need to actually perform the transformation.

The problem is to determine the control $u(t)$ as a functional of past observations $\{y(\tau), t_o < \tau < t\}$ to minimize the performance index J

$$J = E\left\{ \frac{1}{2}\begin{bmatrix} x \\ z \end{bmatrix}' \begin{bmatrix} \Gamma_1 & \mu\Gamma_{12} \\ \mu\Gamma_{12}' & \mu\Gamma_2 \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix}\Bigg|_{t=t_f} + \frac{1}{2}\int_{t_o}^{t_f}\left\{ \begin{bmatrix} x \\ z \end{bmatrix}' \begin{bmatrix} L_1 & L_{12} \\ L_{12}' & L_2 \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} + u'Su \right\} dt \right\}. \tag{5}$$

Motivated by previous results for the regulator [3] and filtering [9] problems, the objective of this paper is to decompose the optimal stochastic control problem into two separate problems for slow and fast subsystems, and to investigate the near optimality and limiting behavior of the solutions.

Depending on the stability of the matrix $A_2(t)$, one of the two following assumptions is fundamental.

Assumption 1: For every fixed $t \in [t_o, t_f]$, the eigenvalues of $A_2(t)$ have negative real parts, $\mathrm{Re}\lambda\{A_2(t)\} < \delta < 0$.

Assumption 2: For every fixed $t \in [t_o, t_f]$, the matrix $A_2(t)$ is nonsingular, the pair $[A_2(t), C_2(t)]$ is detectable, and the pair $[A_2(t), B_2(t)]$ is stabilizable.

In addition, it is assumed that all the matrices appearing in (1)–(5) are continuously differentiable functions of t.

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[1] As it is customary in L-Q-G analysis, the differential equation form is retained to imply an Ito stochastic differential equation. Due to the linearity of the problem, no difficulties arise from this standard practice.

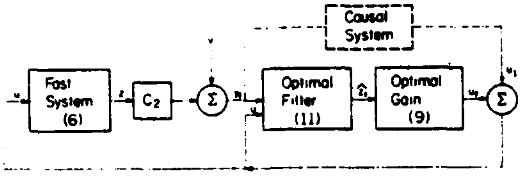

Fig. 1. The definition of u_f, $\hat{z}_f$, and u_1 for the fast system.

In the next section, a fast control problem will be considered first. Then the complete system solution will be discussed and a reduced-order problem will formally be defined. Finally, the study of the limiting behavior of the complete system solution will be made by using both the reduced and the fast problems.

II. A Fast Control Problem

We first formulate a fast control problem as having the system state and observation equations

$$\mu \dot{z} = A_2 z + B_2 u + G_2 w \tag{6}$$

$$y_f = C_2 z + v \tag{7}$$

for which we are to determine the control u as a functional of past observations y_f, minimizing the performance index J_f

$$J_f = E\left\{ \frac{\mu}{2}[z'\Gamma_2 z]_{t=t_f} + \frac{1}{2}\int_{t_o}^{t_f}[z'L_2 z + u'Su]\,dt \right\}. \tag{8}$$

The optimal control u_f is obtained in a straightforward manner by using the separation principle, so that

$$u_f = -S^{-1}B_2'K_2\hat{z}_f \tag{9}$$

where K_2 satisfies the Riccati equation

$$\mu \dot{K}_2 = -[A_2'K_2 + K_2A_2 + L_2] + K_2B_2S^{-1}B_2'K_2, \qquad K_2(t_f) = \Gamma_2. \tag{10}$$

Here $\hat{z}_f$ denotes the optimal estimate of $z(t)$ given the past observations, which for any given $u(t)$ is the output of the filter

$$\mu \dot{\hat{z}}_f = A_2\hat{z}_f + P_2C_2'R^{-1}\left[y_f - C_2\hat{z}_f\right] + B_2u, \hat{z}_f(t_o) = E\{z(t_o)\} \tag{11}$$

where $P_2(t)/\mu$ is the error covariance of $\hat{z}_f$ satisfying

$$\mu\dot{P}_2 = A_2P_2 + P_2A_2' + G_2QG_2' - P_2C_2'R^{-1}C_2P_2, P_2(t_o) = \mu\,\mathrm{cov}(z(t_o)) \tag{12}$$

which does not depend on $u(t)$. The resulting minimum value J_{f_o} of J_f is

$$J_{f_o} = \frac{1}{\mu}\int_{t_o}^{t_f}\mathrm{tr}\left\{\left[\tfrac{1}{2}L_2 + K_2P_2C_2'R^{-1}C_2\right]P_2\right\}dt + \tfrac{1}{2}\mathrm{tr}\left[P_2(t_f)\Gamma_2\right] + \tfrac{1}{2}\mu\hat{z}'(t_o)K_2(t_o)\hat{z}(t_o). \tag{13}$$

If a control u_f^* satisfying

$$u_f^* = u_f + u_1 \equiv -S^{-1}B_2'K_2\hat{z}_f + u_1 \tag{14}$$

is applied to both the system and filter in (6) and (11) as shown in Fig. 1, then the value J_f^* of J_f can be expressed by

$$J_f^* \equiv J_{f_o} + \Delta J = J_{f_o} + \frac{1}{2}\int_{t_o}^{t_f}E\{u_1'Su_1\}\,dt. \tag{15}$$

These results are now used to establish the limiting behavior of the optimal control and the performance index for the fast system (6).

The properties of $\hat{z}_f$ and P_2 as $\mu \to 0+$ have already been considered in [9] and are briefly summarized here. If either Assumption 1 or 2 hold, then the estimate of $\hat{z}_f$ and the covariance P_2/μ of the error $(z - \hat{z}_f)$ for any given u are

$$\hat{z}_f = \hat{z}_o + \tilde{z}_o(\theta) + O(\mu^{1/2}), \qquad \theta = (t - t_o)/\mu \tag{16}$$

$$P_2 = \bar{P}_2 + \tilde{P}_2(\theta) + O(\mu) \tag{17}$$

where $\bar{P}_2$ is the solution of the algebraic Riccati equation for every fixed $t \in [t_o, t_f]$

$$0 = A_2\bar{P}_2 + \bar{P}_2A_2' + G_2QG_2' - \bar{P}_2C_2'R^{-1}C_2\bar{P}_2 \tag{18}$$

and $\tilde{P}_2(\theta)$ [which is not a covariance matrix!] satisfies the boundary layer equation at t_o

$$\frac{d}{d\theta}\tilde{P}_2(\theta) = \bar{A}_2(t_o)\tilde{P}_2 + \tilde{P}_2\bar{A}_2'(t_o) - \tilde{P}_2C_2'(t_o)R^{-1}C_2(t_o)\tilde{P}_2, \qquad \theta > 0$$

$$\tilde{P}_2(0) = \mu\,\mathrm{cov}(z(t_o)) - \bar{P}_2(t_o) \tag{19}$$

$$\bar{A}_2 \triangleq \left[A_2 - \bar{P}_2C_2'R^{-1}C_2\right]. \tag{20}$$

Since $\bar{A}_2$ is a stable matrix, it can be shown that the solution of the unforced Riccati equation (19) tends to zero exponentially fast in θ, even though $\tilde{P}_2(0)$ may be negative semidefinite.

The estimate $\hat{z}_o$ is the output of a filter

$$\frac{d}{d\tau}\hat{z}_o = A_2(t_1)\hat{z}_o + \bar{P}_2(t_1)C_2'(t_1)R^{-1}\left[y_f - C_2\hat{z}_o\right] + B_2u, \qquad \hat{z}_o(0) = z(0) \tag{21}$$

which is stationary in a time scale $\tau = (t - t_1)/\mu$ stretching every small fixed subinterval $[t_1, t_1 + \epsilon] \subset [t_o, t_f]$. The matrices in (21) are all constant with respect to τ and depend on $t_1 \in [0, t_f]$ as a fixed parameter.

The boundary layer term $\tilde{z}_o(\theta)$ satisfies

$$\frac{d}{d\theta}\tilde{z}_o(\theta) = \bar{A}_2(t_o)\tilde{z}_o(\theta) + \tilde{P}_2(\theta)C_2'(t_o)R^{-1}\left[y_f - C_2\hat{z}_o\right], \qquad \tilde{z}_o(0) = 0 \tag{22}$$

which may be considered as the output of a stable time-invariant system (system matrix $\bar{A}_2$) driven by white noise (whose covariance is $O(1/\mu)$ in the θ-variable) which is multiplied by an exponentially decaying gain $\tilde{P}_2(\theta)$. Consequently, the covariance equation of $\tilde{z}_o(\theta)$ may be derived, and it then can be shown to be bounded by

$$\frac{1}{\mu}M_1\exp\left\{-\alpha\frac{(t - t_o)}{\mu}\right\}. \tag{23}$$

It is thus shown in [9] that

$$\hat{z}_f = \hat{z}_o + O(\mu^{1/2}) \tag{24}$$

$$P_2 = \bar{P}_2 + O(\mu) \tag{25}$$

for $t \in [t', t_f]$ where $t' > t_o$.

The limiting behavior of the optimal control is obtained when the expression for K_2 is used in conjunction with the filter. The behavior of K_2 has been analyzed in [3] and under Assumptions 1 or 2 it may be written as

$$K_2 = \bar{K}_2 + \tilde{K}_2(\sigma) + O(\mu), \qquad \sigma = (t_f - t)/\mu. \tag{26}$$

The gain $\bar{K}_2$ is the solution of the algebraic Riccati equation

$$0 = A_2'\bar{K}_2 + \bar{K}_2A_2 + L_2 - \bar{K}_2B_2S^{-1}B_2'\bar{K}_2 \tag{27}$$

and the boundary layer term $\tilde{K}_2(\sigma)$ at t_f satisfies

$$\frac{d}{d\sigma}\tilde{K}_2(\sigma) = \tilde{A}_2'(t_f)\tilde{K}_2 + \tilde{K}_2\tilde{A}_2(t_f) - \tilde{K}_2B_2(t_f)S^{-1}B_2'(t_f)\tilde{K}_2, \qquad \sigma > 0$$

$$\tilde{K}_2(0) = \Gamma_2 - \bar{K}_2(t_f) \tag{28}$$

$$\tilde{A}_2 \triangleq A_2 - B_2S^{-1}B_2'\bar{K}_2. \tag{29}$$

The boundary layer term $\tilde{K}_2$ may be bounded by

$$|\tilde{K}_2| < M_2 \exp - \frac{\beta(t_f - t)}{\mu} \tag{30}$$

where M_2 is of the order of $\Gamma_2 - \bar{K}_2(t_f)$. Hence, for $t \in [t_o, t'']$, where $t'' < t_f$, we may write

$$K_2 = \bar{K}_2 + O(\mu). \tag{31}$$

In this way the optimal control is approximated by

$$u_f = \bar{u}_f + u_{fl}(\theta) + u_{fr}(\sigma) + O(\mu^{1/2}), \qquad t \in [t_o, t_f] \tag{32}$$

where

$$\bar{u}_f = -S^{-1}B_2'\bar{K}_2\hat{z}_o \tag{33}$$

and u_{fl} and u_{fr} are the layer correction terms

$$u_{fl}(\theta) = -S^{-1}B_2'\bar{K}_2\hat{z}_o(\theta), \qquad u_{fr}(\sigma) = -S^{-1}B_2'\tilde{K}_2(\sigma)\hat{z}_o. \tag{34}$$

If these terms are ignored, the approximation

$$u_f = \bar{u}_f + O(\mu^{1/2}) \tag{35}$$

is valid on a subinterval $[t_1, t_2] \subset [t_o, t_f]$.

In analyzing the effect of replacing u_f by its approximations (32) of (35), several cases are of interest. It is easy to verify that

$$E\{u_{fl}'Su_{fl}\} < \frac{1}{\mu}\operatorname{tr}\left[\bar{K}_2(t_o)B_2S^{-1}B_2'\bar{K}_2(t_o)\right]M_1 \exp - \frac{\alpha(t - t_o)}{\mu} \tag{36}$$

so that if $\bar{K}_2(t_o) = O(\mu^i)$, then

$$J_{fl} \triangleq \frac{1}{2}\int_{t_o}^{t_f} E\{u_{fl}'Su_{fl}\}\,dt = O(\mu^{2i}). \tag{37}$$

Similarly,

$$E\{u_{fr}'Su_{fr}\} < \frac{1}{\mu}\operatorname{tr}\{[M_2B_2S^{-1}B_2'M_2]M_3\}\exp - \frac{\beta(t_f - t)}{\mu} \tag{38}$$

where M_3/μ is $E\{\hat{z}_o(t_f)\hat{z}_o'(t_f)\}$, and hence M_3 is finite. Consequently, if $M_2 = O(\mu^j)$, then

$$J_{fr} \triangleq \frac{1}{2}\int_{t_o}^{t_f} E\{u_{fr}'Su_{fr}\}\,dt = O(\mu^{2j}). \tag{39}$$

The parameters i and j in (37) and (39) are included to account for several possible cases of behavior of the weighting matrices L_2 and Γ_2 as $\mu \to 0+$ as discussed below.

To see the effect of replacing the optimal control u_f by the control $u_f^o = \bar{u}_f$, we use (14), (15) with

$$u_1 = -[u_{fl} + u_{fr}] + O(\mu^{1/2})$$

which, when combined with (37), (39) results in the performance index $\bar{J}_f$

$$\bar{J}_f = J_{fo} + J_{fl} + J_{fr} + O(\mu). \tag{40}$$

Two cases need to be considered to determine the relative magnitude of J_{fo}, J_{fl}, and J_{fr}. It can easily be seen from (13) that, in general, $J_{fo} = O(1/\mu)$ which is due to the white-noise behavior of the fast variables in the limit as $\mu \to 0+$. If now $L_2 \neq 0$ as $\mu \to 0+$, then from (27), (37), and (39) $J_{fl} = O(1)$, $J_{fr} = O(1)$, so that the relative error in performance is expressed by

$$\bar{J}_f = J_{fo}[1 + O(\mu)] + O(\mu), \qquad J_{fo} = O\left(\frac{1}{\mu}\right). \tag{41}$$

If now $L_2 = \mu\bar{L}_2$, then from (27) $\bar{K}_2 = O(\mu)$ provided A_2 is stable, otherwise, $\bar{K}_2$ will tend to a nonzero limit. In this case if (28) is also used,

$$J_{fo} = O(1), \qquad J_{fl} = O(\mu^2), \qquad J_{fr} = O(\|\Gamma_2\|^2). \tag{42}$$

Consequently, if in addition, the terminal cost Γ_2 is small, $\Gamma_2 = \mu^{1/2}\bar{\Gamma}_2$, then $J_{fr} = O(\mu)$, and (41) is still valid. It should be noted that while in both cases the relative increments in the performance index resulting from neglecting the boundary layers are $O(\mu)$, the absolute increments are $O(1)$ in the first case and $O(\mu)$ in the second. The importance of the latter case is also evident from the fact that $\bar{u}_f = O(\mu^{1/2})$ so that since the system is stable no control is really necessary due to $L_2 = O(\mu)$. (Note that $J_{fo} = O(1)$ stems from the covariance of the fast states which behave like white noise as $\mu \to 0+$). In the case of finite L_2, in order to achieve an absolute $O(\mu)$ approximation to J_{fo}, the near optimal control should include the sum of $\bar{u}_f$ and the two layer correction terms.

These results are summarized in the following.

Theorem 1: Let Assumption 2 hold, then the limit as $\mu \to 0+$ of the solution of the fast optimal control problem (6)–(8) satisfies (32). Furthermore, the limits of the performance index when u_f or $\bar{u}_f$ are used are given by (40) and (41). If instead, Assumption 1 is satisfied and $L_2 = \mu\bar{L}_2$, $\Gamma_2 = \mu^{1/2}\bar{\Gamma}_2$, then (42) holds.

III. The Complete Control Problem

We now return to the stochastic control problem for the complete system (1)–(5). The solution when $\mu > 0$ is

$$u_o = -S^{-1}\{[B_o'K_1 + B_2'K_{12}']\hat{x}_o + [\mu B_o'K_{12} + B_2'K_2]\hat{z}_o\} \tag{43}$$

where the gains satisfy the Riccati system

$$\dot{K}_1 = -(A_o'K_1 + K_1A_o + L_1) + (K_1B_o + K_{12}B_2)S^{-1}(K_1B_o + K_{12}B_2)',$$
$$K_1(t_f) = \Gamma_1 \tag{44}$$

$$\mu\dot{K}_{12} = -(K_{12}A_2 + \mu A_o'K_{12} + L_{12}) + (K_1B_o + K_{12}B_2)S^{-1}(\mu B_o'K_{12} + B_2'K_2),$$
$$K_{12}(t_f) = \Gamma_{12} \tag{45}$$

$$\mu\dot{K}_2 = -(A_2'K_2 + K_2A_2 + L_2) + (K_2B_2 + \mu K_{12}'B_o)S^{-1}(\mu B_o'K_{12} + B_2'K_2),$$
$$K_2(t_f) = \Gamma_2. \tag{46}$$

Similarly, the optimal estimates $\hat{x}_o$ and $\hat{z}_o$ are obtained from the linear filters which for any given control u are

$$\dot{\hat{x}}_o = A_o\hat{x}_o + (P_1C_o' + P_{12}C_2')R^{-1}\nu + B_ou \tag{47}$$

$$\mu\dot{\hat{z}}_o = A_2\hat{z}_o + (\mu P_{12}'C_o' + P_2C_2')R^{-1}\nu + B_2u \tag{48}$$

where $\nu(t)$ is the innovation

$$\nu(t) = y(t) - C_o\hat{x}_o(t) - C_2\hat{z}_o(t). \tag{49}$$

The error covariances satisfy

$$\dot{P}_1 = A_oP_1 + P_1A_o' + G_oQG_o' - (P_1C_o' + P_{12}C_2')R^{-1}(C_oP_1 + C_2P_{12}'),$$
$$P_1(t_o) = \operatorname{cov}[x(t_o)] \tag{50}$$

$$\mu\dot{P}_{12} = \mu A_oP_{12} + P_{12}A_2' + G_oQG_2' - (P_1C_o' + P_{12}C_2')R^{-1}(\mu C_oP_{12} + C_2P_2),$$
$$P_{12}(t_o) = \operatorname{cov}[x(t_o), z(t_o)] \tag{51}$$

$$\mu\dot{P}_2 = A_2P_2 + P_2A_2' + G_2QG_2' - (\mu P_{12}'C_o' + P_2C_2')R^{-1}(\mu C_oP_{12} + C_2P_2),$$
$$P_2(t_o) = \mu\operatorname{cov}[z(t_o)]. \tag{52}$$

To investigate the limiting behavior of the optimal control, we analyze the limiting behavior of the filter (47), (48) and of the linear regulator (43). We first note that expressions analogous to (14) and (15) are valid for J^* obtained with u^* and J_o obtained with u_o. The dominant contribution to J_o is due to the fast subsystem since it contains a $O(1/\mu)$ term. Therefore, if the slow subsystem is of interest, any approximation or the limiting behavior should include both the $O(1/\mu)$ and $O(1)$ terms. Under the conditions of Theorem 1, the contributions of the fast subsystem may be reduced to $O(1)$, comparable to the contribution of the slow subsystem.

The first step in a singular perturbation analysis is to define a reduced problem. The reduced problem is formally obtained from (1)–(5) by setting $\mu=0$. We point out that this is not a valid limiting operation. If A_2 is stable, it is valid for the purpose of substituting z in a slow system (Theorem 1 in [9]). However, it is not valid, in general, for the purpose of substituting z into (5). The substitution of $N=-C_2A_2^{-1}B_2$, $D=-C_2A_2^{-1}G_2$ and

$$z_r=-A_2^{-1}[B_2u+G_2w] \tag{53}$$

in (1), (3), and (5) results in

$$\dot{x}_r=A_ox_r+B_ou+G_ow \tag{54}$$

$$y=C_ox_r+Nu+Dw+v \tag{55}$$

$$J=E\left\{\frac{1}{2}\int_{t_o}^{t_f}[x_r'L_1x_r-2x_r'L_{12}A_2^{-1}B_2u+u'S_ou]\,dt+\frac{1}{2}[x_r'\Gamma_1x_r]\Big|_{t=t_f}\right\}+J_1 \tag{56}$$

$$S_o=S+B_2'A_2'^{-1}L_2A_2^{-1}B_2. \tag{57}$$

Here J_1 contains terms not influenced by the control, some of which, such as the integral of the variance of the white noise w, are ill-defined. Therefore, only the terms which may be affected by the optimization are given in (56). The optimal control u_r which minimizes (56) for the system (54), (55) is called the reduced control and is given by

$$u_r=-S_o^{-1}[K_oB_o-L_{12}A_2^{-1}B_2]'\hat{x}_r \tag{58}$$

where the reduced filter for any given u satisfies

$$\dot{\hat{x}}_r=A_o\hat{x}_r+[P_oC_o'+G_oQD']R_o^{-1}[y-C_o\hat{x}_r-Nu]+B_ou,$$

$$R_o=(R+DQD'). \tag{59}$$

The gain K_o and error covariance P_o satisfy the usual RE's

$$\dot{K}_o=-(K_oA_o+A_o'K_o+L_1)$$
$$+(K_oB_o-L_{12}A_2^{-1}B_2)S_o^{-1}(K_oB_o-L_{12}A_2^{-1}B_2)' \tag{60}$$

$$\dot{P}_o=A_oP_o+P_oA_o'+B_oQB_o'-(P_oC_o'+G_oQD')R_o^{-1}(C_oP_o+DQG_o'). \tag{61}$$

The asymptotic validity of the reduced problem is established in two steps. First, the limiting behavior of the filters (47), (48) is investigated for any given u. It follows from Theorem 2 of [9] that under Assumptions 1 or 2 for $t\in[t',t_f]$, $t'>t_o$, that

$$\hat{x}_o=\hat{x}_r+O(\mu^{1/2})$$

$$P_1=P_o+O(\mu),\qquad P_2=\bar{P}_2+O(\mu). \tag{62}$$

Similarly, using [3], [11] for $t\in[t_o,t'']$, $t''<t_f$, we have

$$K_1=K_o+O(\mu),\qquad K_2=\bar{K}_2+O(\mu)$$

$$K_{12}=-(L_{12}-K_oB_oS^{-1}B_2'\bar{K}_2)\bar{A}_2^{-1}+O(\mu)\equiv\bar{K}_{12}+O(\mu). \tag{63}$$

Now all that we need to obtain the behavior of u_o as $\mu\to0+$ is to substitute u_o into (48) and then find the limiting behavior of $\hat{z}_o$. The substitution of (43) into (48) yields

$$\mu\dot{\hat{z}}_o=\bar{A}_2\hat{z}_o+P_2C_2'R^{-1}\nu(t)-B_2S^{-1}(B_o'K_o+B_2'\bar{K}_{12}')\hat{x}_o+O(\mu^{1/2}). \tag{64}$$

The use of the basic result in singular perturbation theory (see [9]) allows $\hat{z}_o$ to be written as a sum of two terms

$$\hat{z}_o=\bar{A}_2^{-1}B_2S^{-1}(B_o'K_o+B_2'\bar{K}_{12}')\hat{x}_o+\hat{z}_f+O(\mu^{1/2}) \tag{65}$$

where

$$\mu\dot{\hat{z}}_f=A_2\hat{z}_f+P_2C_2'R^{-1}\nu(t)-B_2S^{-1}B_2'K_2\hat{z}_f. \tag{66}$$

The term with $\hat{x}_o$ in (65) is only needed insofar as its effect on the slow system is concerned, and its relative contribution to the variance of $\hat{z}_o$ is $O(\mu)$. The second term $\hat{z}_f$ is clearly the state of a fast system with white noise input $\nu(t)$. The substitution of (65) in (43) results after derivations similar to [11] in the expression

$$u_o=-S_o^{-1}(K_oB_o-L_{12}A_2^{-1}B_2)'\hat{x}_o-S^{-1}B_2'K_2\hat{z}_f+O(\mu^{1/2})$$
$$=-S_o^{-1}(K_oB_o-L_{12}A_2^{-1}B_2)'\hat{x}_r-S^{-1}B_2'K_2\hat{z}_f+O(\mu^{1/2})$$
$$\equiv u_r+u_f+O(\mu^{1/2}). \tag{67}$$

The definition of u_f in (67) together with $\hat{z}_f$ in (66) imply that u_f and $\hat{z}_f$ are the same as those discussed in Section II if we let $y_f=(y-C_o\hat{x}_r)$. It is seen that the optimal control of the complete system is given approximately as a sum of the reduced control obtained formally, and the fast control, which is performed in a stretched time scale and is composed of a stationary term and two boundary layers.

IV. Near Optimum Performance

The effect of using u_r or $u_r+\bar{u}_f$ in the singularly perturbed system (1), (2) on the performance index will now be considered. Again, it is sufficient to use (14), (15), and (32). The effect of using only the reduced control u_r is

$$J_r=J_o+J_{fl}+J_{fr}+\frac{1}{2\mu}\int_{t_o}^{t_f}\operatorname{tr}\left[\bar{K}_2B_2S^{-1}B_2'\bar{K}_2(\bar{V}_2-\bar{P}_2)\right]dt+O(\mu) \tag{68}$$

where $\bar{V}_2/\mu$ is the unconditional covariance of z. It is seen, therefore, that the error (J_r-J_o) is $O(1/\mu)$ unless the fast system is stable and $L_2=\mu\bar{L}_2$. In such a case, neglecting the fast control causes an error of $O(1)$. If, in addition, $\Gamma_2=\mu^{1/2}\bar{\Gamma}_2$, that is the fast variables are stable and are of little interest, then avoiding the use of the fast control results in $O(\mu)$ approximation to the optimal performance. On the other hand, if the fast subsystem is unstable or if it is of sufficient interest, then using a sum of the reduced control and the fast control results in $O(\mu)$ approximation to the optimal performance.

These results for the case when the fast subsystem is asymptotically stable are summarized as follows.

Theorem 2: Let Assumption 1 hold and let $L_2=O(\mu)$ and $\Gamma_2=O(\mu^{1/2})$, then the solution to the optimal control problem (1)–(5) as $\mu\to0+$ is approximated by

$$u_o=u_r+O(\mu^{1/2}),\qquad t\in[t',t'']\subset[t_o,t_f]$$

where u_r is the optimal control of the reduced problem. If u_r is applied to the system (1), (2), the resulting performance index J_r is near optimal in the sense that

$$J_r=J_o+O(\mu). \tag{69}$$

A more general case is summarized in the following theorem.

Theorem 3: If Assumption 2 holds, then the optimal control u_o for the system (1)–(5) as $\mu\to0+$ is approximated by

$$u_o=u_r+u_f+O(\mu^{1/2})=u_r+\bar{u}_f+u_{fl}+u_{fr}+O(\mu^{1/2})$$
$$\equiv u^*+O(\mu^{1/2}). \tag{70}$$

If u^* is applied to the system (1), (2) the resulting performance index J^* is near optimal in the sense that

$$J^*=J_o+O(\mu). \tag{71}$$

We note that in this case J_o may be $O(1/\mu)$.

The significance of Theorem 3 is that the separation into a two time scales solution is valid for both the filter and the controller, as was the case for each separately. The result may be illustrated in Fig. 2. While

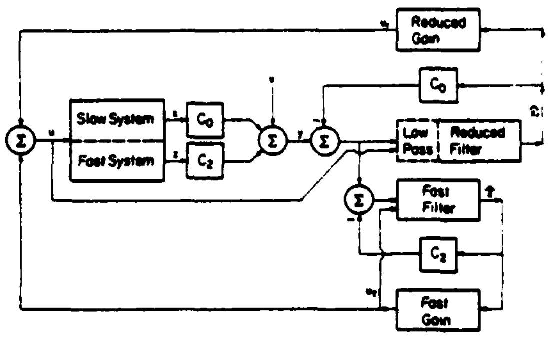

Fig. 2. The near optimal control for the complete system.

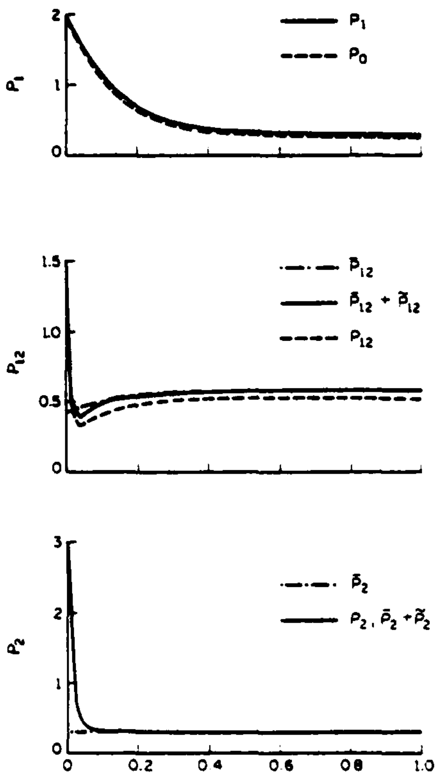

Fig. 3. The exact, reduced, and layer-corrected covariances for the stable case.

the input to the slow filter incorporates the entire control $u_s + u_f$, which includes a fast component, it is possible without loss of information to use a low pass filter so that such inputs can be sampled at the low rate used for the slow filter. This leads to important simplifications in real-time operation where the slow computations are performed by separate dedicated computers.

V. Example

To demonstrate the near optimality of the separate time scales solution, a simple second-order example was considered for both stable and unstable fast subsystems. The system equations for the example are

$$\dot{x} = -2.5x + u + w$$
$$\mu\dot{z} = a_2 z + 2u + w$$
$$y = x + z + v$$

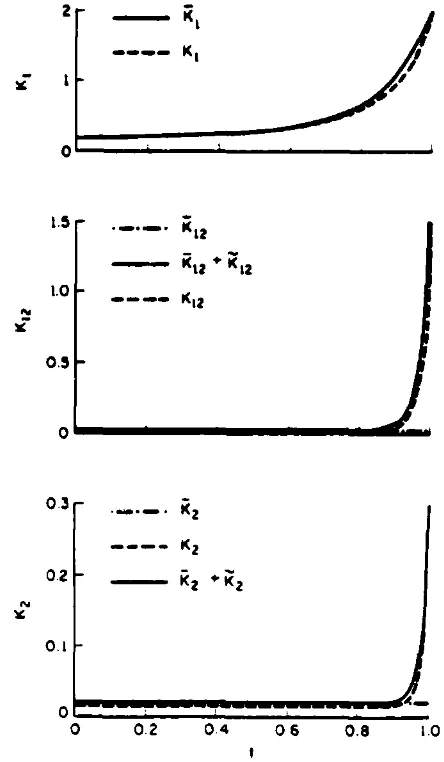

Fig. 4. The exact, reduced, and layer-corrected gains for the stable case.

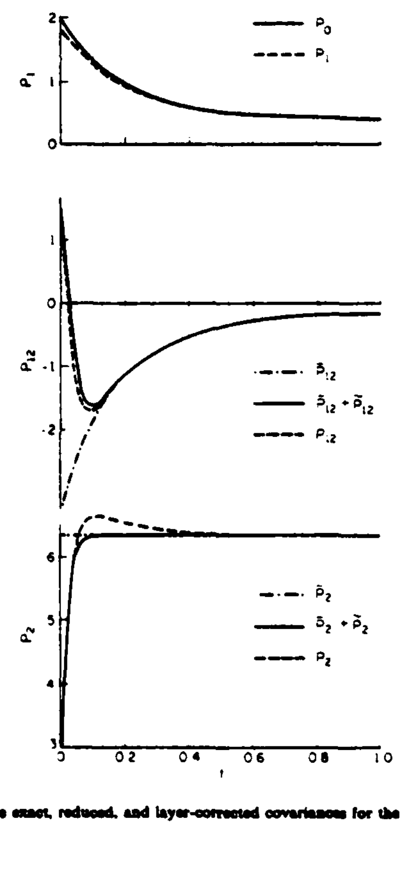

Fig. 5. The exact, reduced, and layer-corrected covariances for the unstable case.

TABLE I
THE OPTIMAL PERFORMANCE INDEX J_o AS A FUNCTION OF μ

μ	J_o for $a_2 = -3$	J_o for $a_2 = +3$
0.01	1.8245	2713.80
0.02	1.8421	1391.74
0.04	1.8806	714.98
0.07	1.8386	421.42
0.1	1.9972	302.87
0.2	2.1973	163.71
0.4	2.6206	93.875
0.7	3.3045	65.14

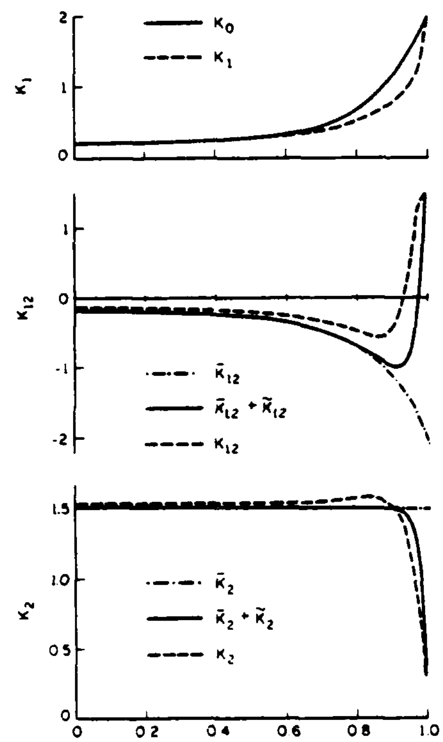

Fig. 6. The exact, reduced, and layer-corrected gains for the unstable case.

$$t_o=0,\quad t_f=1,\quad Q=2,\quad R=1,\quad \hat{x}(0)=3,\quad \hat{z}(0)=2$$

$$P_1(0)=2,\quad P_{12}(0)=1.5,\quad P_2(0)=\frac{3}{\mu}$$

$$\Gamma_1=2,\quad \Gamma_{12}=1.5,\quad \Gamma_2=3\mu,\quad L_1=1,\quad L_{12}=L_2=\mu,\quad S=1.$$

For the stable case, $a_2=-3$ and for the unstable case, $a_2=+3$. The reduced and fast filter gains and covariances can be calculated analytically for this case, while a computer had to be used for the exact solution. In Fig. 3, the exact covarainces for the stable case for $\mu=0.1$ are compared to their approximate values with and without the layer corrections. Similarly, Fig. 4 compares the exact gains to the approximate gains for the stable case and $\mu=0.1$. Figs. 5 and 6 display these results for the unstable case. The performance index J_o was computed for several values of μ and is shown in Table I. The incremental relative error in the performance index when only the reduced control is used and when the boundary layer terms are added, are shown in Figs. 7 and 8 for the stable and unstable case, respectively. These results demonstrate the general behavior of both the control and the performance index discussed in the paper. It is seen that as expected, the relative increments in performance decrease when the boundary layer terms are added, but the decrease is more significant in absolute terms for the unstable case since J_{f_0} is quite large for small μ illustrating the $O(1/\mu)$ behavior.

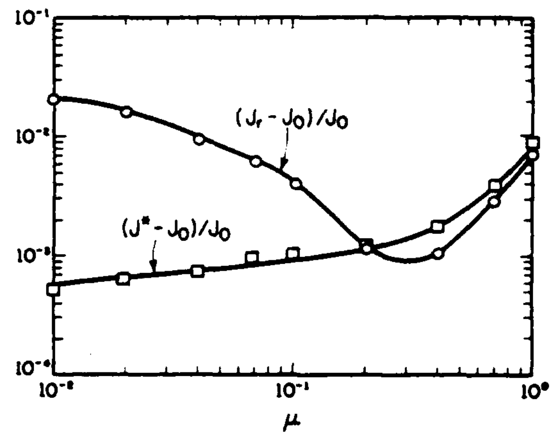

Fig. 7. The relative increments of the performance index $(J_r-J_o)/J_o$ and $(J^*-J_o)/J_o$ for the stable case.

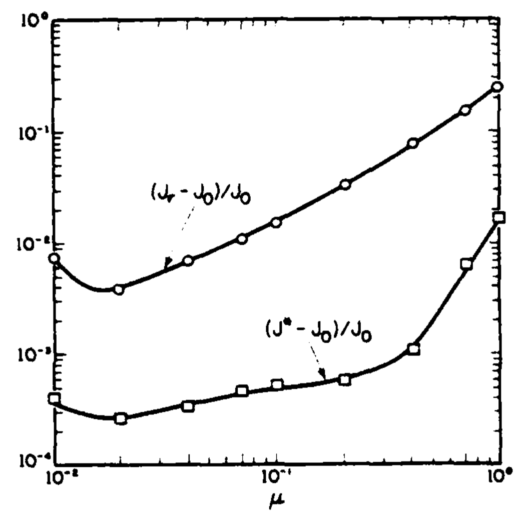

Fig. 8. The relative increments of the performance index $(J_r-J_o)/J_o$ and $(J^*-J_o)/J_o$ for the unstable case.

The choice of $L_2=\mu$ explains the fact that in both cases, the reduced cost J_r is very close to the optimal cost J_o and not much improvement is obtained by adding the fast dynamics. However, a different choice, say $L_2=\alpha\mu$ where α is a larger constant (e.g., 10) may result in a more significant contribution of the fast dynamics, so that J^* will be significantly closer to J_o than J_r. It should be noted that since μ represents the relative ratio of the fast time constants to the slow time constant, values of $\mu>0.1$ need no longer conform to the asymptotic results for small μ. For example for $\mu=0.2$ the ratio of these time constants is 1:6 so that the fast subsystem is really not so fast anymore. Furthermore, for $\mu=0.3$, the interval [0, 1] is equal to ten fast time constants, so that for any larger μ we are getting an overlap of the boundary layers, and, therefore, the asymptotic results are no longer valid. These observations explain the behavior in Fig. 7 of the near optimal performance for $\mu>0.2$. The approximate values of the gains and the covariances for $\mu=0.1$ appear to be a very good approximation to the exact value, especially after the boundary layer corrections are made. The $O(\mu)$ difference in the steady-state values is mostly unnoticeable except for P_{12} in Fig. 3.

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PARAMETER SCALING AND WELL-POSEDNESS OF STOCHASTIC SINGULARLY PERTURBED CONTROL SYSTEMS*

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Abstract

In stochastic singularly perturbed control systems the meaning of the fast variable is not always clear due to the idealized white-noise model. In this paper the linear quadratic stochastic control problem for a singularly perturbed system is reformulated by using appropriate parameter scalings which are fractional powers of the perturbation parameter. The regions of these parameters are determined so that the variables in both time-scales are well-defined, resulting in a meaningful two time-scale near-optimal solution.

I. Introduction

The usefulness of the singular perturbation approach for the analysis and control of deterministic dynamical systems with fast and slow modes is evident from the results surveyed in [1]. One of the problems encountered in extending the deterministic linear regulator results for singularly perturbed systems to filtering [2] or smoothing [3] of linear stochastic systems stems from the idealized behavior of the white noise used in the models. Fast state variables may themselves be asymptotic to white noise as in [4], in which case they are of no interest for estimation purposes. Alternately, they may be included via the usual formulation of singularly perturbed systems; however, in this case they have a negligible effect on the slow subsystem. The extension to the stochastic control problem is even more problematic [5,6], as the finite-time problem becomes an infinite-time one for the fast states. Past attempts to include control of the fast state variables have either permitted a divergent performance index, or used separate performance indices for the fast and slow subsystems. An alternative approach to circumvent such difficulties is to allow colored noise only (as in [7]) which, in a sense, limits the significance of the fast subsystem.

The problems arising in the standard singularly perturbed formulation of linear systems with white noise input may be illustrated as follows. Suppose a fast state z satisfies the following equation

$$\mu\dot{z}(t) = A_2(t)z(t) + G_2(t)w(t) \tag{1}$$

where $w(t)$ is white noise with covariance matrix W and $\mu > 0$ is a small parameter representing the small time constants in the system. If the matrix $A_2(t)$ is a stable matrix (eignevalues for every t have negative real parts), then as $\mu \to 0^+$ the process $z(t)$ tends to a white noise vector, with infinite variance parameter. More precisely, since the limit does not exist in the usual sense, the limit is to be interpreted as follows

$$\int_0^{t_1} z(t)dt = \int_0^{t_1} -A_2^{-1}G_2dU(t) + O(\mu^{\frac{1}{2}}),\ t_1 \geq t_0 > 0. \tag{2}$$

where $U(t)$ is the Wiener-Levy process defined as the integral of $w(t)$. We have used the notation $f(t) = O(\mu^{\alpha})$ to mean that there exist constants c and μ^* such that $E[\|f(t)\|^2] \leq c\mu^{2\alpha}$ for $\mu \in (0,\mu^*]$. The results implied by (2) mean that $z(t)$, while not representing a meaningful physical fast variable in its own right (having infinite variance), does have a finite contribution as an input to a slow system. This contribution to a slow subsystem may be found for small enough μ by replacing z with its white noise limit. In order to allow z to represent a fast stochastic variable with finite variance, one may use the following formulation

$$\mu\dot{z} = A_2z + \mu^{\frac{1}{2}}G_2w \tag{3}$$

In this case, z has a well-defined meaningful limit in the fast time-scale $\tau = t/\mu$, so that

$$\frac{d}{d\tau}\tilde{z} = \tilde{A}_2\tilde{z} + \tilde{G}_2\tilde{w} \tag{4}$$

where

$$\tilde{z}(\tau) = z(\mu\tau) \tag{5a}$$

$$\tilde{w}(\tau) = \mu^{\frac{1}{2}}w(\mu\tau) \tag{5b}$$

It should be noted that $\tilde{w}(\tau)$ is a valid white noise process in the stretched time-scale τ, such that

$$E[\tilde{w}(\tau_1)\tilde{w}'(\tau_2)] = W\delta(\tau_1 - \tau_2). \tag{6}$$

However the contribution of z as an input to a slow system becomes negligible in this case, since

$$\int_0^{t_1} z(t)dt = O(\mu^{\frac{1}{2}}),\quad t_1 \geq t_0 > 0. \tag{7}$$

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The objective of this paper is to explore the possibilities of introducing appropriate weighting parameters as powers of μ into the system formulation of a linear stochastic control problem, so that the resulting two-time scale near-optimal solution is meaningful. The formulation should allow the two extreme cases discussed above in addition to other intermediate cases. The relations among the parameters that result in a well-posed problem will be explored. The term well-posedness is used here to imply that either one or both variables may represent physical variables (finite variances), that the control problems in each time-scale are meaningful, and that the performance index does not become infinite in the limit. The discussion is limited to the case of a stable matrix A_2 (the fundamental matrix of the fast subsystem). For simplicity the heuristic notation of differential equations is used throughout, since the systems are linear; however, stochastic differential calculus with Wiener-Levy processes may be employed to derive the results with minor changes.

In the next section the problem is formulated. The optimal exact solution and its limiting behavior are provided in Sections III and IV. Finally, discussion and analysis of the results are given.

II. Problem Formulation

The singularly perturbed system to be considered is modeled by the following linear state equations

$$\dot{x}_1 = A_1x_1 + A_{12}x_2 + B_1u + G_1w, \qquad x_1(t_0) = x_{10} \quad (8)$$

$$\mu\dot{x}_2 = \mu^{\beta}A_{21}x_1 + A_2x_2 + B_2u + \mu^{\alpha}G_2w, \quad x_2(t_0) = x_{20} \quad (9)$$

observed in additive white Gaussian noise as follows

$$y_1 = C_{11}x_1 + C_{12}x_2 + v_1 \quad (10)$$

$$y_2 = \mu^{\gamma}C_{21}x_1 + C_{22}x_2 + \mu^{\gamma}v_2 \quad (11)$$

Here the state vector $x = [x_1', x_2']'$ is partitioned into its slow (x_1) and fast (x_2) substates with corresponding observation vectors y_1 and y_2. The parameter μ is as defined in Section I, and the processes w, v_1, v_2 are assumed to be independent white Gaussian with covariances W, V_1 and V_2 respectively, with positive definite V_1 and V_2. The initial conditions x_{i0} are assumed to be Gaussian with zero-means and covariances X_i with $\operatorname{cov}(x_{10}, x_{20}) = \mu^{\alpha}X_{12}$. The control variable u is common to both subsystems and is to be chosen as a functional of past observations in order to minimize the performance index:

$$J = E\{\frac{1}{2}\, x'\Gamma x\Big|_{t=t_f} + \frac{1}{2}\int_{t_0}^{t_f}[x'Qx + u'Ru]dt\} \quad (12)$$

where

$$\Gamma = \begin{bmatrix} \Gamma_1 & \mu\Gamma_{12} \\ \mu\Gamma_{12}' & \mu\Gamma_2 \end{bmatrix}, \quad Q = \begin{bmatrix} L_1'L_1 & \mu^{\delta}L_1'L_2 \\ \mu^{\delta}L_2'L_1 & \mu^{2\delta}L_2'L_2 \end{bmatrix} \quad (13)$$

The parameters α, β, γ, δ are chosen to represent the various limiting conditions as discussed in Section I. They represent the relative size of the small parameters within the system, relative to the small time constants of the fast subsystem. Any given system may be partitioned into several fast substates with different parameters, even though we only study a system with one such set of parameters. The inclusion of a separate observation channel y_2 for the fast subsystem is essential in this case, since for $\alpha > 0$, the fast variables cannot be estimated in a meaningful manner from the slow observation channel (signal-to-noise ratio tends to zero). Note that for the previously considered case [5], $\alpha = \beta = \gamma = 0$, y_2 may be combined with y_1. In that case $\delta = \frac{1}{2}$ in order to yield a finite performance index, so that the fast variable was of no interest as far as the control is concerned, and served only as a model for a wide-band disturbance to the slow variables. The objective of this study is to investigate the limiting behavior of the optimal stochastic control problem (8)-(13) as $\mu \to 0^+$. In particular a near optimal two time-scales solution is desired, such that the problem is well-posed in both time-scales. Finally, the regions of the values of the parameters α, β, γ, δ are to be determined so that the resulting problem is well-posed and has a meaningful solution.

From (9) it can be shown that $\operatorname{cov}(x_1, x_2) = O(\mu^a)$, $a = \min(\alpha, \beta)$; hence, in order to insure that the primary contribution to the state x_2 is due to the fast variable we shall use the constraint $\beta \geq \alpha$. Relaxing this constraint should cause no difficulties and may be treated in a similar manner. Similarly from (9) and (11) it is easily seen that if $\gamma > \alpha$, the fast variable will be observed noiselessly in the limit ($\mu \to 0^+$), hence the restriction $0 \leq \gamma \leq \alpha$ is imposed. Finally, if $\alpha > \frac{1}{2}$, the problem becomes deterministic in the limit as $\mu \to 0^+$, and consequently we shall consider the region $0 \leq \alpha \leq \frac{1}{2}$ only. These constraints may be written as

$$0 \leq \gamma \leq \alpha \leq \beta \leq \tfrac{1}{2} \quad (14)$$

since if $\beta > \frac{1}{2}$ the coupling between x_1 and x_2 may be neglected as being of order less than $O(\mu^{\frac{1}{2}})$. It can also be observed from (9) that the covariance of x_2 is $O(\mu^{2\alpha-1})$, a fact which will be utilized in the filtering covariances. Next the solution of the stochastic control problem and its limits will be considered.

III. Optimal Solution

The separation principle may be used to obtain the optimal control solution of the problem posed by (8)-(13). The control gain matrix K and the filter error covariance matrix P may be partitioned as follows:

$$K = \begin{bmatrix} K_1 & \mu K_{12} \\ \mu K_{12}' & \mu K_2 \end{bmatrix}, \quad P = \begin{bmatrix} P_1 & \mu^{\alpha}P_{12} \\ \mu^{\alpha}P_{12}' & \mu^{2\alpha-1}P_2 \end{bmatrix} \quad (15)$$

The optimal control u^* and the filtering equations

for $\hat{x}$ will therefore be given by

$$u^* = -R^{-1}[(B_1'K_1+B_2'K_{12}')\hat{x}_1+(B_2'K_2+\mu B_1'K_{12})\hat{x}_2] \tag{16}$$

$$\dot{\hat{x}}_1 = A_1\hat{x}_1+A_{12}\hat{x}_2+B_1u^*+[P_1C_1'+\mu^{\alpha-\gamma}P_{12}C_2']V^{-1}\nu(t) \tag{17}$$

$$\mu\dot{\hat{x}}_2 = \mu^{\beta}A_{21}\hat{x}_1+A_2\hat{x}_2+B_2u^*+\mu^{\alpha}[\mu P_{12}'C_1'+\mu^{\alpha-\gamma}P_2C_2']V^{-1}\nu(t) \tag{18}$$

where the innovations process $\nu(t)$ is defined by

$$\nu(t) \triangleq \begin{bmatrix} y_1(t) \\ \mu^{-\gamma}y_2(t) \end{bmatrix} - \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & \mu^{-\gamma}C_{22} \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix}$$

$$\triangleq y(t) - [C_1 \quad \mu^{-\gamma}C_2]\,\hat{x}\,. \tag{19}$$

Note that (19) is also used to define y, C_1, and C_2, and the matrix V is defined by

$$V = \begin{bmatrix} V_1 & 0 \\ 0 & V_2 \end{bmatrix} \tag{20}$$

It should be emphasized that $\nu(t)$ is a standard innovations process in the slow time scale (white with finite covariance). Furthermore, the filtering equation for $\hat{x}_2$ has the same form as the original state as far as the μ^{α} term multiplying the white input $\nu(t)$ is concerned. The gain matrices satisfy the usual Riccati equations,

$$-\dot{K}_1 = K_1A_1+\mu^{\beta}K_{12}A_{21}+A_1'K_1+\mu^{\beta}A_{21}'K_{12}'+L_1'L_1$$
$$-(K_{12}B_2+K_1B_1)R^{-1}(B_1'K_1+B_2'K_{12}'),\quad K_1(t_f)=\Gamma_1, \tag{21a}$$

$$-\mu\dot{K}_{12}=K_1A_{12}+K_{12}A_2+\mu A_1'K_{12}+\mu^{\beta}A_{21}'K_2+\mu^{\delta}L_1'L_2$$
$$-(K_{12}B_2+K_1B_1)R^{-1}(B_2'K_2+\mu B_1'K_{12}),\quad K_{12}(t_f)=\Gamma_{12} \tag{21b}$$

$$-\mu\dot{K}_2=K_2A_2+A_2'K_2+\mu K_{12}'A_{12}+\mu A_{12}'K_{12}+\mu^{2\delta}L_2'L_2$$
$$-(K_2B_2+\mu K_{12}'B_1)R^{-1}(B_2'K_2+\mu B_1'K_{12}),\quad K_2(t_f)=\Gamma_2 \tag{21c}$$

It can be shown that as $\mu \to 0^+$ the limiting behavior of the gains become

$$K_1 = K_0 + O(\mu) \tag{22a}$$

$$K_{12} = \bar{K}_{12} + \tilde{K}_{12}(\theta) + O(\mu),\quad \theta = \frac{t_f - t}{\mu} \tag{22b}$$

$$K_2 = \bar{K}_2 + \tilde{K}_2(\theta) + O(\mu) \tag{22c}$$

where $\tilde{K}_1(\theta)$ and $\tilde{K}_2(\theta)$ represent boundary layer terms and will not be given here due to space limitations. The other terms in (22) are

$$-\dot{K}_0=K_0(A_0-B_0R_0^{-1}N_0'L_0)+(A_0-B_0R_0^{-1}N_0'L_0)'K_0$$
$$+ L_0'(I-N_0R_0^{-1}N_0')L_0-K_0B_0R_0^{-1}B_0'K_0,\quad K_0(t_f)=\Gamma_1 \tag{23a}$$

$$0=\bar{K}_2A_2+A_2'\bar{K}_2+\mu^{2\delta}L_2'L_2-\bar{K}_2B_2R^{-1}B_2'\bar{K}_2 \tag{23b}$$

$$0=K_0A_{12}+\bar{K}_{12}A_2+\mu^{\beta}A_{21}'\bar{K}_2+\mu^{\delta}L_1'L_2-(K_0B_1+\bar{K}_{12}B_2)R^{-1}B_2'\bar{K}_2 \tag{23c}$$

where

$$A_0=A_1-\mu^{\beta}A_{12}A_2^{-1}A_{21},\quad B_0=B_1-A_{12}A_2^{-1}B_2,\quad N_0=-\mu^{\delta}L_2A_2^{-1}B_2$$

$$L_0=L_1-\mu^{\beta+\delta}L_2A_2^{-1}A_{21},\quad R_0 = R+N_0'N_0 \tag{24}$$

Note that for the stable case considered here $\bar{K}_2 = O(\mu^{2\delta})$. The gains K_0 and $\bar{K}_2$ represent the gain solutions for the reduced-order slow-control problem and the infinite-time fast-control problem, respectively [5]. Similarly, the Riccati equations for the filter covariances may be written as

$$\dot{P}_1=A_1P_1+P_1A_1'+\mu^{\alpha}A_{12}P_{12}'+\mu^{\alpha}P_{12}A_{12}'+G_1WG_1'$$
$$-(P_1C_1'+\mu^{\alpha-\gamma}P_{12}C_2')V^{-1}(C_1P_1+\mu^{\alpha-\gamma}C_2P_{12}'),\quad P_1(t_0)=X_1 \tag{25a}$$

$$\mu\dot{P}_{12}= \mu A_1P_{12}+\mu^{\alpha}A_{12}P_2+P_{12}A_2'+\mu^{\beta-\alpha}P_1A_{21}'+G_1WG_2'$$
$$-(P_1C_1'+\mu^{\alpha-\gamma}P_{12}C_2')V^{-1}(\mu C_1P_{12}+\mu^{\alpha-\gamma}C_2P_2),$$
$$P_{12}(t_0)=X_{12} \tag{25b}$$

$$\mu\dot{P}_2=A_2P_2+P_2A_2'+\mu^{1-\alpha+\beta}A_{21}P_{12}+\mu^{1-\alpha+\beta}P_{12}'A_{21}'+G_2WG_2'$$
$$-(\mu P_{12}'C_1'+\mu^{\alpha-\gamma}P_2C_2')V^{-1}(\mu C_1P_{12}+\mu^{\alpha-\gamma}C_2P_2),$$
$$P_2(t_0)= \mu^{1-2\alpha}X_2 \tag{25c}$$

The limiting behavior of the resulting covariances may be found by duality to the control case,

$$P_1 = P_0 + O(\mu) \tag{26a}$$

$$P_{12} = \bar{P}_{12} + \tilde{P}_{12}(\lambda) + O(\mu),\quad \lambda = \frac{t-t_0}{\mu} \tag{26b}$$

$$P_2 = \bar{P}_2 + \tilde{P}_2(\lambda) + O(\mu). \tag{26c}$$

Here again the boundary layer terms $\tilde{P}_{12}$ and $\tilde{P}_2$ will not be given, while the remaining variables are

$$\dot{P}_0=A_0P_0+P_0A_0'-(P_0C_0'+G_0WD_0')V_0^{-1}(C_0P_0+D_0WG_0')+G_0WG_0',$$
$$P_0(t_0)=X_1 \tag{27a}$$

$$0=A_2\bar{P}_2+\bar{P}_2A_2'+G_2WG_2'- \mu^{2(\alpha-\gamma)}\bar{P}_2C_2'V^{-1}C_2\bar{P}_2, \tag{27b}$$

$$0= \mu^{\alpha}A_{12}\bar{P}_2+\bar{P}_{12}A_2'+\mu^{\beta-\alpha}P_0A_{21}'$$
$$+G_1WG_2'-\mu^{(\alpha-\gamma)}(P_0C_1'+ \mu^{\alpha-\gamma}\bar{P}_{12}C_2')V^{-1}C_2\bar{P}_2, \tag{27c}$$

where the system matrices are defined by

$$C_0=C_1- \mu^{\beta-\gamma}C_2A_2^{-1}A_{21},\quad D_0= - \mu^{\alpha-\gamma}C_2A_2^{-1}G_2$$
$$G_0=G_1- \mu^{\alpha}A_{12}A_2^{-1}G_2,\quad V_0 = V + D_0WD_0' \tag{28}$$

Again, it is observed that P_0 and $\bar{P}_2$ are the covariances of the filtering problems defined by the reduced-order slow system and the quasisteady-state fast system, respectively.

IV. Near-Optimal Limiting Solution

In order to transform the filter and hence the control into its slow and fast components, the following transformations to (17)-(18) are introduced after first substituting (16):

$$\eta_1 = \hat{x}_1 - \mu^{1-\gamma}M(T\hat{x}_1 + \hat{x}_2)$$
$$\eta_2 = T\hat{x}_1 + \hat{x}_2 \tag{29}$$

where T and M satisfy the differential equations

$$\mu\dot{T} = F_2T - F_{21} - \mu T(F_1 - F_{12}T)$$
$$\mu\dot{M} = -MH_2 + H_1 + \mu(F_1 - F_{12}T)M \qquad (30)$$

The matrices F and H are defined by

$$F_1=A_1-B_1R^{-1}(B_1'K_1+B_2'K_{12}'),\ F_{12}=A_{12}-B_1R^{-1}(B_2'K_2+\mu B_1'K_{12})$$
$$F_{21}=\mu^{\beta}A_{21} - B_2R^{-1}(B_1'K_1+B_2'K_{12}'),$$
$$F_2=A_2-B_2R^{-1}(B_2'K_2+\mu B_1'K_{12}) \qquad (31)$$

$$H_1=\mu^{\gamma}A_{12}-(P_1C_1'+\mu^{\alpha-\gamma}P_{12}C_2')V^{-1}C_2$$
$$H_2=A_2-[\mu^{\alpha-\gamma}(\mu P_{12}'C_1' +\mu^{\alpha-\gamma}P_2C_2')+\mu^{1-\gamma}T(P_1C_1' + \mu^{\alpha-\gamma}P_{12}C_2')]V^{-1}C_2 \qquad (32)$$

The resulting equations for the new variables η_1 and η_2 become after lengthy manipulations:

$$\dot{\eta}_1 = (F_1-F_{12}T)\eta_1-B_1R^{-1}(B_2'K_2 + \mu B_1'K_{12})\eta_2 + \mu^{-\gamma}M[B_2R^{-1}(B_2'K_2 + \mu B_1'K_{12}) - \mu TF_{12}]\eta_2 +\{(P_1C_1'+ \mu^{\alpha-\gamma}P_{12}C_2')- \mu^{\alpha-\gamma}M[(\mu P_{12}'C_1'+ \mu^{\alpha-\gamma}P_2C_2') + \mu^{(1-\alpha)}T(P_1C_1'+ \mu^{\alpha-\gamma}P_{12}C_2')]\}V^{-1}\nu_0 \qquad (33)$$

$$\mu\dot{\eta}_2=H_2\eta_2-[B_2R^{-1}(B_2'K_2+ \mu B_1'K_{12})- \mu TF_{12}]\eta_2 + \mu^{\alpha}[(\mu P_{12}'C_1'+ \mu^{\alpha-\gamma}P_2C_2')+ \mu^{1-\alpha}T(P_1C_1' + \mu^{\alpha-\gamma}P_{12}C_2')]V^{-1}\nu_0 \qquad (34)$$

where ν_0 is defined by

$$\nu_0(t) \triangleq y(t)-(C_1 - \mu^{-\gamma}C_2T)(\eta_1 + \mu^{1-\gamma}M\eta_2) \qquad (35)$$

The near-optimal two time-scale limiting solution is obtained by manipulating the matrices in (33)-(35) and neglecting terms smaller than $O(\mu^{\frac{1}{2}})$. After several detailed manipulations which are omitted here, the resulting approximate filters-controllers are given by

$$\eta_1 = \hat{x}_s + O(\mu^{\frac{1}{2}})$$
$$\eta_2 = \hat{x}_f + O(\mu^{\frac{1}{2}})$$
$$u^* = u_0 + O(\mu^{\frac{1}{2}}) \qquad (36)$$

where

$$u_0 \triangleq u_s+u_f = R_0^{-1}(N_0'L_0+B_0'K_0)\hat{x}_s + R^{-1}B_2'\bar{K}_2\hat{x}_f \qquad (37)$$

The slow and fast filters are given by

$$\dot{\hat{x}}_s=A_0\hat{x}_s+B_0u_0+(P_0C_0'+G_0WD_0')V_0^{-1}[y-C_0\hat{x}_s- \mu^{-\gamma}E_0u_0] \qquad (38)$$

$$\mu\dot{\hat{x}}_f=A_2\hat{x}_f+B_2u_f+\mu^{\alpha}\mu^{\alpha-\gamma}\bar{P}_2C_2'V^{-1}[y-C_0\hat{x}_s-\mu^{-\gamma}C_2\hat{x}_f-\mu^{-\gamma}E_0u_s] \qquad (39)$$

where

$$E_0 = -C_2A_2^{-1}B_2 \qquad (40)$$

Note that the slow filter and control are the same as those obtained by solving the reduced-order problem, by formally setting $\mu = 0$ in the left-hand side of (9). The terms $\mu^{-\gamma}$ multiplying E_0 and C_2 in (38)-(39) stem from our definition of y by multiplying y_2 by $\mu^{-\gamma}$, consequently this term does not diverge even for $\gamma > 0$. It should be noted that the slow filter is driven by the slow innovations only. In view of [5] the performance index when u_0 is used is within $O(\mu)$ of the optimal performance index. The two time-scale near-optimal scheme given by (37)-(39) is shown in Fig. 1.

V. Discussion and Analysis

In this section we analyze the relationships between the scaling parameter exponents and their effect on the various variables in the systems. First, the performance index will be finite if the expression $\mu^{2\delta}\text{cov}(x_2)$ is finite. Hence, in view of (15), we should require that δ satisfy

$$\delta \geq (\tfrac{1}{2} - \alpha) \qquad (41)$$

Note, that this implies full weighting to the fast variable only when $\alpha = \frac{1}{2}$. However, as discussed above, the case $\alpha = \frac{1}{2}$ is the only one for which the fast variable is a well-defined process in the fast-time scale. If $\alpha > \gamma$, then from (27b) as $\mu \to 0^+$ the covariance of the error of the fast variable tends to the covariance of the variable itself, implying that the fast variable is not observed due to the noisy observations. Hence, for a meaningful problem we should require $\alpha = \gamma$, even though $\alpha > \gamma$ causes no difficulties except for the disappearance of the observations terms in (39), as $\mu \to 0^+$. For the case $\alpha = \gamma$, the fast filter (39) formulation is quite simplified as $\mu \to 0^+$, since only the fast observation channel y_2 remains, namely

$$\mu\dot{\hat{x}}_f=A_2\hat{x}_f+B_2u_f+\bar{P}_2C_{22}'\nu_2 \qquad (42)$$
$$\nu_2=\{y_2-C_{22}\hat{x}_f-[\mu^{\alpha}C_{21}- \mu^{\beta}C_{22}A_2^{-1}A_{21}]\hat{x}_s-C_{22}A_2^{-1}B_2u_s\}$$

where the neglected term may be larger than $O(\mu^{\frac{1}{2}})$ but is negligible with respect to the variance of $\hat{x}_f$. Finally, as for the parameter β, the restriction imposed in (14) does not result in any difficulties, however, when $\beta > \alpha$, then the effect of the slow system on the fast is negligible as $\mu \to 0^+$, even though its contribution is still larger than $O(\mu^{\frac{1}{2}})$.

V. Summary

A singularly perturbed stochastic linear control problem is formulated where weighting parameters as powers of the perturbation parameter are used. It is seen that if these parameters are restricted to certain regions, the resulting control problem is well-posed. The most significant parameter value is α which for $\alpha = 0$ yields the previously considered case and for $\alpha = \frac{1}{2}$ yields a well-defined fast variable. These two extreme values represent the extremes of parasitic fast variables and finite variance meaningful fast variables. Other values of α may be used to represent other cases in between.

The formulation may be extended to multiple time-scales problems with more than one perturbation parameter as in [8]. Furthermore any problem involving several partitioned blocks of variables

with different α, β, γ, δ for each subsystem, may be solved in the same manner. This paper considered only a single fast subsystem for simplicity.

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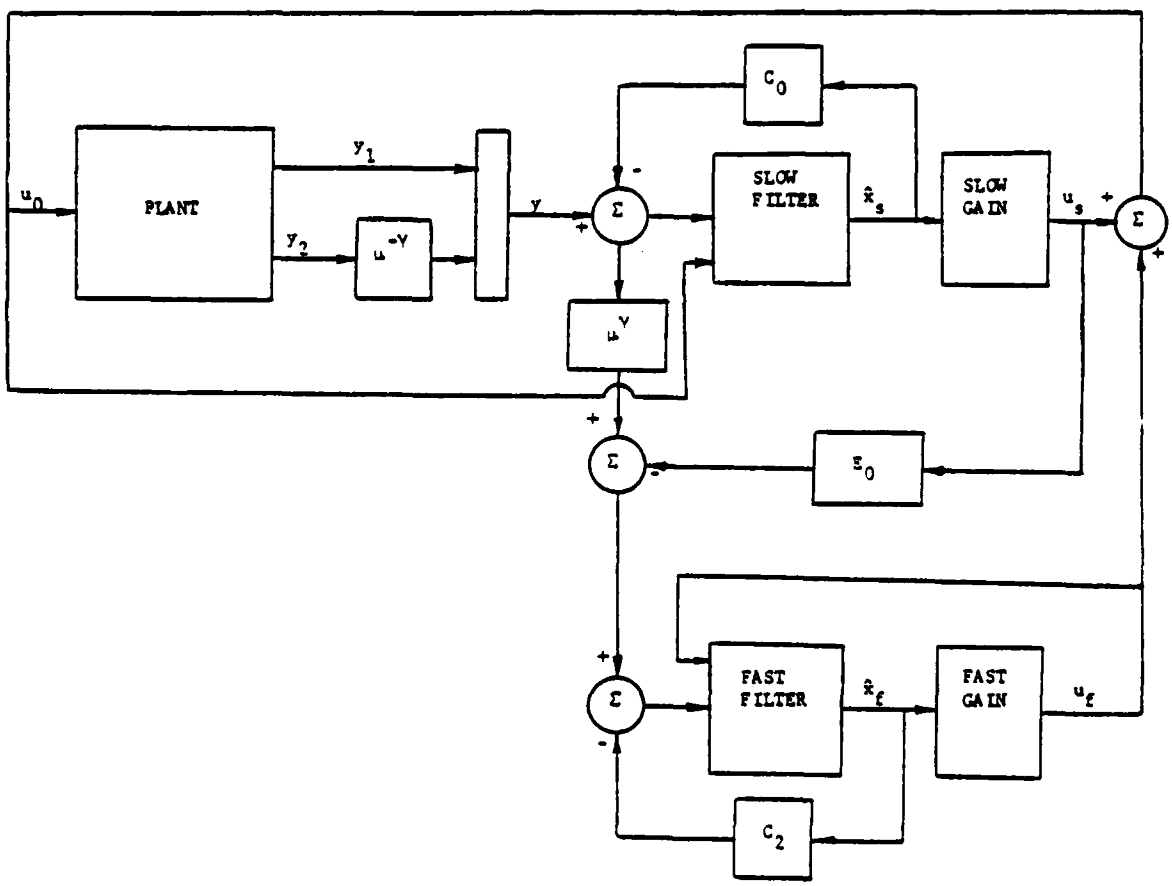

Fig. 1. Block diagram of the two time-scale solution.

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Brief Paper

Control of Linear Singularly Perturbed Systems with Colored Noise Disturbance*

HASSAN K. KHALIL†

Key Word Index—Optimal control; order reduction; singular perturbations; stochastic control.

Summary—This paper considers the stochastic control of linear-quadratic problems for singularly perturbed systems when the input noise is colored. A near optimal linear output feedback control is obtained by optimizing a slow subsystem only.

1. *Introduction*

SINGULAR perturbation methods are physically motivated tools for order reduction, separation of time scales and other simplifications in control system analysis and design[1]. Recently Haddad and Kokotovic have applied the singular perturbation theory to the stochastic control for Linear-Quadratic-Gaussian problems for systems with fast and slow modes[2]. In their approach both the input noise and observation noise processes are modeled as white noise. For the same problem, Teneketzis and Sandell have developed a hierarchical control design method to reduce the on-line computations[3]. However, singularly perturbed systems incorporate fast dynamics; so if the correlation time of the system disturbances is not short compared to the small times of interest of the fast modes, modeling input noise as white is not appropriate, and does not take advantage of singular perturbation techniques. As an illustration consider a circuit with parasitic capacitors. If we model the input noise as white noise, these capacitors are not negligible regardless of how small their time constants are. Thus whenever the correlation time of the input noise is not short compared to times of fast modes of the system, it should be modeled as colored noise. Hence in this paper we consider the case when the input noise is colored. Since colored noise is modeled as the output of a system driven by white noise[4], our problem is a special case of[2] when the fast modes equations are noise free. We will keep the assumption that observation noise is modeled as white noise. The derivation of a control algorithm will show that this assumption does not degrade the results.

The assumption of colored input noise, or equivalently white input noise with noise free fast dynamics equations, will enable us to give a clearer presentation of the problem. We avoid complications of the more general problem, such as ill-defined quadratic performance indices in the limiting procedure[2]. Moreover we get an explicit result (Theorem 1) which cannot be obtained as a special case of the corresponding result of [2]. The methodology here is simpler than that in previous treatments of linear-quadratic singularly perturbed systems. We approximate a performance index directly, thus avoiding consideration of boundary layers. Moreover the solution is analogous to the compact form of the deterministic case[5].

2. *Problem statement*

Consider the singularly perturbed linear system

$$\dot{\mathbf{x}}_1(t)=\tilde{A}_{11}(t)\mathbf{x}_1(t)+\tilde{A}_{12}(t)\mathbf{x}_2(t)+\tilde{B}_1(t)\mathbf{u}(t)+\tilde{G}_1(t)\mathbf{v}_1(t),\quad \mathbf{x}_1(t_0)=\mathbf{x}_{10} \tag{1}$$

$$\mu\dot{\mathbf{x}}_2(t)=\tilde{A}_{21}(t)\mathbf{x}_1(t)+\tilde{A}_{22}(t)\mathbf{x}_2(t)+\tilde{B}_2(t)\mathbf{u}(t)+\tilde{G}_2(t)\mathbf{v}_1(t),\quad \mathbf{x}_2(t_0)=\mathbf{x}_{20} \tag{2}$$

with observation equation

$$\mathbf{y}(t)=\tilde{C}_1(t)\mathbf{x}_1(t)+\tilde{C}_2(t)\mathbf{x}_2(t)+\mathbf{w}_2(t) \tag{3}$$

and performance index:

$$J=E\left\{\frac{1}{2}\begin{pmatrix}\mathbf{x}_1\\ \mathbf{x}_2\end{pmatrix}'\begin{pmatrix}\tilde{T}_1 & \mu\tilde{T}_{12}\\ \mu\tilde{T}'_{12} & \mu\tilde{T}_2\end{pmatrix}\begin{pmatrix}\mathbf{x}_1\\ \mathbf{x}_2\end{pmatrix}\Bigg|_{t=t_f}+\frac{1}{2}\int_{t_0}^{t_f}\begin{pmatrix}\mathbf{x}_1\\ \mathbf{x}_2\end{pmatrix}'\begin{pmatrix}\tilde{C}'_1\tilde{C}_1 & \tilde{C}'_1\tilde{C}_2\\ \tilde{C}'_2\tilde{C}_1 & \tilde{C}'_2\tilde{C}_2\end{pmatrix}\begin{pmatrix}\mathbf{x}_1\\ \mathbf{x}_2\end{pmatrix}+\mathbf{u}'R\mathbf{u}\,dt\right\} \tag{4}$$

where dimensions of $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{u}$, $\mathbf{y}$ and $\mathbf{v}_1$ are n, m, r, l and s respectively, μ is a sufficiently small positive scalar, and $\mathbf{v}_1(t)$ is colored noise modeled by

$$\dot{\mathbf{v}}_1(t)=H(t)\mathbf{v}_1(t)+\mathbf{w}_1(t),\quad \mathbf{v}_1(t_0)=\mathbf{v}_{10} \tag{5}$$

$\mathbf{w}_1(t)$ and $\mathbf{w}_2(t)$ are zero mean white Gaussian noise processes that need not be uncorrelated. The covariance matrix of $\mathbf{w}'(t)=(\mathbf{w}'_1(t),\mathbf{w}'_2(t))$ is

$$E\{\mathbf{w}(t)\mathbf{w}'(s)\}=\begin{pmatrix}V_1(t) & V_{12}(t)\\ V'_{12}(t) & V_2(t)\end{pmatrix}\delta(t-s). \tag{6}$$

Also, $\mathbf{x}_{10}$, $\mathbf{x}_{20}$ and $\mathbf{v}_{10}$ are Gaussian random vectors with mean values $\bar{\mathbf{x}}_{10}$, $\bar{\mathbf{x}}_{20}$ and $\bar{\mathbf{v}}_{10}$ respectively. There may be, in general, correlation between them, but each of them is uncorrelated with $\{\mathbf{w}(t), t\geq t_0\}$. The assumed form of the terminal cost in (4) is usual in the treatment of singularly perturbed linear quadratic problems[6]. A more general case, when $\mu\tilde{T}_{12}$, $\mu\tilde{T}_2$ are replaced by $\tilde{T}_{12}$, $\tilde{T}_2$ respectively, has been treated in [7]. It results in singularly perturbed problems with infinitely large boundary conditions.

The problem is to determine the control $\mathbf{u}(t)$ as a function of past observation $\{\mathbf{y}(s), t_0\leq s<t\}$ which minimizes (4) under the following conditions for $t\in[t_0,t_f]$:

(1) All the matrices are continuous, bounded and have bounded first derivatives.

(2) $\tilde{T}_1\geq 0$, $R(t)>0$, $V_2(t)>0$.

(3) $\tilde{A}_{22}(t)$ is a stable matrix $(\mathrm{Re}\{\lambda(\tilde{A}_{22}(t))\}\leq -\sigma$ for some $\sigma>0)$.

Combining (5) with (1), (2), (3), and (4), and putting $\mathbf{x}'=(\mathbf{x}'_1,\mathbf{v}'_1)$ and $\mathbf{z}=\mathbf{x}_2$, we obtain the augmented system

$$\dot{\mathbf{x}}(t)=A_{11}(t)\mathbf{x}(t)+A_{12}(t)\mathbf{z}(t)+B_1(t)\mathbf{u}(t)+G_1(t)\mathbf{w}_1(t),\quad \mathbf{x}(t_0)=\mathbf{x}_0 \tag{7}$$

$$\mu\dot{\mathbf{z}}(t)=A_{21}(t)\mathbf{x}(t)+A_{22}(t)\mathbf{z}(t)+B_2(t)\mathbf{u}(t),\quad \mathbf{z}(t_0)=\mathbf{z}_0 \tag{8}$$

$$\mathbf{y}(t)=C_1(t)\mathbf{x}(t)+C_2(t)\mathbf{z}(t)+\mathbf{w}_2(t) \tag{9}$$

with performance index

$$J=E\left\{\frac{1}{2}\begin{pmatrix}\mathbf{x}\\ \mathbf{z}\end{pmatrix}'\begin{pmatrix}T_1 & \mu T_{12}\\ \mu T'_{12} & \mu T_2\end{pmatrix}\begin{pmatrix}\mathbf{x}\\ \mathbf{z}\end{pmatrix}\Bigg|_{t=t_f}+\frac{1}{2}\int_{t_0}^{t_f}\begin{pmatrix}\mathbf{x}\\ \mathbf{z}\end{pmatrix}'\begin{pmatrix}C'_1C_1 & C'_1C_2\\ C'_2C_1 & C'_2C_2\end{pmatrix}\begin{pmatrix}\mathbf{x}\\ \mathbf{z}\end{pmatrix}+\mathbf{u}'R\mathbf{u}\,dt\right\}. \tag{10}$$

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We define the reduced problem, formally obtained by setting $\mu = 0$ in (7)–(10), by

$$\dot{x}_r(t) = A_0(t)x_r(t) + B_0(t)u(t) + G_1(t)w_1(t),\ x_r(t_0) = x_0 \tag{11}$$

$$y(t) = C_0(t)x_r(t) + D_0(t)u(t) + w_2(t) \tag{12}$$

with performance index

$$J_r = E\{\tfrac{1}{2}x_r'(t_f)T_1x_r(t_f) + \tfrac{1}{2}\int_{t_0}^{t_f}[x_r'C_0'C_0x_r + 2u'D_0'C_0x_r + u'(R + D_0'D_0)u]\mathrm{d}t\} \tag{13}$$

where

$$A_0 = A_{11} - A_{12}A_{22}^{-1}A_{21}, \qquad B_0 = B_1 - A_{12}A_{22}^{-1}B_2$$

$$C_0 = C_1 - C_2A_{22}^{-1}A_{21}, \qquad D_0 = -C_2A_{22}^{-1}B_2.$$

3. *Near optimal linear output feedback control*

A transformation due to Chang[8], which separates the fast and slow modes, is

$$\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} I_1 - \mu ML & -\mu M \\ L & I_2 \end{pmatrix}\begin{pmatrix} x \\ z \end{pmatrix} \tag{14}$$

where $M(t)$ and $L(t)$ matrices satisfy

$$\mu\dot{L} = A_{22}L - A_{21} - \mu L(A_{11} - A_{12}L) \tag{15}$$

$$\mu\dot{M} = -M(A_{22} + \mu LA_{12}) + A_{12} + \mu(A_{11} - A_{12}L)M. \tag{16}$$

Since the initial conditions for $L(t)$ and $M(t)$ are arbitrary, we choose them to be

$$L(t_0) = A_{22}^{-1}(t_0)A_{21}(t_0) \tag{17}$$

$$M(t_0) = A_{12}(t_0)A_{22}^{-1}(t_0). \tag{18}$$

Applying the usual singular perturbation techniques[1], we obtain for all $t \in [t_0, t_f]$

$$L(t) = A_{22}^{-1}A_{21}(t) + 0(\mu) \tag{19}$$

$$M(t) = A_{12}(t)A_{22}^{-1}(t) + 0(\mu). \tag{20}$$

Then (7), (8) and (9) becomes

$$\dot{p}(t) = (A_0(t) + 0(\mu))p(t) + (B_0(t) + 0(\mu))u(t) + (G_1(t) + 0(\mu))w_1(t),\quad p(t_0) = p_0 \tag{21}$$

$$\mu\dot{q}(t) = (A_{22}(t) + 0(\mu))q(t) + (B_2(t) + 0(\mu))u(t) + \mu G_2(t)w_1(t),\quad q(t_0) = q_0 \tag{22}$$

$$y(t) = (C_0(t) + 0(\mu))p(t) + (C_2(t) + 0(\mu))q(t) + w_2(t). \tag{23}$$

Letting $\eta(t)$ and $\rho(t)$ satisfy the equations

$$\dot{\eta}(t) = A_0(t)\eta(t) + B_0(t)u(t) + G_1(t)w_1(t),\quad \eta(t_0) = \eta_0 = p_0 \tag{24}$$

$$\mu\dot{\rho}(t) = A_{22}(t)\rho(t) + B_2(t)u(t) + \mu G_2(t)w_1(t),\quad \rho(t_0) = \rho_0 = q_0 \tag{25}$$

it can be easily shown that

$$\eta(t) = p(t) + 0(\mu) \tag{26}$$

$$\rho(t) = q(t) + 0(\mu). \tag{27}$$

Since $A_{22}(t)$ is a stable matrix for all $t \in [t_0, t_f]$, using Lemma 2 of [9] it follows that $\rho(t)$ has bounded covariance as $\mu \to 0$. Then from (14), (26) and (27) we have

$$x(t) = \eta(t) + 0(\mu) \tag{28}$$

$$z(t) = -A_{22}^{-1}(t)A_{21}(t)\eta(t) + \rho(t) + 0(\mu). \tag{29}$$

Lemma 1. As an input to slow linear systems, $\rho(t)$ can be approximated to $0(\mu)$ by

$$-A_{22}^{-1}(t)B_2(t)u(t),$$

for all $t \in [t_0, t_f]$, that is

$$\int_{t_0}^{t_1}\rho(t)\mathrm{d}t = \int_{t_0}^{t_1}[-A_{22}^{-1}(t)B_2(t)u(t)]\mathrm{d}t + 0(\mu),\quad t_1 \in [t_0 + \varepsilon, t_f] \tag{30}$$

for some fixed $\varepsilon > 0$.

Proof. From (25) and Lemma 2 of [9], we have

$$\rho(t) = \exp\left[A_{22}(t_0)\left(\frac{t - t_0}{\mu}\right)\right]\rho_0 + \frac{1}{\mu}\int_{t_0}^{t}\exp\left[A_{22}(\sigma)\left(\frac{t - \sigma}{\mu}\right)\right] \times [B_2(\sigma)u(\sigma) + \mu G_2(\sigma)w_1(\sigma)]\mathrm{d}\sigma + 0(\mu). \tag{31}$$

It can be shown, by reversing the order of integration, that

$$\begin{aligned}\int_{t_0}^{t_1}\rho(\tau)\mathrm{d}\tau = {} & \mu\left\{\exp\left[A_{22}(t_0)\left(\frac{t_1 - t_0}{\mu}\right)\right] - I\right\}A_{22}^{-1}(t_0)\rho_0 \\ & + \mu\int_{t_0}^{t_1}\left\{\exp\left[A_{22}(\sigma)\left(\frac{t_1 - \sigma}{\mu}\right)\right] - I\right\} \times A_{22}^{-1}(\sigma)G_2(\sigma)w_1(\sigma)\mathrm{d}\sigma \\ & + \int_{t_0}^{t_1}\exp\left[A_{22}(\sigma)\left(\frac{t_1 - \sigma}{\mu}\right)\right]A_{22}^{-1}(\sigma)B_2(\sigma)u(\sigma)\mathrm{d}\sigma \\ & - \int_{t_0}^{t_1}A_{22}^{-1}(\sigma)B_2(\sigma)u(\sigma)\mathrm{d}\sigma + 0(\mu) \\ = {} & -\int_{t_0}^{t_1}A_{22}^{-1}(t)B_2(t)u(t)\mathrm{d}t + 0(\mu),\end{aligned}$$

$$\text{for all } t_1 \in [t_0 + \varepsilon, t_f]. \tag{32}$$

Lemma 2.

$$J = J_S + 0(\mu) \tag{33}$$

where

$$J_S = E\{\tfrac{1}{2}\eta'(t_f)T_1\eta(t_f) + \tfrac{1}{2}\int_{t_0}^{t_f}[\eta'C_0'C_0\eta + 2u'D_0'C_0\eta + u'(R + D_0'D_0)u]\mathrm{d}t\}. \tag{34}$$

Proof. It follows from (29) and (30), that for an $o(\mu)$ approximation of J the quantity

$$-A_{22}^{-1}(t)A_{21}(t)\eta(t) - A_{22}^{-1}(t)B_2(t)u(t) \tag{35}$$

can be used instead of z in (10). Then, using (28) and (35), (10) reduces to

$$J = J_S + 0(\mu).$$

We define

$$y_1(t) = C_0(t)\eta(t) + D_0(t)u(t) + w_2(t). \tag{36}$$

Lemma 3. As an input to slow linear systems $y(t)$ can be approximated to $o(\mu)$ by $y_1(t)$ for all $t \in [t_0, t_f]$, that is

$$\int_{t_0}^{t_1}y(t)\mathrm{d}t = \int_{t_0}^{t_1}y_1(t)\mathrm{d}t + 0(\mu),\quad t_1 \in [t_0 + \varepsilon, t_f] \tag{37}$$

for some fixed $\varepsilon > 0$.

Proof. From (23), we have

$$\begin{aligned}\int_{t_0}^{t_1}y(t)\mathrm{d}t &= \int_{t_0}^{t_1}[(C_0(t) + 0(\mu))p(t) + (C_2(t) + 0(\mu))q(t) + w_2(t)]\mathrm{d}t \\ &= \int_{t_0}^{t_1}[C_0(t)p(t) + C_2(t)q(t) + w_2(t)]\mathrm{d}t + 0(\mu)\end{aligned} \tag{38}$$

since $\mathbf{p}(t)$ and $\mathbf{q}(t)$ have bounded covariances as $\mu \to 0$. Using (26), (27), and Lemma 1, (38) becomes

$$\int_{t_0}^{t_f} \mathbf{y}(t)\,dt = \int_{t_0}^{t_f} [C_0(t)\boldsymbol{\eta}(t) + D_0(t)\mathbf{u}(t) + \mathbf{w}_2(t)]\,dt + 0(\mu).$$

Theorem 1. The solution of the stochastic linear quadratic problem for the singularly perturbed system defined in (7)–(10), can be replaced to an $o(\mu)$ approximation in J, by the solution of the stochastic linear quadratic problem for the reduced problem defined in (11)–(13).

Proof. From Lemma 2, we know that, to an $o(\mu)$ error, the minimization of J reduces to the minimization of J_S with respect to (24). Assuming that $\mathbf{y}_1(t)$ is available for measurement, (24), (36) and (34) define a stochastic linear quadratic problem for the slow variable $\boldsymbol{\eta}(t)$, whose solution is given in [4]. Since in that solution $\mathbf{y}_1(t)$ is used as an input to a slow filter, from Lemma 3, $\mathbf{y}_1(t)$ can be replaced by $\mathbf{y}(t)$ giving an $o(\mu)$ error in the estimate $\hat{\boldsymbol{\eta}}(t)$. An $o(\mu)$ error in $\hat{\boldsymbol{\eta}}(t)$ gives an $o(\mu)$ error in J. When $\mathbf{y}_1(t)$ is replaced by $\mathbf{y}(t)$, the solution of this problem is identical to the solution of the reduced problem defined in (11)–(13). This can be seen by comparing (11), (12) and (13) to (24), (36) and (34) respectively with $\mathbf{x}_r$ replaced by $\boldsymbol{\eta}$.

Remark 1. Theorem 1, besides establishing the main result for systems with colored noise inputs, is also true for systems with white noise inputs when the noise does not appear in the equations of the fast variables, that is when the system equations takes the form of the augmented system defined in (7)–(9).

Thus from Theorem 1 and [4], we obtain the near optimal linear output feedback control

$$\mathbf{u} = -(R + D_0'D_0)^{-1}(D_0'C_0 + B_0'N)\hat{\mathbf{x}}_r, \quad t \in [t_0, t_f] \qquad (39)$$

where $N(t)$ is the solution of the matrix Riccati equation

$$\begin{aligned} \dot{N} = &-N(A_0 - B_0(R + D_0'D_0)^{-1}D_0'C_0) \\ &-(A_0 - B_0(R + D_0'D_0)^{-1}D_0'C_0)'N \\ &+NB_0(R + D_0'D_0)^{-1}B_0'N \\ &-(C_0'C_0 - C_0'D_0(R + D_0'D_0)^{-1}D_0'C_0) \end{aligned} \qquad (40)$$

with terminal condition $N(t_f) = T_1$, and $\hat{\mathbf{x}}_r$ is the optimal observer given by

$$\begin{aligned} \dot{\hat{\mathbf{x}}}_r = {} & A_0(t)\hat{\mathbf{x}}_r + B_0(t)\mathbf{u}(t) \\ & + K(t)[\mathbf{y}(t) - C_0(t)\hat{\mathbf{x}}_r(t) - D_0(t)\mathbf{u}(t)], \quad t \in [t_0, t_f] \end{aligned} \qquad (41)$$

with initial condition $\hat{\mathbf{x}}_r(t_0) = E[\mathbf{x}(t_0)] = \bar{\mathbf{x}}_0$, where

$$K(t) = [\Lambda(t)C_0'(t) + G_1(t)V_{12}(t)]V_2^{-1}(t), \quad t \in [t_0, t_f] \qquad (42)$$

and $\Lambda(t)$ is the solution of the matrix Riccati equation

$$\begin{aligned} \dot{\Lambda} = {} & (A_0 - G_1V_{12}V_2^{-1}C_0)\Lambda + \Lambda(A_0 - G_1V_{12}V_2^{-1}C_0)' \\ & - \Lambda C_0'V_2^{-1}C_0\Lambda + G_1V_1G_1' - G_1V_{12}V_2^{-1}V_{12}'G_1' \end{aligned} \qquad (43)$$

with initial condition $\Lambda(t_0) = P_{x_0} = E\{(\mathbf{x}_0 - \bar{\mathbf{x}}_0)(\mathbf{x}_0 - \bar{\mathbf{x}}_0)'\}$.

4. *Realization of the control algorithm*

A realization of the near optimal control algorithm is given in terms of the matrices of the original system defined in (1)–(5). Let

$$N = \begin{pmatrix} N_1 & N_2 \\ N_2' & N_3 \end{pmatrix}, \qquad \Lambda = \begin{pmatrix} \Lambda_1 & \Lambda_2 \\ \Lambda_2' & \Lambda_3 \end{pmatrix}$$

$$\begin{aligned} \tilde{A}_0 &= \tilde{A}_{11} - \tilde{A}_{12}\tilde{A}_{22}^{-1}\tilde{A}_{21}, & \tilde{B}_0 &= \tilde{B}_1 - \tilde{A}_{12}\tilde{A}_{22}^{-1}\tilde{B}_2, \\ \tilde{G}_0 &= \tilde{G}_1 - \tilde{A}_{12}\tilde{A}_{22}^{-1}\tilde{G}_2 & \tilde{C}_0 &= \tilde{C}_1 - \tilde{C}_2\tilde{A}_{22}^{-1}\tilde{A}_{21}, \\ \tilde{D}_0 &= -\tilde{C}_2\tilde{A}_{22}^{-1}\tilde{B}_2, & \tilde{E}_0 &= -\tilde{C}_2\tilde{A}_{22}^{-1}\tilde{G}_2, \\ \psi_1 &= \Lambda_1\tilde{C}_0' + \Lambda_2\tilde{E}_0', & \psi_2 &= \Lambda_2'\tilde{C}_0' + \Lambda_3\tilde{E}_0'. \end{aligned}$$

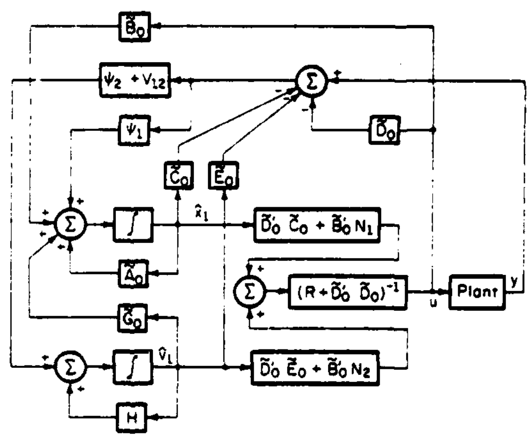

FIG. 1.

Then the near optimal linear control is given by

$$\mathbf{u} = (R + \tilde{D}_0'\tilde{D}_0)^{-1}[(\tilde{D}_0'\tilde{C}_0 + \tilde{B}_0'N_1)\hat{\mathbf{x}}_1 + (\tilde{D}_0'\tilde{E}_0 + \tilde{B}_0'N_2)\hat{\mathbf{v}}_1] \qquad (44)$$

where $\hat{\mathbf{x}}_1$ and $\hat{\mathbf{v}}_1$ are given by the filters

$$\dot{\hat{\mathbf{x}}}_1 = \tilde{A}_0\hat{\mathbf{x}}_1 + \tilde{G}_0\hat{\mathbf{v}}_1 + \tilde{B}_0\mathbf{u} + \psi_1 V_2^{-1}(\mathbf{y} - \tilde{c}_0\hat{\mathbf{x}}_1 - \tilde{E}_0\hat{\mathbf{v}}_1 - \tilde{D}_0\mathbf{u}), \qquad \hat{\mathbf{x}}_1(t_0) = \bar{\mathbf{x}}_{10} \qquad (45)$$

$$\dot{\hat{\mathbf{v}}}_1 = H\hat{\mathbf{v}}_1 + (\psi_2 + V_{12})V_2^{-1}(\mathbf{y} - \tilde{c}_0\hat{\mathbf{x}}_1 - \tilde{E}_0\mathbf{v}_1 - \tilde{D}_0\mathbf{u}), \qquad \hat{\mathbf{v}}_1(t_0) = \bar{\mathbf{v}}_{10}. \qquad (46)$$

5. *Example*

A familiar example in singular perturbation literature is the speed control of a dc motor. We suppose that the motor is disturbed by a stochastically varying torque operating on its shaft. For a typical dc motor characteristics [10, pp. 12–14] the state equations are

$$\dot{x}_1 = (K/J)x_2 + (1/J)w_1 \qquad (47)$$

$$\mu\dot{x}_2 = -(C/RT_m)x_1 - (1/T_m)x_2 + (1/RT_m)u \qquad (48)$$

where x_1, x_2, and u are speed, current, and voltage deviations from their respective nominal values 712 rad/s, 0.75 A, 27 V, and w_1 is the disturbing torque. The motor constants are $R = 16\Omega$, $C = 0.02$ V.S/rad, $J = 10^{-6}$ Kg·m². The small parameter $\mu = T_e/T_m$ where

$$T_e = \text{electrical time constant} = L/R$$

and

$$T_m = \text{mechanical time constant} = JR/CK.$$

Scaling the time in (47) and (48) so that its unit is ms, our stochastic problem is given by

$$\dot{x}_1 = 20x_2 + 10^3 w_1 \qquad (49)$$

$$\mu\dot{x}_2 = -3\,125 \times 10^{-5}x_1 - 25 \times 10^{-3}x_2 + 1.5625 \times 10^{-3}u \qquad (50)$$

$$y = \begin{bmatrix} 1 \\ 0 \end{bmatrix} x_1 + \begin{bmatrix} 0 \\ 80 \end{bmatrix} x_2 + w_2 \qquad (51)$$

$$J = \frac{1}{2} \lim_{\substack{t_f \to \infty \\ t_0 \to -\infty}} \frac{1}{(t_f - t_0)} E\left\{\int_{t_0}^{t_f} (x_1^2 + 6400x_2^2 + 25u^2)\,dt\right\} \qquad (52)$$

TABLE 1

μ	J_{opt}	$\Delta J = J - J_{opt}$	$\frac{\Delta J}{J_{opt}} \times 100$
10^{-1}	19.48375	0.432824	2.22146
10^{-2}	14.58385	0.02166	0.14852
10^{-3}	13.8648	0.0017	0.01226
10^{-4}	13.787885	0.000119	0.00086

where w_1 and w_2 are uncorrelated zero mena white Gaussian noise processes with

$$V_1 = 4 \times 10^{-4}, \quad V_2 = \begin{bmatrix} 49 & 0 \\ 0 & 0.36 \end{bmatrix}.$$

This is a realistic example of the special case of Remark 1, and is used to illustrate Theorem 1. We compare the performance index J resulting from applying Theorem 1 with the optimal performance index J_{opt} resulting from applying the exact control for different values of μ. The results are shown in Table 1.

6. *Conclusions*

We have applied the singular perturbation theory to the stochastic control for Linear–Quadratic–Gaussian problems for systems with fast and slow modes. The input noise has been modeled as colored noise and the observation noise as white noise. Our treatment incorporates another important problem in which the input noise is white but the fast dynamic equations are noise free. In the resulting control algorithm, the output measurement $y(t)$ is used as an input to a slow filter only. No filtering of fast variables is required. This permitted the modeling of the observation noise as white noise. As an illustration, in determining the required speed of response of the measuring instruments only slow variables are of importance. With respect to such low pass instruments, it is justified to model the observation noise as white noise. The assumption of colored input noise and white observation noise is thus consistent with the separation of time scales in singularly perturbed systems.

The main result of this paper (Theorem 1) is that the optimal solution of the stochastic control problem can be approximated by the optimal solution of the reduced problem. This leads to saving in both on-line and off-line computations. Moreover, the Riccati equations that are solved are better conditioned than the equations of the original problem. It is significant that the approximate control does not require the knowledge of the value of the singular perturbation parameter μ. Hence this control algorithm is applicable to systems where μ represents small uncertain parameters. The approach in this paper differs from [2] in that it represents a direct generalization of the deterministic problem [5] and preserves its form. It is important to notice that Theorem 1 cannot be obtained as a special case of Theorem 2 of [2] unless an additional assumption is made that the fast variables are of little interest, that is

$$\tilde{c}_2 = o(\mu^{1/2}) \quad \text{and} \quad \hat{T}_2 = o(\mu^{1/2}).$$

Another advantage of this approach is that no familiarity with the singular perturbation literature is required. Only the standard facts of the Linear–Quadratic–Gaussian problems are needed.

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SECTION 8

MULTIMODELING OF LARGE SCALE SYSTEMS

Control Strategies for Decision Makers Using Different Models of the Same System

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Abstract—Situations in which strategies of various decision makers are designed using different models of the same system are a characteristic of large-scale systems practice. For interconnected systems with slow and fast dynamics we develop a design algorithm which takes into account such multimodel situations. Conditions for the validity of this approximate design are formulated and illustrated by a power system example.

I. Introduction

IT IS COMMON in systems with several decision makers that the decision makers assume different simplified models of the same system. As a consequence, even if decision makers have the same overall goal, their individual objective functionals may have different analytical expressions. Thus, large scale system problems should, in general, be characterized by multiple decision makers having different models, different information sets, and different objective functionals.

This multimodel situation is illustrated by a multi-area power system. The decision maker in an area employs a detailed model of his area only and a "dynamic equivalent" of the remainder of the system. Other decision makers behave similarly in their own areas. Thus, the same power system appears in different forms to different decision makers. Present power system practice suggests that even if the decision makers were given a complete model of the system. they would *still use different simplified models* to match their individual needs.

In this paper singular perturbations [1], [2] are employed to capture the multimodel nature of interconnected systems with slow and fast dynamics. We consider systems strongly coupled through their slow parts and weakly coupled through their fast parts. In [3] such systems have been treated from a periodic coordination point of view. For this class of systems a new multiparameter perturbation method is developed and used in the design of regulators and Pareto strategies. It is illustrated by a load-frequency control problem for a two-area power system.

II. A Multimodel Representation

A linear system consisting of strongly coupled slow subsystems and weakly coupled fast subsystems is modeled by

$$\dot{x} = A_o x + \sum_{j=1}^{N} A_{oj} z_j + \sum_{j=1}^{N} B_{oj} u_j, \qquad x(0) = x_o \tag{1a}$$

$$\mathcal{E}_i \dot{z}_i = A_{io} x + A_{ii} z_i + \sum_{j \neq i} \mathcal{E}_{ij} A_{ij} z_j + B_{ii} u_i, \qquad z_i(0) = z_{io} \tag{1b}$$

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where $\dim x = n_o$, $\dim z_i = n_i$, $\dim u_i = m_i$, $i = 1, \cdots, N$. The small singular perturbation parameters $\mathcal{E}_i > 0$, one per subsystem, represent time constants, inertias, masses etc., while the small regular perturbation parameters $\mathcal{E}_{ij}$, $i \neq j$, represent weak coupling between the subsystems. The states z_i are "fast" since their derivatives $\dot{z}_i$ are of order $1/\mathcal{E}_i$ which are large.

Linearized models of many real systems appear in this form. A well documented case is the load-frequency control of a two-area power system in which each area is represented by one steam plant. A model of such a system based on [4], [5] is given in Appendix A. It is apparent from this model that the time constants of the speed governor and the turbine are much smaller than the time constants of the system inertia and the integral control. Hence, we select the ratios of small versus large time constants as the singular perturbation parameters $\mathcal{E}_1$ and $\mathcal{E}_2$,

$$\mathcal{E}_i = \frac{\max(T_{Gi}, T_{ti})}{T_i}. \tag{2}$$

We then substitute T_{Gi} and T_{ti} in terms of $\mathcal{E}_i$ and identify

$$x' = (v_1, v_2, \Delta f_1, \Delta f_2, \Delta p_{12}) \tag{3a}$$

as the slow states, and

$$z_i' = (\Delta p_{Gi}, \Delta a_i), \qquad i = 1, 2 \tag{3b}$$

as the fast states. Then, taking into account that the controls act upon speed changers, $u_i = \Delta p_{c_i}$, we obtain the model in the form of (1), where the subsystems are uncoupled, $\mathcal{E}_{ij} = 0$.

Let us now assume that the decision maker of the kth subsystem neglects the weak coupling parameters and the fast dynamics of all other subsystems, but retains the exact model of his own subsystem. In the model (1) this simplification is equivalent to the assumption that $\mathcal{E}_j = 0$, $j \neq k$, and $\mathcal{E}_{ij} = 0$, that is to

$$\dot{x}_k = A_o x_k + \sum_{j=1}^{N} A_{oj} z_j + \sum_{j=1}^{N} B_{oj} u_j \tag{4a}$$

$$\mathcal{E}_k \dot{z}_k = A_{ko} x_k + A_{kk} z_k + B_{kk} u_k \tag{4b}$$

$$0 = A_{io} x_k + A_{ii} z_i + B_{ii} u_i, \qquad i \neq k, i = 1, \cdots, N.$$

Now, if A_{ii} are nonsingular, the substitution of

$$z_i = -A_{ii}^{-1}(A_{io} x_k + B_{ii} u_i), \qquad i \neq k, i = 1, \cdots, N \tag{5}$$

into (4) results in the kth simplified model

$$\dot{x}_k = A_k x_k + A_{ok} z_k + B_{ok} u_k + \sum_{j \neq k} B_{kj} u_j \tag{6a}$$

$$\mathcal{E}_k \dot{z}_k = A_{ko} x_k + A_{kk} z_k + B_{kk} u_k \tag{6b}$$

where

$$A_k = A_o - \sum_{j \neq k} A_{oj} A_{jj}^{-1} A_{jo}, \quad B_{kj} = B_{oj} - A_{oj} A_{jj}^{-1} B_{jj}.$$

We point out that this "kth model simplification" is achieved by the "kth parameter perturbation," that is, when the only parameter assumed to be different from zero is $\mathcal{E}_k$. In our power system example the first perturbation ($k = 1$) means that the decision maker of the Area 1 neglects the small time constants of the Area 2, and the second perturbation ($k = 2$) means that the decision maker of the Area 2 neglects the small time constants of the Area 1. If there were interactions among the fast subsystems, they would have been neglected by both decision makers.

III. The Design Problem

In the above presentation we have viewed the model (6) as a result of an intentional model simplification, that is as if the original model (1) had been available to all decision makers. We now make a more realistic assumption that

each decision maker knows only his simplified model (6). (7)

The earlier interpretation remains valid as a description of the relationship between the original model (1) and its simplifications (6), but need not imply the decision maker's knowledge of the original model.

The main purpose of this paper is to analyze the impact of such multimodel assumptions on the design of control strategies. We consider that the kth decision maker will base the design of his control strategy on the kth model (6) to meet his set of design specifications. To be specific let the design specifications of the kth decision maker be expressed in terms of a cost functional $J_k = J_k(x, z_k, u_k)$. The cost functional J_i of the ith decision maker is known to the kth decision maker in the form $\bar{J}_i(x, u_i)$ since the z_i variable does not appear in his model (6). For the well-posedness of this design problem it is assumed that $\bar{J}_i(x, u_i)$ is consistent with $J_i(x, z_i, u_i)$ in the sense that it can be obtained from J_i using (5). The kth decision maker problem is characterized by (6) and the cost functionals

$$J_k = J_k(x, z_k, u_k), \quad \bar{J}_i = \bar{J}_i(x, u_i), \qquad i \neq k, i = 1, \cdots, N. \tag{8}$$

It will be viewed as a perturbation of an original problem characterized by (1) and the cost functionals

$$J_k = J_k(x, z_k, u_k), \qquad k = 1, \cdots, N. \tag{9}$$

This original problem will be helpful in the analysis of the impact of the multimodel situation on the design of Pareto optimal strategies.

Motivated by the single-parameter singular perturbation approach, we propose that each decision maker will use the two-time-scale design method [1]. He would then have to solve two separate subproblems for the fast and slow subsystems of (6).

In the fast time scale $\tau_k = t/\mathcal{E}_k$ he would have to design a fast control u_{kf} for the fast subsystem

$$\frac{dz_{kf}}{d\tau_k}=A_{kk}z_{kf}+B_{kk}u_{kf} \tag{10}$$

subject to the initial condition

$$z_{kf}(0)=z_{ko}+A_{kk}^{-1}(A_{ko}x_o+B_{kk}u_{ks}(0)). \tag{11}$$

Since the fast subsystem (10) is completely uncoupled from other states and controls, the design of u_{kf} can be approached by each decision maker as a separate state regulator problem.

In the slow time scale, the slow subproblem of the kth decision maker is obtained by setting $\mathcal{E}_k=0$ in (6) and using (5) to eliminate z_k from (6) and from the cost functional J_k. The relationship between the simplified model (6) and the original model (1) is such that setting $\mathcal{E}_k=0$ in (6) is equivalent to neglecting all the perturbation parameters in (1). Together with the assumption that $\bar{J}_i(x,u_i)$ is consistent with $J_i(x,z_i,u_i)$, this implies the existence of a common slow problem for all decision makers.

The reduced order model for the slow state x_s,

$$\dot{x}_s=A_sx_s+\sum_{k=1}^{N}B_{ks}u_{ks}, \qquad x_s(0)=x_o \tag{12}$$

where

$$A_s=A_o-\sum_{i=1}^{N}A_{oi}A_{ii}^{-1}A_{io}, \qquad B_{ks}=B_{ok}-A_{ok}A_{kk}^{-1}B_{kk}$$

involves all the slow controls u_{ks} and thus the slow problem has to be solved as a problem with multiple decision makers. To summarize, the design problem is approximately decomposed into N fast subsystem regulator problems and a slow game type problem.

Let us assume for the moment that all these subproblems have been solved and that as a result we know the feedback matrices in

$$u_{ks}=G_{ks}x_s, \quad u_{kf}=G_{kf}z_{kf}, \qquad k=1,\cdots,N. \tag{13}$$

According to the two-time scale method [1] the control law of the kth decision maker using the feedback matrices G_{ks}, G_{kf} will be composed as follows

$$u_k=\left[(I+G_{kf}A_{kk}^{-1}B_{kk})G_{ks}+G_{kf}A_{kk}^{-1}A_{ko}\right]x+G_{kf}z_k. \tag{14}$$

At this point, this control law is an ad hoc transplant of an earlier state regulator result into the new multimodel environment. Our task is now to study the properties of the actual system (1) controlled by the control law (14). For this purpose we analyze the relationship between the response $x(t)$, $z_k(t)$ of the actual system and the response $x_s(t)$, $z_{kf}(t)$ of the designed subsystems.

IV. Multiparameter Perturbations

When the proposed control law (14) is applied to the actual system (1) the resulting feedback system is

$$\dot{x}=\left[A_o+\sum_{i=1}^{2}\left\{B_{oi}(I+G_{if}A_{ii}^{-1}B_{ii})G_{is}+B_{oi}G_{if}A_{ii}^{-1}A_{io}\right\}\right]x+\sum_{i=1}^{2}(A_{oi}+B_{oi}G_{if})z_i \tag{15a}$$

$$\mathcal{E}_1\dot{z}_1=(A_{11}+B_{11}G_{1f})A_{11}^{-1}(A_{1o}+B_{11}G_{1s})x+(A_{11}+B_{11}G_{1f})z_1+\mathcal{E}_3A_{12}z_2 \tag{15b}$$

$$\mathcal{E}_2\dot{z}_2=(A_{22}+B_{22}G_{2f})A_{22}^{-1}(A_{2o}+B_{22}G_{2s})x+\mathcal{E}_4A_{21}z_1+(A_{22}+B_{22}G_{2f})z_2 \tag{15c}$$

where, without loss of generality we consider the case of only two decision makers ($k=1,2$) and denote $\mathcal{E}_{12}=\mathcal{E}_3$, $\mathcal{E}_{21}=\mathcal{E}_4$. In this system there are four perturbation parameters which are now ordered as components of a vector $\mathcal{E}$ in a set H of R^4. The coupling parameters $\mathcal{E}_3$, $\mathcal{E}_4$ can be positive, negative or zero, while the singular perturbation parameters $\mathcal{E}_1$, $\mathcal{E}_2$ are strictly positive. Systems of the type (15) have been investigated in [6] under the additional assumption that in the limit, as the norm $\|\mathcal{E}\|\to 0$, we have either $\mathcal{E}_1/\mathcal{E}_2\to 0$ or $\mathcal{E}_2/\mathcal{E}_1\to 0$. Since, under this assumption, one of the fast subsystems is much faster than the other, this problem is treated by two nested single parameter perturbations and is referred to as the multitime scale problem. Therefore, if it is known that $\mathcal{E}_1\ll\mathcal{E}_2$ or $\mathcal{E}_2\ll\mathcal{E}_1$ then the existing multi-time scale results can be used to analyze (15). We now consider the new problem when $\mathcal{E}_1$ and $\mathcal{E}_2$ are of the same order of magnitude, that is when their ratio is bounded by some positive constants m and M,

$$m<\frac{\mathcal{E}_2}{\mathcal{E}_1}<M. \tag{16}$$

Thus, the set H to which we restrict the possible values of $\mathcal{E}$ is a cylinder in R^4 whose base is a conical sector in R^2. In contrast to the multitime scale problem we call this case the multiparameter problem. Our power system example is an illustration of this new problem since the small parameters of Area 1 are of the same order as those of Area 2. In many other situations the subsystems have similar speeds and do not allow the multitime scale assumption.

We are now interested in predicting the behavior of the actual system (15) for all small values of $\mathcal{E}$ in H. We base this prediction on our knowledge of the slow response $x_s(t)$ and the fast response $z_{1f}(t/\mathcal{E}_1)$, $z_{2f}(t/\mathcal{E}_2)$ of the subproblems in the preceding section.

Theorem 1: If $u_{1f}=G_{1f}z_{1f}$ and $u_{2f}=G_{2f}z_{2f}$ are designed to stabilize the fast subsystems (10), that is if

$$\mathrm{Re}\,\lambda(A_{kk}+B_{kk}G_{kf})<0, \qquad k=1,2; \tag{17}$$

then, for every finite $T>0$, there exists a positive scalar σ such that

$$x(t)=x_s(t)+O(\|\mathcal{E}\|) \tag{18a}$$

$$z_1(t)=-A_{11}^{-1}(A_{1o}+B_{11}G_{1s})x_s(t)+z_{1f}(t/\mathcal{E}_1)+O(\|\mathcal{E}\|) \tag{18b}$$

$$z_2(t) = -A_{22}^{-1}(A_{2o}+B_{22}G_{2s})x_s + z_{2f}(t/\mathcal{E}_2) + O(\|\mathcal{E}\|) \tag{18c}$$

hold for all $0<t<T$ and all $\mathcal{E}\in H$, $0<\|\mathcal{E}\|<\sigma$. If in addition $u_{1s}=G_{1s}x_s$ and $u_{2s}=G_{2s}x_s$ stabilize the slow subsystem (12), that is if

$$\mathrm{Re}\lambda\left(A_s+\sum_{k=1}^{2}B_{ks}G_{ks}\right)<0 \tag{19}$$

then (18) hold for all $t\in[0,\infty)$.

Remark: In other words this theorem claims that on the respective intervals of t the expressions (18) provide a uniform approximation of $x(t)$, $z_1(t)$, and $z_2(t)$ *as* $\mathcal{E}\to 0$ *along any trajectory in* H.

Proof: We first transform (15) into the separate slow and fast parts

$$\dot{y} = E_o y \tag{20a}$$

$$\mathcal{E}_1\dot{v}_1 = E_1 v_1 + \left[\mathcal{E}_3 A_{12} + \mathcal{E}_1 L_1(A_{o2}+B_{o2}G_{2f})\right]v_2 \tag{20b}$$

$$\mathcal{E}_2\dot{v}_2 = \left[\mathcal{E}_4 A_{21} + \mathcal{E}_2 L_2(A_{o1}+B_{o1}G_{1f})\right]v_1 + E_2 v_2 \tag{20c}$$

where

$$E_o = A_o + \sum_{i=1}^{2}\left\{B_{oi}(I+G_{if}A_{ii}^{-1}B_{ii})G_{is} + B_{oi}G_{if}A_{ii}^{-1}A_{io} - (A_{oi}+B_{oi}G_{if})L_i\right\},$$

$$E_k = (A_{kk}+B_{kk}G_{kf}) + \mathcal{E}_k L_k(A_{ok}+B_{ok}G_{kf}), \qquad k=1,2.$$

The transformation used is

$$\begin{bmatrix} y \\ v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} I_o - \sum_{i=1}^{2}\mathcal{E}_i M_i L_i & -\mathcal{E}_1 M_1 & -\mathcal{E}_2 M_2 \\ L_1 & I_1 & 0 \\ L_2 & 0 & I_2 \end{bmatrix}\begin{bmatrix} x \\ z_1 \\ z_2 \end{bmatrix} \tag{21}$$

where I_o, I_1, and I_2 are the identity matrices of the appropriate dimensions and L_1, L_2, M_1, and M_2 satisfy the matrix algebraic equations

$$P_1 \equiv (A_{11}+B_{11}G_{1f})L_1 - (A_{11}+B_{11}G_{1f})A_{11}^{-1}(A_{1o}+B_{11}G_{1s}) - \mathcal{E}_1 L_1 E_o + \mathcal{E}_3 A_{12} L_2 = 0 \tag{22a}$$

$$P_2 \equiv (A_{22}+B_{22}G_{2f})L_2 - (A_{22}+B_{22}G_{2f})A_{22}^{-1}(A_{2o}+B_{22}G_{2s}) - \mathcal{E}_2 L_2 E_o + \mathcal{E}_4 A_{21} L_1 = 0 \tag{22b}$$

$$P_3 \equiv M_1 E_1 + \mathcal{E}_2 M_2 L_2(A_{o1}+B_{o1}G_{1f}) - (A_{o1}+B_{o1}G_{1f}) - \mathcal{E}_1 E_o M_1 + \mathcal{E}_4 M_2 A_{21} = 0 \tag{22c}$$

$$P_4 \equiv M_2 E_2 + \mathcal{E}_1 M_1 L_1(A_{o2}+B_{o2}G_{2f}) - (A_{o2}+B_{o2}G_{2f}) - \mathcal{E}_2 E_o M_2 + \mathcal{E}_3 M_1 A_{12} = 0. \tag{22d}$$

When bounded L_1, L_2, M_1, M_2 exist the transformation (21) is obviously nonsingular for all $\mathcal{E}$ in a sphere around $\mathcal{E}=0$. The existence and differentiability of L_1, L_2, M_1, M_2 with respect to $\mathcal{E}$ is established as follows. First note that in view of (17) the unique solution of (22) at $\mathcal{E}=0$ is

$$L_k(0) = A_{kk}^{-1}(A_{ko}+B_{kk}G_{ks}), \qquad k=1,2 \tag{23a}$$

$$M_k(0) = (A_{ok}+B_{ok}G_{kf})(A_{kk}+B_{kk}G_{kf})^{-1}, \qquad k=1,2. \tag{23b}$$

Then consider the operator $P(L_1,L_2,M_1,M_2,\mathcal{E})$ whose components are P_1, P_2, P_3, P_4. This operator is analytic in all of its arguments. At $\mathcal{E}=0$ its partial derivatives, with respect to L_1, M_1 and L_2, M_2 are $(A_{11}+B_{11}G_{1f})$ and $(A_{22}+B_{22}G_{2f})$, respectively. They are invertible because of (17). Thus, by the implicit function theorem, L_1, L_2, M_1, and M_2 are analytic in $\mathcal{E}$ at $\mathcal{E}=0$. Using this result the matrices and the initial conditions of the transformed system (20) are uniformly approximated by the matrices and initial conditions of the subproblems (10), (12) in the preceding section, that is,

$$\dot{y} = \left[A_s + B_{1s}G_{1s} + B_{2s}G_{2s} + O(\|\mathcal{E}\|)\right]y, \qquad y(0) = x_o + O(\|\mathcal{E}\|) \tag{24a}$$

$$\mathcal{E}_1\dot{v}_1 = \left[(A_{11}+B_{11}G_{1f}) + O(\|\mathcal{E}\|)\right]v_1 + O(\|\mathcal{E}\|)v_2, \qquad v_1(0) = z_{1o} + A_{11}^{-1}(A_{1o}+B_{11}G_{1s})x_o + O(\|\mathcal{E}\|) \tag{24b}$$

$$\mathcal{E}_2\dot{v}_2 = O(\|\mathcal{E}\|)v_1 + \left[(A_{22}+B_{22}G_{2f}) + O(\|\mathcal{E}\|)\right]v_2, \qquad v_2(0) = z_{2o} + A_{22}^{-1}(A_{2o}+B_{22}G_{2s})x_o + O(\|\mathcal{E}\|). \tag{24c}$$

The uniform convergence $y(t)\to x_s(t)$ as $\|\mathcal{E}\|\to 0$ immediately follows from the continuous dependence of (24a) on its right hand side and the initial conditions. To prove the convergence of the fast variables we introduce a joint fast time scale $\tau = t/\sqrt{\mathcal{E}_1\mathcal{E}_2}$ and obtain

$$\begin{bmatrix} \dfrac{d\tilde{v}_1}{d\tau} \\[2mm] \dfrac{d\tilde{v}_2}{d\tau} \end{bmatrix} = \begin{bmatrix} \sqrt{\dfrac{\mathcal{E}_2}{\mathcal{E}_1}} & 0 \\ 0 & \sqrt{\dfrac{\mathcal{E}_1}{\mathcal{E}_2}} \end{bmatrix} \cdot \begin{bmatrix} (A_{11}+B_{11}G_{1f}) + O(\|\mathcal{E}\|) & O(\|\mathcal{E}\|) \\ O(\|\mathcal{E}\|) & (A_{22}+B_{22}G_{2f}) + O(\|\mathcal{E}\|) \end{bmatrix}\begin{bmatrix} \tilde{v}_1(\tau) \\ \tilde{v}_2(\tau) \end{bmatrix} \tag{25}$$

where $v_1(t)=\tilde{v}_1(\tau)$, $v_2(t)=\tilde{v}_2(\tau)$. The limit $\|\mathcal{E}\|\to 0$ expands every finite t-interval to infinity in the τ-scale, a characteristic of the "stretched" time scales in most singular perturbation problems [6]. For infinite intervals $\tau\in[0,\infty)$ the stability property (17) and the bounds (16) on $\mathcal{E}_1/\mathcal{E}_2$, $\mathcal{E}_2/\mathcal{E}_1$, guarantee the uniform convergence $\tilde{v}_1(\tau)\to\tilde{v}_{10}(\tau)$, $\tilde{v}_2(\tau)\to\tilde{v}_{20}(\tau)$, where $\tilde{v}_{10}(\tau)$, $\tilde{v}_{20}(\tau)$ are the solutions of

$$\frac{d\tilde{v}_{10}}{d\tau}=\sqrt{\frac{\mathcal{E}_2}{\mathcal{E}_1}}\,(A_{11}+B_{11}G_{1f})\tilde{v}_{10}(\tau),$$

$$\tilde{v}_{10}(0)=z_{1o}+A_{11}^{-1}(A_{1o}+B_{11}G_{1s})x_o, \tag{26a}$$

$$\frac{d\tilde{v}_{20}}{d\tau}=\sqrt{\frac{\mathcal{E}_1}{\mathcal{E}_2}}\,(A_{22}+B_{22}G_{2f})\tilde{v}_{20}(\tau),$$

$$\tilde{v}_{20}(0)=z_{2o}+A_{22}^{-1}(A_{2o}+B_{22}G_{2s})x_o. \tag{26b}$$

Now the nonsingular rescaling of τ into τ_1 and τ_2

$$\tau_1=\sqrt{\frac{\mathcal{E}_2}{\mathcal{E}_1}}\,\tau,\quad \tau_2=\sqrt{\frac{\mathcal{E}_1}{\mathcal{E}_2}}\,\tau$$

and the comparison with (10) and (11) shows that

$$\tilde{v}_{10}(\tau)=z_{1f}(\tau_1), \tag{27a}$$

$$\tilde{v}_{20}(\tau)=z_{2f}(\tau_2). \tag{27b}$$

Using the inverse transformation of (21)

$$\begin{bmatrix} x\\ z_1\\ z_2\end{bmatrix}=\begin{bmatrix} I_o & \mathcal{E}_1M_1 & \mathcal{E}_2M_2\\ -L_1 & I_1-\mathcal{E}_1L_1M_1 & -\mathcal{E}_2L_1M_2\\ -L_2 & -\mathcal{E}_1L_2M_1 & I_2-\mathcal{E}_2L_2M_2\end{bmatrix}\begin{bmatrix} y\\ v_1\\ v_2\end{bmatrix} \tag{28}$$

proves (18).

As a simple application of Theorem 1 consider the pole-placement design of the load-frequency control in Appendix B. Each area designs its controller matrix G_{kf} to place the fast eigenvalues. The slow eigenvalues are placed by the centralized design of G_{1s}, G_{2s}.

V. Pareto Optimal Strategy

We consider the situation in which decision makers decide on their strategies through mutual cooperation. The solution of such problems is found in the class of Pareto optimal strategies [7], the essence of which is that no variation from a Pareto optimal strategy can decrease the costs of both decision makers. Let each decision maker have a quadratic cost functional

$$J_k=\frac{1}{2}\int_o^\infty (y_k'y_k+u_k'R_ku_k)\,dt,\qquad R_k>0 \tag{29}$$

$$y_k=C_{ok}x+C_{kk}z_k. \tag{30}$$

A Pareto solution is a pair u_1, u_2 which minimizes

$$J=\gamma_1J_1+\gamma_2J_2,\quad 0<\gamma_k<1,\quad \gamma_1+\gamma_2=1 \tag{31}$$

for some γ_1 and γ_2. The optimal state regulator is a special case of this problem when the decision makers agree on a choice of γ_1 and γ_2 as weighting factors.

The specific form of the subproblems of Section III is as follows. *The slow subproblem* is characterized by (12) and the cost functional

$$J_s=\gamma_1J_{1s}+\gamma_2J_{2s} \tag{32}$$

where

$$J_{ks}=\frac{1}{2}\int_o^\infty (x_s'C_{ks}'C_{ks}x_s+2u_{ks}'D_{ks}'C_{ks}x_s+u_{ks}'R_{ks}u_{ks})\,dt,$$

$$C_{ks}=C_{ok}-C_{kk}A_{kk}^{-1}A_{ko},\quad D_{ks}=-C_{kk}A_{kk}^{-1}B_{kk},$$

$$R_{ks}=R_k+D_{ks}'D_{ks}. \tag{33}$$

Its solution is

$$u_{ks}=-R_{ks}^{-1}\left(D_{ks}'C_{ks}+\frac{1}{\gamma_k}B_{ks}'K_s\right)x_s \tag{34}$$

where K_s is the positive semidefinite stabilizing solution of the Riccati equation

$$K_s\tilde{A}_s+\tilde{A}_s'K_s+\sum_{i=1}^{2}\left[\frac{-1}{\gamma_i}K_sB_{is}R_{is}^{-1}B_{is}'K_s\right.$$
$$\left.+\gamma_iC_{is}'(I-D_{is}R_{is}^{-1}D_{is}')C_{is}\right]=0 \tag{35}$$

where

$$\tilde{A}_s=A_s-\sum_{i=1}^{2}B_{is}R_{is}^{-1}D_{is}'C_{is}.$$

Denoting $B_s=(B_{1s}\ B_{2s})$ and $C_s'=(C_{1s}',\ C_{2s}')$, a necessary and sufficient condition [1] for the existence and uniqueness of K_s is that

$$(A_s,B_s,C_s)\ \text{is stabilizable-detectable.} \tag{36}$$

This condition does not depend on the weighting factors γ_1, γ_2.

The *fast subproblem* (k) is characterized by (10) and the cost functional

$$J_{kf}=\frac{1}{2}\int_o^\infty (z_{kf}'C_{kk}'C_{kk}z_{kf}+u_{kf}'R_ku_{kf})\,d\tau_k. \tag{37}$$

Its solution is

$$u_{kf}=-R_k^{-1}B_{kk}'K_{kf}z_{kf} \tag{38}$$

where K_{kf} is the positive semidefinite stabilizing solution of the Riccati equation

$$K_{kf}A_{kk}+A_{kk}'K_{kf}+C_{kk}'C_{kk}-K_{kf}B_{kk}R_k^{-1}B_{kk}'K_{kf}=0, \tag{39}$$

which, as is well known, exists iff

the triples (A_{kk}, B_{kk}, C_{kk}) are stabilizable-detectable. (40)

The specific form of the control law (14) is now

$$u_k = -\left[\left(I - R_k^{-1}B'_{kk}K_{kf}A_{kk}^{-1}B_{kk}\right)R_{ks}^{-1}\left(D'_{ks}C_{ks} + \frac{1}{\gamma_k}B'_{ks}K_s\right) + R_k^{-1}B'_{kk}K_{kf}A_{kk}^{-1}A_{ko}\right]x - R_k^{-1}B'_{kk}K_{kf}z_k, \qquad k=1,2. \tag{41}$$

When u_1 and u_2 are applied to the actual system (1), we know from Theorem 1 that for sufficiently small $\mathcal{E}$ the resulting response will be close to the predicted one. In an optimization problem it is of interest to check whether the resulting values of the cost functionals will be near their optimal values. The optimal values J_1^* and J_2^* are obtained with the strategies u_1^*, u_2^* which optimize the costs for the actual system (1).

Theorem 2: Under conditions (36) and (40), the application of u_1 and u_2 of (41) to system (1) results in J_1 and J_2 satisfying the relations

$$\lim_{\mathcal{E}\to 0}(J_k - J_k^*) = 0, \qquad k=1,2 \tag{42}$$

where $\mathcal{E}\to 0$ is taken along any trajectory in H.

Proof: The optimal strategy for the actual system (1) and (31) is

$$u_k^* = -\frac{1}{\gamma_k}R_k^{-1}B_k'K\hat{x} \tag{43}$$

where K is the stabilizing solution of the Riccati equation

$$KA + A'K + Q - KSK = 0 \tag{44}$$

and

$$A = \begin{bmatrix} A_o & A_{o1} & A_{o2} \\ \dfrac{A_{1o}}{\mathcal{E}_1} & \dfrac{A_{11}}{\mathcal{E}_1} & \dfrac{\mathcal{E}_3 A_{12}}{\mathcal{E}_1} \\ \dfrac{A_{2o}}{\mathcal{E}_2} & \dfrac{\mathcal{E}_4 A_{21}}{\mathcal{E}_2} & \dfrac{A_{22}}{\mathcal{E}_2} \end{bmatrix}, \quad B_1 = \begin{bmatrix} B_{o1} \\ \dfrac{B_{11}}{\mathcal{E}_1} \\ 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} B_{o2} \\ 0 \\ \dfrac{B_{22}}{\mathcal{E}_2} \end{bmatrix},$$

$$Q = \gamma_1 Q_1 + \gamma_2 Q_2, \quad S = \frac{1}{\gamma_1}S_1 + \frac{1}{\gamma_2}S_2,$$

$$Q_1 = \begin{bmatrix} C'_{o1}C_{o1} & C'_{o1}C_{11} & 0 \\ C'_{11}C_{o1} & C'_{11}C_{11} & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad Q_2 = \begin{bmatrix} C'_{o2}C_{o2} & 0 & C'_{o2}C_{22} \\ 0 & 0 & 0 \\ C'_{22}C_{o2} & 0 & C'_{22}C_{22} \end{bmatrix},$$

$$S_k = B_k R_k^{-1} B_k', \quad \text{and} \quad \hat{x}' = (x' \;\; z_1' \;\; z_2').$$

To avoid unboundedness as $\mathcal{E}\to 0$ the solution K of (44) is sought in the form

$$K = \begin{bmatrix} K_{oo} & \mathcal{E}_1 K_{o1} & \mathcal{E}_2 K_{o2} \\ \mathcal{E}_1 K'_{o1} & \mathcal{E}_1 K_{11} & \sqrt{\mathcal{E}_1\mathcal{E}_2}\, K_{12} \\ \mathcal{E}_2 K'_{o2} & \sqrt{\mathcal{E}_1\mathcal{E}_2}\, K'_{12} & \mathcal{E}_2 K_{22} \end{bmatrix}. \tag{45}$$

In this form all the coefficients in (44) have well defined limits as $\mathcal{E}\to 0$ along any trajectory in H. In particular let $\alpha_{ij} = \lim(\mathcal{E}_i/\mathcal{E}_j)$, $i,j=1,2$, then the limiting process in (44) with (16) results in

$$\begin{aligned} &K_{oo}(0)A_o + K_{o1}(0)A_{1o} + K_{o2}(0)A_{2o} + A_o'K_{oo}(0) \\ &\quad + A'_{1o}K'_{o1}(0) + A'_{2o}K'_{o2}(0) + \gamma_1 C'_{o1}C_{o1} \\ &\quad + \gamma_2 C'_{o2}C_{o2} - \frac{1}{\gamma_1}K_{oo}(0)S_{o1}K_{oo}(0) \\ &\quad - \frac{1}{\gamma_2}K_{oo}(0)S_{o2}K_{oo}(0) - \frac{1}{\gamma_1}K_{o1}(0)\bar{S}'_{o1}K_{oo}(0) \\ &\quad - \frac{1}{\gamma_2}K_{o2}(0)\bar{S}'_{o2}K_{oo}(0) - \frac{1}{\gamma_1}K_{oo}(0)\bar{S}_{o1}K'_{o1}(0) \\ &\quad - \frac{1}{\gamma_1}K_{o1}(0)S_{11}K'_{o1}(0) - \frac{1}{\gamma_2}K_{oo}(0)\bar{S}_{o2}K'_{o2}(0) \\ &\quad - \frac{1}{\gamma_2}K_{o2}(0)S_{22}K'_{o2}(0) = 0 \end{aligned} \tag{46a}$$

$$\begin{aligned} &K_{oo}(0)A_{o1} + K_{o1}(0)A_{11} + A'_{1o}K_{11}(0) \\ &\quad + \sqrt{\alpha_{12}}\, A'_{2o}K'_{12}(0) + \gamma_1 C'_{o1}C_{11} \\ &\quad - \frac{1}{\gamma_1}K_{oo}(0)\bar{S}_{o1}K_{11}(0) - \frac{1}{\gamma_1}K_{o1}(0)S_{11}K_{11}(0) \\ &\quad - \frac{\sqrt{\alpha_{12}}}{\gamma_2}\left(K_{oo}(0)\bar{S}_{o2} + K_{o2}(0)S_{22}\right)K'_{12}(0) = 0 \end{aligned} \tag{46b}$$

$$\begin{aligned} &K_{oo}(0)A_{o2} + K_{o2}(0)A_{22} + \sqrt{\alpha_{21}}\, A'_{1o}K_{12}(0) \\ &\quad + A'_{2o}K_{22}(0) + \gamma_2 C'_{o2}C_{22} \\ &\quad - \frac{\sqrt{\alpha_{21}}}{\gamma_1}\left(K_{oo}(0)\bar{S}_{o1} + K_{o1}(0)S_{11}\right)K_{12}(0) \\ &\quad - \frac{1}{\gamma_2}K_{oo}(0)\bar{S}_{o2}K_{22}(0) \\ &\quad - \frac{1}{\gamma_2}K_{o2}(0)S_{22}K_{22}(0) = 0 \end{aligned} \tag{46c}$$

$$\begin{aligned} &K_{11}(0)A_{11} + A'_{11}K_{11}(0) + \gamma_1 C'_{11}C_{11} - \frac{1}{\gamma_1}K_{11}(0)S_{11}K_{11}(0) \\ &\quad - \frac{\alpha_{12}}{\gamma_2}K_{12}(0)S_{22}K'_{12}(0) = 0 \end{aligned} \tag{46d}$$

$$\begin{aligned} &\sqrt{\alpha_{12}}\, K_{12}(0)A_{22} + \sqrt{\alpha_{21}}\, A'_{11}K_{12}(0) - \frac{\sqrt{\alpha_{21}}}{\gamma_1}K_{11}(0)S_{11}K_{12}(0) \\ &\quad - \frac{\sqrt{\alpha_{12}}}{\gamma_2}K_{12}(0)S_{22}K_{22}(0) = 0 \end{aligned} \tag{46e}$$

$$\begin{aligned} &K_{22}(0)A_{22} + A'_{22}K_{22}(0) + \gamma_2 C'_{22}C_{22} - \frac{\alpha_{21}}{\gamma_1}K'_{12}(0)S_{11}K_{12}(0) \\ &\quad - \frac{1}{\gamma_2}K_{22}(0)S_{22}K_{22}(0) = 0 \end{aligned} \tag{46f}$$

where

$$S_{ok} = B_{ok} R_k^{-1} B'_{ok},\ \bar{S}_{ok} = B_{ok} R_k^{-1} B'_{kk},\ S_{kk} = B_{kk} R_k^{-1} B'_{kk}.$$

The unique positive semidefinite solution of (46d), (46e), (46f) is

$$K_{11}(0) = \gamma_1 K_{1f},\ K_{12}(0) = 0,\ K_{22}(0) = \gamma_2 K_{2f}. \tag{47}$$

Then (46b) and (46c) yield

$$K_{ok}(0) = K_{oo}(0)\hat{E}_k - \gamma_k \tilde{E}_k \tag{48}$$

where

$$\hat{E}_k = (\bar{S}_{ok} K_{kf} - A_{ok})(A_{kk} - S_{kk} K_{kf})^{-1},$$
$$\tilde{E}_k = (A'_{ko} K_{kf} + C'_{ok} C_{kk})(A_{kk} - S_{kk} K_{kf})^{-1}$$

and the substitution into (46a) results (after lengthy calculations) in

$$K_{oo}(0)\tilde{A}_s + \tilde{A}'_s K_{oo}(0) + \sum_{i=1}^{2}\Big[-\frac{1}{\gamma_i} K_{oo}(0) B_{is} R_{is}^{-1} B'_{is} K_{oo}(0) + \gamma_i C'_{is}(I - D_{is} R_{is}^{-1} D'_{is}) C_{is}\Big] = 0. \tag{49}$$

This equation in $K_{oo}(0)$ is identical to (35) in K_s for the slow subproblem. The uniqueness of the positive semidefinite stabilizing solution implies that

$$K_{oo}(0) = K_s. \tag{50}$$

Since the above solution of (46) does not depend on α_{12} and α_{21}, the limits $K_{ij}(0)$ of K_{ij} are uniquely defined as $\mathcal{E} \to 0$ along any trajectory in H. We now use this result to evaluate the limit of J_k^* as $\mathcal{E} \to 0$ in H.

$$J_k^* = \frac{1}{2}\hat{x}'_o M^{(k)} \hat{x}_o, \qquad k = 1,2 \tag{51}$$

where $M^{(k)}$ satisfies the Lyapunov equation

$$M^{(k)}(A - SK) + (A - SK)' M^{(k)} + Q_k + \frac{1}{\gamma_k^2} K S_k K = 0. \tag{52}$$

Assuming for $M^{(1)}$ and $M^{(2)}$ the form (45), we obtain, by an analogous argument, the limits

$$M_{11}^{(1)}(0) = K_{1f},\ M_{12}^{(1)}(0) = 0,\ M_{22}^{(1)}(0) = 0,$$
$$M_{01}^{(1)}(0) = M_{oo}^{(1)}(0)\hat{E}_1 - \tilde{E}_1,\ M_{o2}^{(1)}(0) = M_{oo}^{(1)}(0)\hat{E}_2 \tag{53}$$
$$M_{11}^{(2)}(0) = 0,\ M_{12}^{(2)}(0) = 0,\ M_{22}^{(2)}(0) = K_{2f},$$
$$M_{01}^{(2)}(0) = M_{oo}^{(2)}(0)\hat{E}_1,\ M_{o2}^{(2)}(0) = M_{oo}^{(2)}(0)\hat{E}_2 - \tilde{E}_2 \tag{54}$$

where $M_{oo}^{(k)}(0)$ satisfies the Lyapunov equation

$$M_{oo}^{(k)}(0)\hat{A}_s + \hat{A}'_s M_{oo}^{(k)}(0) + C'_{ks}(I - D_{ks} R_{ks}^{-1} D'_{ks}) C_{ks} + \frac{1}{\gamma_k^2} K_s B_{ks} R_{ks}^{-1} B'_{ks} K_s = 0 \tag{55}$$

with

$$\hat{A}_s = \tilde{A}_s - \sum_{i=1}^{2} \frac{1}{\gamma_i} B_{is} R_{is}^{-1} B'_{is} K_s.$$

To evaluate the actual cost J_k we express u_1 and u_2 of (41) as

$$u_k = -\frac{1}{\gamma_k} R_k^{-1} B'_k \begin{bmatrix} K_s & 0 & 0 \\ \mathcal{E}_1 K'_{1m} & \mathcal{E}_1 \gamma_1 K_{1f} & 0 \\ \mathcal{E}_2 K'_{2m} & 0 & \mathcal{E}_2 \gamma_2 K_{2f} \end{bmatrix} \hat{x}$$
$$= -\frac{1}{\gamma_k} R_k^{-1} B'_k L \hat{x} \tag{56}$$

where

$$K_{km} = K_s \hat{E}_k - \gamma_k \tilde{E}_k = K_{ko}(0).$$

When u_1 and u_2 are applied to (1), they result in

$$J_k = \frac{1}{2}\hat{x}'_o N^{(k)} \hat{x}_o, \qquad k = 1,2 \tag{57}$$

where $N^{(k)}$ satisfies the Lyapunov equation

$$N^{(k)}(A - SL) + (A - SL)' N^{(k)} + Q_k + \frac{1}{\gamma_k^2} L' S_k L = 0. \tag{58}$$

To calculate the loss of performance $(J_k - J_k^*)$ we subtract (52) from (58) and obtain the equation for $N^{(k)} - M^{(k)} = W^{(k)}$

$$W^{(k)}(A - SL) + (A - SL)' W^{(k)} + \frac{1}{\gamma_k^2} L' S_k L - \frac{1}{\gamma_k^2} K S_k K + M^{(k)} S(K - L) + (K - L)' S M^{(k)} = 0. \tag{59}$$

We again assume the form (45) for $W^{(k)}$. This allows the limit $\mathcal{E} \to 0$ to be taken in (59) along any trajectory in H. Knowing the stability of $\hat{A}_s$, $(A_{11} - S_{11} K_{1f})$ and $(A_{22} - S_{22} K_{2f})$ it can be shown that

$$\lim_{\mathcal{E} \to 0} W_{ij}^k = 0, \qquad \mathcal{E} \in H \tag{60}$$

which proves Theorem 2.

An interpretation of the proposed design is that a Pareto game played on the full system is replaced by a Pareto game played on the slow part of the system and two regulator problems for the fast subsystems. The slow Pareto game has the same weighting factors as the original game. The two regulator problems do not depend on the weighting factors. This means that each decision maker optimizes his own fast dynamics independently from the other decision maker. They need to agree only on the optimization of the slow dynamics.

A Pareto game problem for our power system example is solved in Appendix C.

VI. Conclusion

A characteristic of large scale systems with multiple decision makers is that the decision makers use different simplified models of the same system. Each decision maker models his own subsystem in detail and assumes a certain "equivalent" reduced order model of the rest of the system. In this paper an attempt is made to interpret this practical multimodel situation as a perturbation problem. In present formulation the two basic assumptions are that the fast subsystems are weakly coupled and that the model of the slow subsystem is common for all decision makers. The latter assumption is less realistic, but has greatly simplified the treatment. Our multiparameter perturbation analysis establishes sufficient conditions for the multimodel response to be close to the actual system response. This analysis serves as a basis for a decomposed design approach, as illustrated by a power system example.

Appendix A

We consider a power system consisting of two interconnected identical areas. Following [4], [5] the model of each area consists of the governor equation (A-1), the non-reheat steam turbine equation (A-2), and the power balance equation (A-3).

$$T_G\Delta\dot{a} = -\Delta a + \Delta P_c - \frac{1}{r}\Delta f \tag{A-1}$$

$$T_t\Delta\dot{P}_G = -\Delta P_G + \Delta a \tag{A-2}$$

$$T\Delta\dot{f} = -\Delta f + \frac{1}{D}(\Delta P_G - \Delta P_d - \Delta P_{tie}). \tag{A-3}$$

Taking $\Delta P_{12} = \Delta P_{tie_1}$, the tie-line equation is

$$\Delta\dot{P}_{12} = T_{12}(\Delta f_1 - \Delta f_2). \tag{A-4}$$

To fulfill all design requirements [4] the integral of the area control error (ACE) is incorporated into the state vector

$$v_k = \int (\mathrm{ACE})_k\,dt = \int (\Delta P_{tie_k} + b_{sk}\Delta f_k)\,dt. \tag{A-5}$$

The system variables entering (A-1)–(A-5) are: Δa = turbine valve position variation; ΔP_G = turbine output variation; Δf = frequency variation; ΔP_c = speed changer variation; ΔP_{12} = tie-line power flow variation. Typical numerical values of the system parameters are r = speed regulation = 0.25; T_G = governor time constant = 0.1; T_t = turbine time constant = 0.2; T = system inertia time constant = 20; T_{12} = synchronizing power flow coefficient = 32.7; D = 0.5, b_s = 4.5. All quantities are in per unit and time is in seconds. Assuming a constant load disturbance ΔP_d and redefining the state and control variables to be deviation from their steady state values, the disturbance inputs can be omitted from the equations. According to (2) we choose $\epsilon_1 = \epsilon_2 = 0.2/20 = 0.01$, that is $T_{G_k} = 10\epsilon_k$, $T_{tk} = 20\epsilon_k$. Then

$$A_0 = \begin{bmatrix} 0 & 0 & 4.5 & 0 & 1 \\ 0 & 0 & 0 & 4.5 & -1 \\ 0 & 0 & -0.05 & 0 & -0.1 \\ 0 & 0 & 0 & -0.05 & 0.1 \\ 0 & 0 & 32.7 & -32.7 & 0 \end{bmatrix},$$

$$A_{01} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0.1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad A_{02} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0.1 & 0 \\ 0 & 0 \end{bmatrix},$$

$$A_{10} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -0.4 & 0 & 0 \end{bmatrix}, \quad A_{20} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.4 & 0 \end{bmatrix}$$

$$A_{kk} = \begin{bmatrix} -0.05 & 0.05 \\ 0 & -0.1 \end{bmatrix}, \quad B_{0k} = 0, \quad B_{kk} = \begin{bmatrix} 0 \\ 0.1 \end{bmatrix}.$$

$$A_s = \begin{bmatrix} 0 & 0 & 4.5 & 0 & 1 \\ 0 & 0 & 0 & 4.5 & -1 \\ 0 & 0 & -.45 & 0 & -0.1 \\ 0 & 0 & 0 & -0.45 & 0.1 \\ 0 & 0 & 32.7 & -32.7 & 0 \end{bmatrix},$$

$$B_{1s} = \begin{bmatrix} 0 \\ 0 \\ 0.1 \\ 0 \\ 0 \end{bmatrix}, \quad B_{2s} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0.1 \\ 0 \end{bmatrix}.$$

Appendix B

In pole-placement design of the power system defined in Appendix A, we solve three subproblems, one for the slow subsystem and one for each fast subsystem. For the slow subsystem the eigenvalues are placed at

$$-0.2 \pm j.1, \quad -0.25 \pm j2.5, \quad -0.2 \tag{B-1}$$

with the feedback gains

$$G_{1s} = [-0.2486 \;\; 0.1375 \;\; -2.5 \;\; 3.0 \;\; -0.9268] \tag{B-2}$$

$$G_{2s} = [-0.0556 \;\; -0.0556 \;\; 0 \;\; 0.5 \;\; -0.9924]. \tag{B-3}$$

For convenience we assume that the desired eigenvalues of the two fast subproblems are the same. The kth fast subsystem eigenvalues are placed at

$$-8, \; -12 \tag{B-4}$$

with the feedback gain

$$G_{kf} = [-0.42 \;\; -0.5]. \tag{B-5}$$

Therefore, from (14), the feedback control is

$$u_1 = [-0.4773\ 0.264\ -8.48\ 5.76\ -1.78]x + [-0.42\ -0.5]z_1 \quad \text{(B-6)}$$

$$u_2 = [-0.1067\ -0.1067\ 0\ -2.72\ -1.91]x + [-0.42\ -0.5]z_2. \quad \text{(B-7)}$$

When u_1 and u_2 are applied to actual system (1) the resulting eigenvalues are

$$-12.964,\ -12.577,\ -7.05,\ -6.37,\ -0.252 \pm j2.634,\ -0.21 \pm j.099,\ -0.2095, \quad \text{(B-8)}$$

which are close to their desired locations,

$$(-12,\ -12,\ -8,\ -8,\ -0.25 \pm j2.5,\ -0.2 \pm j.1,\ -0.2).$$

Appendix C

Let the cost functionals J_1, J_2 of the area decision makers in the power system of Appendix A be defined by

$$Q_1 = \mathrm{diag}(1,1,1,1,1,1,1,0,0) \quad \text{(C-1)}$$

$$Q_2 = \mathrm{diag}(1,1,1,1,1,0,0,1,1) \quad \text{(C-2)}$$

$$R_1 = R_2 = 20. \quad \text{(C-3)}$$

Suppose that they agree on a Pareto game with weighting factors

$$\gamma_1 = \gamma_2 = 0.5. \quad \text{(C-4)}$$

The solution of the slow subsystem Pareto problem is

$$u_{1s} = [-0.3015\ 0\ -4.079\ 1.8219\ -0.0478]x_s \quad \text{(C-5)}$$

$$u_{2s} = [0\ -0.3015\ 1.8219\ -4.079\ -0.0478]x_s. \quad \text{(C-6)}$$

The kth fast subsystem is optimized by

$$u_{kf} = [-0.0162\ -0.0326]z_{kf}. \quad \text{(C-7)}$$

Then, according to (41), the control law is

$$u_1 = [-0.3162\ 0\ -4.473\ 1.911\ -0.05]x + [-0.0162\ -0.0326]z_1 \quad \text{(C-8)}$$

$$u_2 = [0\ -0.3162\ 1.911\ -4.473\ 0.05]x + [-0.0162\ -0.0326]z_2. \quad \text{(C-9)}$$

To evaluate the cost functionals using (57), we assume that the initial conditions are zero mean independent random vector with covariance matrix

$$E\{\hat{x}_o \hat{x}_o'\} = 10^{-4}\mathrm{diag}(1,1,.01,.01,1,1,1,1,1). \quad \text{(C-10)}$$

This choice is typical, given the physical meaning of the state variables [4]. Then the average value of the cost functionals are

$$E\{J_1\} = E\{J_2\} = 12.75 \times 10^{-4}. \quad \text{(C-11)}$$

If this Pareto problem had been solved for the actual system (1), the optimal solution would have been

$$u_1^* = [-0.3162\ 0\ -4.5806\ 1.8694\ 0.053]x + [-0.1079\ -0.0759]z_1 + [0.0378\ 0.0176]z_2 \quad \text{(C-12)}$$

$$u_2^* = [0\ -0.3162\ 1.8694\ -4.5806\ -0.053]x + [0.0378\ 0.0176]z_1 + [-0.1079\ -0.0759]z_2 \quad \text{(C-13)}$$

resulting in the average values

$$E\{J_1^*\} = E\{J_2^*\} = 11.7 \times 10^{-4}. \quad \text{(C-14)}$$

Thus, from (C-11) and (C-14), we find that each cost functional has a loss of nine percent, that is

$$\frac{E\{J_1\} - E\{J_1^*\}}{E\{J_1^*\}} = \frac{E\{J_2\} - E\{J_2^*\}}{E\{J_2^*\}} = 0.09. \quad \text{(C-15)}$$

In a more general case, when the two control areas are different, the application of the control law (41) may result in one of the decision makers benefiting instead of having a loss.

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D-STABILITY AND MULTI-PARAMETER SINGULAR PERTURBATION*

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Abstract. A new multi-parameter singular perturbation problem is formulated. Sufficient conditions for uniform asymptotic stability are derived, and asymptotic behavior of solution is investigated.

1. Introduction. Single parameter singular perturbations have been extensively used in analysis and control of dynamic systems [1]. Even if they possess several small parameters, electrical networks with parasitics and control systems with small time constants, masses, etc., are modeled as single parameter problems. This is done by expressing small parameters as known multiples of a particular parameter μ, such as $m = \alpha_1\mu$, $T = \alpha_2\mu$, where m is a small mass and T is a small time constant. A characteristic of this approach is that its results depend on the scaling coefficients α_i which are assumed to be known. In many cases of practical interest such an assumption cannot be justified. In multi-controller problems and differential game problems small parameters may represent different independent ways in which individual control agents simplify the model of the overall system, and therefore the relation between the small parameters must remain arbitrary [2]. It may be argued that a more realistic study of parasitics should also allow for the ignorance of the ratios of small parameters.

The purpose of this paper is to examine the vector singular perturbation problem when all the small parameters are of the same order of magnitude, but can have arbitrary bounded ratios. This problem is different from the multiple time scale problem [3], [4] when the parameters are of different orders of magnitude. We treat the uniform asymptotic stability and initial value problems for multi-parameter singular perturbations. In contrast to the boundary layer system stability requirement of the single parameter case [1], we employ a generalization of D-stability. Several tests are given delineating important classes of systems satisfying this condition.

2. Multiparameter perturbations. Linear systems with N singular perturbation parameters $\varepsilon_1, \cdots, \varepsilon_N$ have the general form

$$\dot{x} = A_0(t)x + \sum_{j=1}^{N} A_{0j}(t)z_j, \qquad x(t_0) = x_0, \tag{1a}$$

$$\varepsilon_i\dot{z}_i = A_{i0}(t)x + \sum_{j=1}^{N} A_{ij}(t)z_j, \qquad z_i(t_0) = z_{i0}, \tag{1b}$$

where $x \in R^{n_0}$, $z_i \in R^{n_i}$, that is the system dimension is $n = n_0 + \sum_{i=1}^{N} n_i$. The small positive scalars $\varepsilon_1, \cdots, \varepsilon_N$ represent time constants, inertias, masses and similar physical parameters [1]. They are ordered as components of a vector $\varepsilon \in R^N$. System (1) satisfies

Assumption I. For all $t \geq t_0$, all the matrices on the right hand side of (1) are continuous, bounded and have bounded first derivatives.

A characteristic of singularly perturbed systems is that the variables z_i are fast since their derivatives are $1/\varepsilon_i$ large. Under the additional assumption that $\varepsilon_{i+1}/\varepsilon_i \to 0$

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as $\varepsilon_i \to 0$, the system (1) exhibits N time scales, that is z_{i+1} is fast relative to z_i. In [3], [4] such multi-time scale systems are treated by nested single parameter perturbations. However, in many real systems the parameters are of the same order and do not allow the multi-time scale assumption. We therefore assume that the ratios of $\varepsilon_1, \cdots, \varepsilon_N$ are bounded by some positive constants $\bar{m}$ and $\bar{M}$

$$\bar{m} \leq \frac{\varepsilon_i}{\varepsilon_j} \leq \bar{M}, \qquad i, j = 1, \cdots, N, \tag{2}$$

that is the possible values of ε are restricted to a cone $H \subset R^N$. In contrast to the multi-time scale systems, in our case all z_i's are in the same time scale. We call this case the multi-parameter problem. A fundamental requirement for every multi-parameter perturbation result is to hold for all sufficiently small $\varepsilon \in H$, that is as $\varepsilon \to 0$ along any arbitrary path in H.

System (1) is rewritten in a form resembling a single parameter perturbation problem

$$\dot{x} = A_0(t)x + A_{0f}(t)z, \qquad x(t_0) = x_0, \tag{3a}$$

$$\mu\dot{z} = DA_{f0}(t)x + DA_f(t)z, \qquad z(t_0) = z_0. \tag{3b}$$

However, it is not a single parameter problem because both

$$\mu = (\varepsilon_1\varepsilon_2 \cdots \varepsilon_N)^{1/N} \tag{4}$$

and

$$D = \text{Block diag}\left[\frac{\mu}{\varepsilon_1}I_1, \cdots, \frac{\mu}{\varepsilon_N}I_N\right] \tag{5}$$

depend on all ε_i's. The above form is convenient since, in view of (2), the matrix D is bounded for all $\varepsilon \in H$,

$$m \leq \frac{\mu}{\varepsilon_i} \leq M \tag{6}$$

where m, M depend on $\bar{m}$, $\bar{M}$. The matrices A_{0f}, A_{f0} and A_f are formed of the submatrices A_{0i}, A_{i0} and A_{ij}, $i, j = 1, \cdots, N$, respectively, and $z' = [z'_1, \cdots, z'_N]$. A reduced system is now formally obtained by setting $\varepsilon = 0$ in (3),

$$\dot{\bar{x}} = A_0(t)\bar{x} + A_{0f}(t)\bar{z}, \quad \geq \quad \bar{x}(t_0) = x_0, \tag{7a}$$

$$0 = A_{f0}(t)\bar{x} + A_f(t)\bar{z}. \tag{7b}$$

Assuming that $\det A_f(t) \geq k > 0$ for all $t \geq t_0$, (7) can be rewritten as

$$\dot{\bar{x}} = [A_0(t) - A_{0f}(t)A_f^{-1}(t)A_{f0}(t)]\bar{x} \triangleq A_r(t)\bar{x}, \qquad \bar{x}(t_0) = x_0. \tag{8}$$

We also define a boundary layer system

$$\frac{d\hat{z}}{d\tau} = DA_f(t_0)\hat{z}(\tau), \qquad \hat{z}(0) = z_0 - \bar{z}(t_0), \tag{9}$$

where $\tau = (t - t_0)/\mu$ is the "stretched" time scale.

We are concerned with two problems. First, we seek conditions for the uniform asymptotic stability of (1) for all sufficiently small $\varepsilon \in H$. Second, we want to approximate the solution of the initial value problem (1) in terms of the solution of the reduced problem (8) and the boundary layer problem (9).

For the first problem we make the following

Assumption II. The reduced system (8) is uniformly asymptotically stable.

3. Main results. Our crucial assumption is a generalization of the so called D-stability property of the boundary layer system.

Assumption III. For all $t \geq t_0$, the matrix $A_f(t)$ has the property that

$$\operatorname{Re} \lambda \{DA_f(t)\} \leq -2\sigma < 0 \tag{10}$$

where σ is a fixed scalar independent of ε, possibly depending on the bounds m and M.

The main results of this paper are summarized in the following

THEOREM 1. *Under Assumptions* I, II *and* III *there exists a positive scalar* ν *such that for all* $\varepsilon \in H$, $0 < \|\varepsilon\| \leq \nu$, *system* (1) *is uniformly asymptotically stable.*

THEOREM 2. *If Assumptions* I *and* III *are satisfied then for every finite* $T > t_0$ *there exists a positive scalar* ν *such that for all* $t \in [t_0, T]$ *and all* $\varepsilon \in H$, $0 < \|\varepsilon\| \leq \nu$, *the solution of the initial value problem* (1) *is approximated by the solution of the reduced problem* (8) *and the boundary layer problem* (9), *that is,*

$$x(t) = \bar{x}(t) + O(\|\varepsilon\|) \tag{11a}$$

$$z(t) = -A_f^{-1}(t)A_{f0}(t)\bar{x}(t) + \hat{z}(\tau) + O(\|\varepsilon\|). \tag{11b}$$

Moreover, for all $t \in [t_1, T]$, $t_0 < t_1 < T$

$$x(t) = \bar{x}(t) + O(\|\varepsilon\|) \tag{12a}$$

$$z(t) = -A_f^{-1}(t)A_{f0}(t)\bar{x}(t) + O(\|\varepsilon\|). \tag{12b}$$

If in addition Assumption II *is satisfied then* (11) *and* (12) *hold for all* $T \in (t_0, \infty)$.

Our Assumption III has a general form, but it is not verifiable by an algorithm with a finite number of steps. It is satisfied in special cases such as when $A_f(t)$ is block diagonal or block triangular with the on-diagonal matrices satisfying the condition

$$\operatorname{Re} \lambda \{A_{ii}(t)\} \leq -c_i, \quad \text{for all } t \geq t_0, \quad i = 1, \cdots, N. \tag{13}^{1}$$

Another special case is when A_f is constant and the z_i's are scalars. Then Assumption III means that A_f is D-stable, that is DA_f is a stable matrix for all diagonal matrices D with positive elements. Several D-stability conditions have been investigated in the economic literature [5]. Recently this concept has been used in large scale system analysis [6], [7].

Our Assumption III can be considered as an extension of the notion of D-stability to matrices depending on t and to vector rather than scalar subsystems, that is when $n_i > 1$. In this more general framework we now examine several conditions allowing us to test Assumption III. The first condition is the following:

(i) There exists a block diagonal positive definite matrix $P(t)$,

$$P(t) = \text{Block diag}\,[P_1(t), \cdots, P_N(t)] \tag{14}$$

satisfying

$$c_2\|x\|^2 \leq x'P(t)x \leq c_3\|x\|^2 \quad \text{for all } x \in R^{\Sigma_i n_i}, \quad t \geq t_0, \tag{15}$$

such that $Q(t)$ given by

$$P(t)A_f(t) + A_f'(t)P(t) = -Q(t) \tag{16}$$

[1] In this section $c_1, c_2, \cdots$ are used to denote various fixed positive constant scalars.

is bounded from below by

$$x'Q(t)x \geqq c_4\|x\|^2, \quad \text{for all } x \in R^{\Sigma_i n_i}, \quad t \geqq t_0. \tag{17}$$

This condition implies (10) since the Lyapunov function $v(x) = x'P(t)D^{-1}x$ for the system $dx/ds = DA_f(t)x$ has the negative definite derivative $dv/ds = -x'Q(t)x$. Although this condition does not require the knowledge of D, it is still not finitely verifiable. However, it can be used to generate classes of matrices satisfying (10). An example is the case when $A_f(t)$ is symmetric with $\lambda\{A_f(t)\} \leqq -c_5$ for all $t \geqq t_0$. Then condition (i) is satisfied by $P = I$, while $c_2 = c_3 = 1$, $c_4 = 2c_5$ satisfy (15), (17).

The next condition involves two different conditions introduced in [8], [9] as sufficient conditions for stability of matrices with dominating diagonal blocks.

(ii) The matrices $A_{ii}(t)$ are symmetric with

$$\lambda\{A_{ii}(t)\} \leqq -c_6 \quad \text{for all } t \geqq t_0, \qquad i = 1, \cdots, N, \tag{18}$$

and either

$$\sum_{k \neq i} \|A_{ik}(t)\| < c_6 \quad \text{for all } t \geqq t_0, \qquad i = 1, \cdots, N. \tag{19}$$

or

$$\sum_{k \neq i} \|A_{ii}^{-1}(t)A_{ik}(t)\| < 1 \quad \text{for all } t \geqq t_0, \qquad i = 1, \cdots, N. \tag{20^2}$$

If $A_f(t)$ satisfies (18) with (19) or (20) then $DA_f(t)$ satisfies the same condition with c_6 replaced by mc_6 where m is the lower bound in (6).

The last two conditions are due to Siljak [6] and Michel [7] who derived them using the decomposition aggregation method to test the stability of interconnected systems when the isolated subsystems are stable. In these conditions the matrices $A_{ii}(t)$ satisfy (13) and symmetric positive definite $P_i(t)$, $Q_i(t)$ are such that

$$P_i(t)A_{ii}(t) + A_{ii}'(t)P_i(t) = -Q_i(t), \qquad i = 1, \cdots, N. \tag{21}$$

Then there exist positive constants ξ_{ij}, π_{i1}, π_{i2}, π_{i3} and π_{i4} satisfying

$$\|A_{ij}(t)\| \leqq \xi_{ij}, \quad \text{for all } t \geqq t_0, \tag{22}$$

$$\pi_{i1}\|x\|^2 \leqq x'P_i(t)x \leqq \pi_{i2}\|x\|^2, \quad \text{for all } x \in R^{n_i}, \ t \geqq t_0, \tag{23}$$

$$\pi_{i3}\|x\|^2 \leqq x'Q_i(t)x \leqq \pi_{i4}\|x\|^2, \quad \text{for all } x \in R^{n_i}, \ t \geqq t_0. \tag{24}$$

In both Siljak's and Michel's condition an $N \times N$ aggregation is formed and tested for the stability of $A_f(t)$. The elements of Siljak's aggregation matrix S are

$$s_{ij} = \begin{cases} -\eta_{i3}, & i = j, \\ \xi_{ij}\eta_{j1}^{-1}\eta_{i4}, & i \neq j \end{cases} \tag{25}$$

where

$$\eta_{i1} = \sqrt{\pi_{i1}}, \qquad \eta_{i3} = \frac{\pi_{i3}}{2\pi_{i2}}, \qquad \eta_{i4} = \frac{\pi_{i2}}{\sqrt{\pi_{i1}}}$$

and those of Michel's matrix T are

$$t_{ij} = \begin{cases} -d_i\pi_{i3}, & i = j, \\ d_i\pi_{i2}\xi_{ij} + d_j\pi_{j2}\xi_{ji}, & i \neq j, \end{cases} \tag{26}$$

[2] The matrix norm in (20) is defined as $\|A\| = [\lambda_{\max}(AA')]^{1/2}$.

for some positive numbers $d_1, \cdots, d_N$.

The Siljak condition is the following:

(iii) The matrices $A_{ii}(t)$ satisfy (13) and the principal minors M_k of S have alternating signs, that is

$$M_k = (-1)^k \det \begin{bmatrix} s_{11} & \cdots & s_{1k} \\ \vdots & & \\ s_{k1} & \cdots & s_{kk} \end{bmatrix} > 0, \qquad k = 1, \cdots, N. \tag{27}$$

To show that his condition implies Assumption III we consider the Lyapunov function

$$v(x) = \sum_{i=1}^{N} \frac{\varepsilon_i \delta_i}{\mu} v_i(x_i), \qquad x' = (x_1', \cdots, x_N'), \quad x_i \in R^{n_i} \tag{28}$$

with

$$v_i(x_i) = (x_i' P_i(t) x_i)^{1/2}, \tag{29}$$

where $\delta_i > 0$, $i = 1, \cdots, N$, are yet unspecified numbers. By derivation similar to that in [6] it can be shown that the derivative of v with respect to the system

$$\frac{dx}{ds} = DA_f(t)x \tag{30}$$

satisfies the inequality

$$\frac{dv}{ds} \leq \sum_{i=1}^{N} \delta_i \sum_{j=1}^{N} s_{ij} v_j. \tag{31}$$

It is shown in [6] that when inequalities (27) are satisfied there exist numbers $\delta_i > 0$ ($i = 1, \cdots, N$) and $\pi > 0$ such that

$$\frac{dv}{ds} \leq -\pi \sum_{i=1}^{N} \delta_i v_i. \tag{32}$$

Hence

$$\frac{dv}{ds} \leq -\pi m v \triangleq -c_7 v. \tag{33}$$

The last condition is that of Michel:

(iv) The matrices $A_{ii}(t)$ satisfy (13) and there exist numbers d_i, $i = 1, \cdots, N$, such that the matrix T is negative definite.

To show that this condition implies Assumption III we consider the Lyapunov function (28) with δ_i replaced by d_i and $v_i(x_i)$ given by

$$v_i(x_i) = x_i' P_i(t) x_i. \tag{34}$$

In a way similar to [7] it can be shown that its derivative with respect to (30) satisfies the inequality

$$\frac{dv}{ds} \leq \sum_{i,j} t_{ij} \|x_i\| \|x_j\|. \tag{35}$$

Since T is negative definite, let $\lambda = \lambda_{\max}(T) < 0$; thus

$$\frac{dv}{ds} \leq -\lambda \sum_{i=1}^{N} \|x_i\|^2 = -\lambda \|x\|^2. \tag{36}$$

Using (22) and (36) we obtain

$$\frac{dv}{ds} \leq \frac{-\lambda m}{\max_i d_i \max_i \pi_{i2}} v \triangleq -c_8 v. \tag{37}$$

Michel's condition is not finitely verifiable since it requires the existence of positive numbers $d_1, \cdots, d_N$. However, a more conservative, finitely verifiable, condition implying Michel's condition can be obtained [7] by writing the matrix T as

$$T = \bar{D}W + W\bar{D} \tag{38}$$

where

$$\bar{D} = \operatorname{diag}[d_1, \cdots, d_N], \qquad d_i > 0, \tag{39}$$

and W is given by

$$w_{ij} = \begin{cases} -\frac{1}{2}\pi_{i3}, & i = i, \\ \pi_{i2}\xi_{ij}, & i \neq j. \end{cases} \tag{40}$$

Then, if the principal minors of W have alternating signs, that is, satisfy (27), there exists matrix $\bar{D}$ such that T is negative definite.

It is important to notice that Siljak's and Michel's conditions are not equivalent. In fact examples can be constructed for matrices which satisfy one of them and do not satisfy the other, and vice versa [9]. These two conditions are particularly important since they are applicable to large scale systems. They are also applicable to nonlinear systems. Grujic [11] has used the decomposition-aggregation method to test the stability of a class of nonlinear singularity perturbed systems. The motive to look at these two conditions and study their implication to our Assumption III was that the aggregation matrices S and T which satisfy the respective condition are D-stable. However, as we have shown, the proof that Siljak's or Michel's condition implies Assumption III does not rely upon the D-stability of S or T, because we have chosen the Lyapunov function in either case in such a way that we obtain the aggregation matrix independent of D.

The above discussion of Assumption III shows that the class of matrices $A_f(t)$ satisfying Assumption III contains important subclasses. However a complete characterization of that class is yet to be made by further studies.

4. Proof. We follow [12] to separate the fast and slow modes of (3). Using

$$\begin{bmatrix} y \\ v \end{bmatrix} = \begin{bmatrix} I - \mu M D^{-1} L & -\mu M D^{-1} \\ L & I \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} \tag{41}$$

the system (3) is transformed into

$$\dot{y} = (A_0(t) - A_{0f}(t)L(t))y, \tag{42a}$$

$$\mu \dot{v} = (DA_f(t) + \mu L(t)A_{0f}(t))v, \tag{42b}$$

where $L(t)$ and $M(t)$ satisfy

$$\mu \dot{L} = DA_f L - DA_{f0} - \mu L A_0 + \mu L A_{0f} L, \tag{43}$$

$$\mu \dot{M} D^{-1} = -MA_f + A_{0f} - \mu M D^{-1} L A_{0f} + \mu A_0 M D^{-1} - \mu A_{0f} L M D^{-1}. \tag{44}$$

with the initial conditions

$$L(t_0)=A_f^{-1}(t_0)A_{f0}(t_0), \tag{45}$$

$$M(t_0)=A_{0f}(t_0)A_f^{-1}(t_0). \tag{46}$$

We first observe that the fast subsystem (42b) is of the form

$$\mu\dot{z}=(DA_f(t)+\mu\Gamma(t,\varepsilon))z \tag{47}$$

whose properties we examine in Lemmas 1 and 2. Then in Lemmas 3 and 4 we establish the existence and convergence of solutions $L(t)$ and $M(t)$ of (43) and (44). Lemmas 1, 2, 3 and 4 are stated under the Assumptions I and III.

LEMMA 1. *There exist positive scalars ν, K_1 and γ_1 such that for all $\varepsilon\in H$, $0<\|\varepsilon\|\leq\nu$ and $t\geq s$, the state transition matrix $\varphi_1(t,s)$ of the system* (47) *with $\Gamma=0$ has the property that*

$$\|\varphi_1(t,s)\|\leq K_1\exp\left[-\frac{\gamma_1}{\mu}(t-s)\right]. \tag{48}$$

Proof. By Assumption I and (6) we have $\|DA_f(t)\|\leq K_2$, for all $t\geq t_0$ and $\varepsilon\in H$. Using (10) and Lemma 4 of [13, p. 116] we get for all $\theta\geq 0$, $\varepsilon\in H$

$$\left\|\exp\left[\frac{\theta}{\mu}DA_f(t)\right]\right\|\leq K_3\exp\left(-\frac{\sigma}{\mu}\theta\right), \tag{49}$$

where K_3 depends only on σ and K_2. Also there exists $\beta>0$ such that $\|DA_f(t_2)-DA_f(t_1)\|\leq\beta|t_2-t_1|$, $t_1,t_2\geq t_0$. Then by Theorem 12 of [13, p. 117] there exists $\mu^*>0$ such that for all $\mu<\mu^*$, $\varphi_1(t,s)$ satisfies (48) with $K_1=K_3^2$ and $\gamma_1<\sigma$; and ν can be chosen to be the radius of the largest ball centered at the origin with $\mu<\mu^*$.

LEMMA 2. *If $\|\Gamma(t,\varepsilon)\|\leq K_4$, for all $t\geq t_0$, $\varepsilon\in H$, then there exist positive scalars ν, $\gamma_2<\gamma_1$, such that for all $\varepsilon\in H$, $0<\|\varepsilon\|\leq\nu$ and $t\geq s$, the state transition matrix $\varphi_2(t,s)$ of* (47) *satisfies*

$$\|\varphi_2(t,s)\|\leq K_1\exp\left[-\frac{\gamma_2}{\mu}(t-s)\right]. \tag{50}$$

Moreover, there exists $\nu>0$, $K_5>0$ such that for all $\varepsilon\in H$, $0<\|\varepsilon\|\leq\nu$, $t\geq t_0$, the matrix $\varphi_3(t,t_0)=\varphi_2(t,t_0)-\exp[DA_f(t_0)((t-t_0)/\mu)]$ satisfies

$$\|\varphi_3(t,t_0)\|\leq K_5\|\varepsilon\|. \tag{51}$$

Proof. Inequality (50) follows from Lemma 1 and Theorem 9 of [13, p. 70]. To prove (51) we notice that $\varphi_3(t,t_0)$ satisfies the equation

$$\dot{\varphi}_3(t,t_0)=\frac{1}{\mu}[DA_f(t)+\mu\Gamma(t,\varepsilon)]\varphi_3(t,t_0)$$
$$+\frac{1}{\mu}[DA_f(t)-DA_f(t_0)+\mu\Gamma(t,\varepsilon)]\exp\left[DA_f(t_0)\left(\frac{t-t_0}{\mu}\right)\right].$$

Noting that $\varphi_3(t_0,t_0)=0$, we obtain

$$\varphi_3(t,t_0)=\int_{t_0}^{t}\varphi_2(t,\tau)\frac{1}{\mu}[DA_f(\tau)-DA_f(t_0)+\mu\Gamma(\tau,\varepsilon)]\exp\left[DA_f(t_0)\frac{(\tau-t_0)}{\mu}\right]d\tau.$$

Using Lemma 1, (50), and the fact that $\|DA_f(t)-DA_f(t_0)\| \leqq \beta(t-t_0)$, we obtain

$$\begin{aligned}\|\varphi_3(t,t_0)\| &\leqq K_1^2 \int_{t_0}^{t} e^{-(\gamma_2/\mu)(t-\tau)} \frac{1}{\mu}[\beta(\tau-t_0)+\mu K_4]\, e^{-(\gamma_1/\mu)(\tau-t_0)}\, d\tau \\ &\leqq \frac{K_1^2}{\mu} e^{-(\gamma_2/\mu)(t-t_0)} \int_{t_0}^{t} [\beta(\tau-t_0)+\mu K_4]\, d\tau \\ &\leqq \mu K_1^2 e^{-(\gamma_2/\mu)(t-t_0)} \left[\frac{\beta}{2}\left(\frac{t-t_0}{\mu}\right)^2 + K_4\left(\frac{t-t_0}{\mu}\right)\right] \\ &\leqq \mu K_1^2 \left[\frac{2\beta}{\gamma_2^2 e^2} + \frac{K_4}{\gamma_2 e}\right] \\ &\leqq \|\varepsilon\| \frac{K_1^2}{m\sqrt{N}} \left[\frac{2\beta}{\gamma_2^2 e^2} + \frac{K_4}{\gamma_2 e}\right] = K_5\|\varepsilon\|.\end{aligned}$$

Next we establish the existence of solutions of (43) and (44). Let us first remark that the state transition matrix of the system $\dot{\eta} = A_0(t)\eta$ satisfies

$$\|\varphi_0(t,s)\| \leqq K_6 \exp[\gamma_3|t-s|], \quad \text{for all } t, s \geqq t_0, \tag{52}$$

for some positive constants K_6, γ_3, (see [14, p. 287]).

LEMMA 3. *There exists a positive scalar ν such that for all $\varepsilon \in H$, $0<\|\varepsilon\| \leqq \nu$, $t \geqq t_0$, there exists a continuously differentiable bounded solution $L(t)$ of* (43) *and* (45), *satisfying*

$$L(t) = A_f^{-1}(t)A_{f0}(t) + O(\|\varepsilon\|). \tag{53}$$

Proof.[3] Every solution of the integral equation $L(t) = SL(t)$, where

$$\begin{aligned}SL(t) &= \varphi_1(t,t_0)A_f^{-1}(t_0)A_{f0}(t_0)\varphi_0(t_0,t) \\ &\quad + \int_{t_0}^{t} \varphi_1(t,s)\left[\frac{-1}{\mu}DA_{f0}(s) + L(s)A_{0f}(s)L(s)\right]\varphi_0(s,t)\, ds\end{aligned} \tag{54}$$

is a solution of (43) with initial condition (45). Hence it is sufficient to prove the existence of a solution of this integral equation. Using the identity

$$LA_{0f}L - \bar{L}A_{0f}\bar{L} = (L-\bar{L})A_{0f}L + \bar{L}A_{0f}(L-\bar{L}) \tag{55}$$

and expressions (48) and (52) we obtain

$$\|SL(t) - S\bar{L}(t)\| \leqq \frac{K_1K_6\mu}{\gamma_1-\mu\gamma_3}\|L-\bar{L}\|\,\|A_{0f}\|(\|L\|+\|\bar{L}\|), \tag{56}$$

and

$$\begin{aligned}\|SL(t)\| &\leqq \frac{K_1K_6}{\gamma_1-\mu\gamma_3}(\|DA_{f0}\| + \mu\|A_{0f}\|\,\|L\|^2) \\ &\quad + K_1K_6\|A_f^{-1}(t_0)\|\,\|A_{f0}\| \exp\left[-\left(\frac{\gamma_1}{\mu}-\gamma_3\right)(t-t_0)\right].\end{aligned} \tag{57}$$

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[3] In this proof L belongs to the space of bounded continuous $\sum_i n_i \times n_0$ matrix functions on the interval $[t_0, \infty)$ with the norm $\|L\| = \sup_{t \geqq t_0}\|L(t)\|$ where the matrix norm can be any norm. This space is a Banach space.

Letting

$$\rho = 2K_1K_6\left(\frac{2}{\gamma_1}\|DA_{f0}\| + \|A_f^{-1}(t_0)\|\,\|A_{f0}\|\right) \tag{58}$$

we choose $\mu^* > 0$ so small that

$$\mu^*\gamma_5 \le \frac{\gamma_1}{2} \quad \text{and} \quad \frac{4K_1K_6\mu^*}{\gamma_1}\|A_{0f}\|\rho \le \tfrac{1}{2}. \tag{59}$$

If $\|L\| \le \rho$, $\|\hat{L}\| \le \rho$, then for $0 < \mu \le \mu^*$ we get

$$\|SL - S\hat{L}\| \le \tfrac{1}{2}\|L - \hat{L}\| \tag{60}$$

and

$$\|SL(t)\| \le \tfrac{3}{4}\rho. \tag{61}$$

By the contraction principle the solution $L(t)$ exists and is unique in $\|L\| \le \rho$. To prove (53) we let

$$L(t) = A_f^{-1}(t)A_{f0}(t) + \Delta L(t) \triangleq L_0(t) + \Delta L(t). \tag{62}$$

We note that $\Delta L(t_0) = 0$ and that $\Delta L(t)$ satisfies

$$\Delta\dot{L} = \frac{1}{\mu}(DA_f + \mu L_0A_0)\Delta L - \Delta LA_r + \Delta LA_{0f}\Delta L + R_1, \tag{63}$$

where $R_1 = L_0A_{0f}L_0 - L_0A_0 - \dot{L}_0$. Let $\varphi_r(t,s)$ and $\varphi_4(t,s)$ be the state transition matrices of equation (8) and $\mu\dot{\xi} = (DA_f(t) + \mu A_f^{-1}(t)A_{f0}(t)A_{0f}(t))\xi$, respectively. The norm of $\varphi_r(t,s)$ satisfies an inequality similar to (52) with constants K_7 and γ_4. By Lemma 2, the norm of $\varphi_4(t,s)$ satisfies an inequality similar to (50) with constants K_1 and $\gamma_5 < \gamma_1$. Then from the form of the solution of (63)

$$\Delta L(t) = \int_{t_0}^{t} \varphi_4(t,s)[\Delta L(s)A_{0f}(s)\Delta L(s) + R_1(s)]\varphi_r(s,t)\,ds, \tag{64}$$

it follows that

$$\begin{aligned}
\|\Delta L(t)\| &\le \frac{K_1K_7\mu}{\gamma_5 - \mu\gamma_4}(\|A_{0f}\|\,\|\Delta L\|^2 + \|R_1\|) \\
&\le \mu\frac{K_1K_7}{\gamma_5}(\|A_{0f}\|(\|L_0\| + \|L\|)^2 + \|R_1\|) \le \mu K_8 \\
&\le \frac{K_8}{m\sqrt{N}}\|\varepsilon\|,
\end{aligned} \tag{65}$$

for some positive constant K_8, which proves (53), and ν can be chosen in a way similar to that in Lemma 1.

LEMMA 4. *There exists a positive scalar ν such that for all $\varepsilon \in H$, $0 < \|\varepsilon\| \le \nu$, $t \ge t_0$; there exists a continuously differentiable bounded solution $M(t)$ of* (44) *and* (46), *satisfying*

$$M(t) = A_{0f}(t)A_f^{-1}(t) + O(\|\varepsilon\|). \tag{66}$$

The proof of this lemma is similar to that of Lemma 3. Based on Lemmas 3 and 4 the matrices of the transformed system (42) can be written as $O(\|\varepsilon\|)$ perturbations of

$A_r(t)$, $DA_f(t)$, that is (42) becomes

(67a) $$\dot{y} = (A_r(t) + O(\|\varepsilon\|))y, \qquad y(t_0) = x_0 + O(\|\varepsilon\|)$$

(67b) $$\mu\dot{v} = (DA_f(t) + O(\|\varepsilon\|))v, \qquad v(t_0) = z_0 + L_0(t_0)x_0 + O(\|\varepsilon\|).$$

Proof of Theorem 1. Since the transformation (41) is nonsingular for all sufficiently small $\varepsilon \in H$ and $L(t)$, $M(t)$ are bounded for all $t \geqq t_0$, it is sufficient to show that each subsystem (67a) and (67b) is uniformly asymptotically stable. This immediately follows from Lemma 2 and Theorem 9 of [13, p. 70].

Proof of Theorem 2. The uniform convergence $y(t) \to \bar{x}(t)$ as $\|\varepsilon\| \to 0$ follows from the continuous dependence of the solution of (67a) on the right-hand side and the initial conditions. Lemma 2 guarantees the uniform convergence $v(t) \to \hat{z}((t-t_0)/\mu) = \hat{z}(\tau)$. Using the inverse transformation of (41), we obtain

(68a) $$x(t) = y(t) + \mu M(t)D^{-1}v(t) = \bar{x}(t) + O(\|\varepsilon\|),$$

(68b) $$z(t) = -L(t)y(t) + (I - \mu L(t)M(t)D^{-1})v(t) = -A_f^{-1}(t)A_{f0}(t)\bar{x}(t) + \hat{z}(\tau) + O(\|\varepsilon\|)$$

which proves (11).

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Control of Linear Systems with Multiparameter Singular Perturbations*

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The boundary layer stability of multiparameter singularly perturbed systems is the major problem in the design of a near optimal control which disregards the perturbation parameters.

Key Word Index—Perturbation techniques; approximation theory; system order reduction; stability; optimal control; differential games.

Abstract—The singular perturbation theory is extended to systems with several small parameters which can change the system order. Difficulties arising in testing the boundary layer stability in multiparameter linear problems are discussed. The theory is applied to linear quadratic optimal control and Nash game problems.

1. INTRODUCTION

WHEN several small singular perturbation parameters of the same order of magnitude are present in the dynamic model of a physical system, the analysis and control problems surveyed in Kokotovic, O'Malley and Sannuti (1976) are usually approached as single parameter perturbation problems. This is done by expressing small parameters as known multiples of a particular parameter μ, such as $m=\beta_1\mu$, $T=\beta_2\mu$, where m is a small mass and T is a small time constant. A limitation of that approach is that the scaling coefficients β_i are assumed to be known. Thus, it is not applicable to a large class of problems where the parameters represent small unknown parasitics whose values are not known exactly, although they are limited to lie within a certain set. Such a situation arises also in multicontroller design problems when small parameters represent different independent ways to simplify the model of the overall system by individual control agents (Khalil and Kokotovic, 1978).

In Khalil and Kokotovic (1979) we have formulated the multiparameter singular perturbation problem and stressed its difference from the multitime scale problem (Hoppensteadt, 1969). We have shown that the block D-stability property of matrices is a sufficient condition for the asymptotic stability of the boundary layer system. Several block D-stability conditions have been derived.

In this paper we discuss the difficulties involved in testing the stability of the boundary layer system in the multiparameter linear problem. We investigate the block D-stability property of matrices and establish its relationship with the stability criteria for the multitime scale problem. We also show how the decomposition-aggregation stability tests can be used as block D-stability tests. Electrical networks are given as examples of physical systems having the block D-stability property. Since for many problems of interest the block D-stability is restrictive, an alternative way to verify the boundary layer stability is proposed. We then proceed to the multiparameter singularly perturbed linear optimal control problem, discuss its well-posedness and design a near optimal control which does not depend on the values of the small parameters. Hence this control is applicable to systems where the parameters are unknown. Finally, we discuss the asymptotic behavior of a closed-loop Nash solution of infinite-time linear-quadratic differential games for multiparameter singularly perturbed systems.

2. MULTIPARAMETER SINGULAR PERTURBATIONS

We consider the linear time-invariant singularly perturbed system

$$\dot{x} = A_0 x + \sum_{j=1}^{N} A_{0j} z_j, \quad x(0) = x_0 \tag{1}$$

$$\varepsilon_i \dot{z}_i = A_{i0} x + \sum_{j=1}^{N} A_{ij} z_j, \quad z_i(0) = z_{i0} \tag{2}$$

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where $x \in R^{n_0}$, $z_i \in R^{n_i}$. The small positive scalars $\varepsilon_1, \ldots, \varepsilon_N$ are ordered as components of a vector $\varepsilon \in R^N$. A reduced order system is formally obtained by setting $\varepsilon = 0$ in (2),

$$\dot{\bar{x}} = A_0 \bar{x} + A_{0f} \bar{z}, \quad \bar{x}(0) = x_0 \tag{3}$$

$$0 = A_{f0} \bar{x} + A_f \bar{z}, \tag{4}$$

where the matrices A_{0f}, A_{f0} and A_f are formed of the submatrices A_{0i}, A_{i0} and A_{ij}, $i, j = 1, \ldots, N$, respectively, and $z' = (z'_1, \ldots, z'_N)$. Assuming that A_f is nonsingular, (4) has a unique root

$$\bar{z} = -A_f^{-1} A_{f0} \bar{x} \tag{5}$$

whose substitution into (3) yields the reduced system

$$\dot{\bar{x}} = (A_0 - A_{0f} A_f^{-1} A_{f0}) \bar{x} = A_r \bar{x}, \quad \bar{x}(0) = x_0. \tag{6}$$

The task of singular perturbation is to find under what conditions can the properties of the solution of the original system (1), (2) be deduced from the properties of the solution of the reduced system (6). Under the additional assumption that $\varepsilon_{i+1}/\varepsilon_i \to 0$ as $\|\varepsilon\| \to 0$, the system (1), (2) exhibits N fast time scales, that is z_{i+1} is fast relative to z_i. Such multitime scale systems are treated by nested single parameter perturbations (Hopensteadt, 1969). In this paper we treat systems in which the parameters are of the same order and do not allow the multitime scale assumption. We therefore assume that the ratios of $\varepsilon_1, \ldots, \varepsilon_N$ are bounded by some positive constants $\bar{m}_{ij}$, $\bar{M}_{ij}$,

$$\bar{m}_{ij} \leq \frac{\varepsilon_i}{\varepsilon_j} \leq \bar{M}_{ij}, \quad i, j = 1, \ldots, N, \tag{7}$$

that is the possible values of ε are restricted to a cone $H \subset R^N$. Previously, singularly perturbed control problems with several small parameters of the same order have been treated (Kokotovic, O'Malley and Sannuti, 1976) by reformulating the problem as a single parameter problem. This is done by expressing the small parameters $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_N$ as known multiples of a single parameter μ, $\varepsilon_i = \beta_i \mu$. The main limitation of such scalarization is that the results depend on the scaling coefficients β_i, which are often unknown. The validity of the result is restricted to a ray in the space of $\varepsilon_1, \ldots, \varepsilon_N$, defined by a particular set of values $\beta_1, \ldots, \beta_N$. Even if the solution of the full problem (1), (2) converges to the solution of the reduced problem (6) as $\varepsilon \to 0$ along every ray in the cone H, this is not sufficient for convergence as $\varepsilon \to 0$ along any arbitrary path in H.

In a truly multiparameter problem the perturbations of the vector ε are not limited to a particular ray. Convergence results are sought as $\|\varepsilon\| \to 0$ in H, which guarantees that they hold for all sufficiently small $\varepsilon \in H$. We start by rewriting system (1), (2) in a form resembling a single parameter system

$$\dot{x} = A_0 x + A_{0f} z, \quad x(0) = x_0 \tag{8}$$

$$\mu(\varepsilon) \dot{z} = D(\varepsilon)[A_{f0} x + A_f z], \quad z(0) = z_0 \tag{9}$$

but which differs from a single parameter problem because both

$$\mu = \mu(\varepsilon) = (\varepsilon_1 \varepsilon_2 \ldots \varepsilon_N)^{1/N} \tag{10}$$

$$D = D(\varepsilon) = \text{block-diag}\left[\frac{k}{e_1} I_1, \ldots, \frac{\mu}{\varepsilon_N} I_N\right] \tag{11}$$

depend on all ε_i. It is crucial that in view of (7) the elements of D are bounded

$$m_i \leq \frac{\mu}{\varepsilon_i} \leq M_i, \quad \forall \varepsilon \in H, \quad i = 1, \ldots, N \tag{12}$$

where m_i, M_i depend on $\bar{m}_{ij}$ and $\bar{M}_{ij}$ in (7). From single parameter problems we know that we need a boundary layer system to correct the effect of the discrepancy between $\bar{z}(0)$ and the initial value z_0 of z. In the multiparameter problem we define the boundary layer system as

$$\frac{d\tilde{z}}{d\tau} = D(\varepsilon) A_f \tilde{z}(\tau), \quad \tilde{z}(0) = z_0 - \bar{z}(0) \tag{13}$$

where $\tau = t/\mu$ is a 'stretched' time scale. The right hand side of (13) depends on ε through $D(\varepsilon)$ whose elements are bounded by (12) but otherwise arbitrary. In contrast to the multitime scale problem, in this case the fast modes of all z_i's are in the same time scale. As in the single parameter problem here also we need to guarantee the asymptotic stability of the fast modes for all sufficiently small $\varepsilon \in H$, that is

$$\text{Re}\lambda(D(\varepsilon) A_f) < 0, \quad \forall \varepsilon \in H. \tag{14}$$

However, in the single parameter problem $D(\varepsilon) = I$ and only one test of $\lambda(A_f)$ is needed. The main difficulty of the multiparameter problem is that (14) must be tested for every $\varepsilon \in H$ which requires solving infinite number of eigenvalue problems. Our approach is to find classes of matrices A_f for which (14) is satisfied for all diagonal matrices D of the type (11), (12). In Khalil and Kokotovic (1979) we have identified block D-stable matrices as an important class

satisfying (14) for arbitrary bounds $m_i > 0$, $M_i < \infty$. A matrix A_f is said to be block D-stable if $\mathrm{Re}\lambda(DA_f) < 0$ for all $D = \text{block-diag}\ [\alpha_1 I_1, \ldots, \alpha_N I_N]$ with arbitrary positive scalars α_i. Further discussion of this notion and an alternative characterization of a class of matrices satisfying (14) is given in Section 4.

In the following theorem we establish stability properties of the original system (1), (2) from the stability of the reduced system (6) and the approximation of the original solution x, z by the reduced solution $\bar{x}$, $\bar{z}$.

Theorem 1: If (14) is satisfied, then for every finite $T > 0$ there exists a positive scalar v such that, for all $t \in [0, T]$ and all $\varepsilon \in H$, $0 < \|\varepsilon\| \leq v$,

$$x(t) = \bar{x}(t) + 0(\|\varepsilon\|) \tag{15}$$

$$z(t) = -A_f^{-1} A_{f0} \bar{x}(t) + \tilde{z}(\tau) + 0(\|\varepsilon\|), \tag{16}$$

that is the solution of the original problem (1), (2) is approximated by the solution of the reduced problem (6) and the boundary layer problem (13). If, in addition, $\mathrm{Re}\lambda(A_r) < 0$ then (15), (16) hold for all $T > 0$, and the original system is asymptotically stable.

A proof of this result is given in Khalil and Kokotovic (1979).

3. BLOCK D-STABILITY AND MULTITIME SCALES

A block D-stable matrix A_f satisfies (14) for arbitrary bounds m_i, M_i. Hence, we can extend the set H to the cone $\varepsilon_i > 0$ by letting $m_i \to 0$ and $M_i \to \infty$, while still having asymptotically stable boundary layer system (13). Since the set $\varepsilon_i > 0$ covers the regions of multitime scale perturbations, one would think of block D-stability as a more restrictive condition than the boundary layer stability in a multitime scale problem. Let us examine this relationship.

To state the sufficient conditions for asymptotic stability of the multitime scale problem derived in Hoppensteadt (1969), we assume that the parameters $\varepsilon_1, \ldots, \varepsilon_N$ in (1), (2) are ordered such that $\varepsilon_j + 1/\varepsilon_j \to 0$ as $\|\varepsilon\| \to 0$, that is ε_N is the smallest parameter. This ordering shows which matrices must be nonsingular to eliminate the variables z_j as the parameters ε_j are successively set equal to zero. For the adopted ordering ε_N, $\varepsilon_{N-1}, \ldots, \varepsilon_1$ the nonsingularity is required for the matrices

$$E_k = A_{kk} - E_{1k} E_{2k}^{-1} E_{3k}, \quad k = 1, \ldots, N-1 \tag{17}$$

$$E_N = A_{NN} \tag{18}$$

where

$$E_{1k} = (A_{k,k+1} A_{k,k+2} \cdots A_{k,N}), \tag{19}$$

$$E_{2k} = \begin{bmatrix} A_{k+1,k+1} & \cdots & A_{k+1,N} \\ \vdots & & \vdots \\ A_{N,k+1} & \cdots & A_{N,N} \end{bmatrix}, \tag{20}$$

$$E'_{3k} = (A'_{k+1,k} A'_{k+2,k} \cdots A'_{N,k}). \tag{21}$$

To clarify the fact that E_k can be nonsingular for one ordering and singular for another ordering we consider for example the case $N = 2$, $n_1 = n_2 = 1$ with $A_{11} = 0$, $A_{12} = 1$, $A_{21} = -1$, $A_{22} = -1$. When $\varepsilon_2/\varepsilon_1 \to 0$ the system (1), (2) possesses two fast time scales. In contrast the limit $\varepsilon_1/\varepsilon_2 \to 0$ does not result in two fast time scales since z_1 cannot be eliminated from the equations when $\varepsilon_1 = 0$.

With the assumed ordering, $\mathrm{Re}\lambda(A_r) < 0$ and

$$\mathrm{Re}\lambda(E_k) < 0, \quad k = 1, \ldots, N \tag{22}$$

form a set of sufficient conditions for asymptotic stability of (1), (2). The following theorem gives a relation between block D-stability and condition (22).

Theorem 2. If A_f is block D-stable, then condition (22) holds for every block permutation of A_f for which $\mathrm{Re}\lambda(E_k) \neq 0$, $k = 1, \ldots, N$.

Proof. We first recall an eigenvalue property of the single parameter singularly perturbed system with a small scalar parameter ν

$$\dot{y}_1 = F_{11} y_1 + F_{12} y_2 \tag{23}$$

$$\nu \dot{y}_2 = F_{21} y_1 + F_{22} y_2 \tag{24}$$

Lemma: If $\mathrm{Re}\lambda(F_{11} - F_{12} F_{22}^{-1} F_{21}) \neq 0$, $\mathrm{Re}\lambda(F_{22}) \neq 0$ then there exists $\nu^* > 0$ such that for all $\nu \in (0, \nu^*]$ the eigenvalues of (23), (24) have negative real parts *if and only if*

$$\mathrm{Re}\lambda(F_{11} - F_{12} F_{22}^{-1} F_{21}) < 0, \quad \mathrm{Re}\lambda(F_{22}) < 0. \tag{25}$$

To prove Theorem 2 we first consider the two-parameter case $N = 2$. Block D-stability implies that

$$\mathrm{Re}\lambda \begin{bmatrix} A_{11} & A_{12} \\ \dfrac{\varepsilon_1}{\varepsilon_2} A_{21} & \dfrac{\varepsilon_1}{\varepsilon_2} A_{22} \end{bmatrix} < 0 \tag{26}$$

holds for sufficiently small $\varepsilon_2/\varepsilon_1$. When $(A_{11} - A_{12} A_{22}^{-1} A_{21})$, A_{22} have no eigenvalues with zero real parts, it follows from the lemma that

$$\mathrm{Re}\lambda(A_{11} - A_{12} A_{22}^{-1} A_{21}) < 0, \ \mathrm{Re}\lambda(A_{22}) < 0, \tag{27}$$

which proves Theorem 2 for $N = 2$. For $N > 2$, we can show that block D-stability implies (22) by nested application of the lemma. Noticing that if A_f is block D-stable so is $P' A_f P$ for all block

permutation matrices P of comparable block dimensions, it follows that (22) must be satisfied for each ordering, provided $\mathrm{Re}\lambda(E_k) \neq 0$, $k = 1, \ldots, N$.

Remark 1: Asymptotic stability of the multiparameter problem with arbitrary bounds $m_i > 0$, $M_i < \infty$ implies asymptotic stability of the multitime scale problem for all possible orderings of the smallness of the parameters.

Remark 2. Condition (22) is a necessary condition for block D-stability. That (22) is not sufficient is clear from

$$A_f = \begin{bmatrix} -1 & 0 & -27 \\ -1 & -1 & 0 \\ 0 & -1 & -1 \end{bmatrix}, \quad n_1 = n_2 = n_3 = 1.$$

The matrix A_f satisfies (22) but it is unstable.

Remark 3. If for some ordering some of the matrices E_k have zero real part eigenvalues, then (22) is no longer necessary. Consider again

$$A_f = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}, \quad N = 2, \quad n_1 = n_2 = 1$$

in which A_f is D-stable, while (22) is not satisfied.

4. CONDITIONS FOR BOUNDARY LAYER STABILITY

The most difficult part in the multiparameter problem is verifying that A_f satisfies (14) for some given bounds m_i, M_i in (12). Since Block D-stable matrices satisfy (14) with arbitrary $m_i > 0$, $M_i < \infty$, one may start by testing whether A_f satisfies some block D-stability condition. If A_f is block D-stable, then the problem is solved. However, when A_f does not satisfy the necessary condition for block D-stability given in Section 3, or if it satisfies the necessary condition but does not satisfy any of the known sufficient conditions, this does not mean that A_f does not satisfy (14) for the given bounds. In fact, in most multiparameter problems of interest the small parameters $\varepsilon_1, \ldots, \varepsilon_N$ are of the same order of magnitude. Hence the bounds m_i, M_i are close to one. In this case the class of block D-stable matrices is a very conservative subclass of the class of matrices satisfying (14) for the bounds given in the problem.

Our task in studying conditions for boundary layer stability is two-fold. On one hand we need sufficient conditions for block D-stability. The more we know about block D-stability, the more readily we can employ it to verify the boundary layer stability of A_f. On the other hand when block D-stability tests fail we need conditions that make use of the particular bounds given in the problem. For the rest of this section we discuss these two approaches and illustrate them by electrical network examples.

The block D-stability property of matrices is an extension of the concept of D-stability. If A_f is partitioned in scalar blocks, that is if $n_i = 1$, then the block D-stability is merely D-stability. Economists have studied this notion for several years and have formulated several sufficient conditions which are surveyed in Johnson (1974). In Khalil and Kokotovic (1979) we examined Siljak's (Siljak, 1977) and Michel's (Michel and Miller, 1977) interconnected asymptotically stable subsystems. Here we show that their decomposition-aggregation method can be used to test block D-stability and generate sufficient conditions which encompass as special cases the two tests reported in Khalil and Kokotovic, (1979). The matrix A_f is viewed as an interconnection of the isolated subsystem matrices A_{ii} through the interconnections A_{ij}, $i \neq j$. When $\mathrm{Re}\lambda(A_{ii}) < 0$ one may construct a sufficient condition for $\mathrm{Re}\lambda(A_f) < 0$ by limiting in some sense the interconnections A_{ij}. In large scale system literature, Lyapunov methods have been used to construct such conditions. Let v_i be a Lyapunov function for the i-th subsystem. Siljak's approach is to form a vector Lyapunov function whose components are v_i. Michel's approach is to use a weighted sum of $v_1, \ldots, v_N$ as a new scalar Lyapunov function. We are going to show that if a matrix A_f with $\mathrm{Re}\lambda(A_{ii}) < 0$, $i = 1, \ldots, N$, can be shown to be stable using either Siljak's or Michel's approach, then A_f is also block D-stable.

Suppose that the time derivative of v_i along the trajectory of the interconnected system can be majored by the inequality

$$\dot{v}_i \leq \sum_{j=1}^{N} s_{ij} v_j, \quad s_{ii} < 0, \quad s_{ij} \geq 0 \tag{28}$$

for some scalars s_{ij}. Introducing the vector Lyapunov function v we rewrite (28) as $\dot{v} \leqq Sv$. Then using the comparison principle (Siljak, 1977) it follows that a sufficient condition for $\mathrm{Re}\lambda\{A_f\} < 0$ is that S be an M-matrix. Now suppose that A_f is multiplied by $D = \text{block-diag}[\alpha_1 I_1, \ldots, \alpha_N I_N]$. Choosing the same v as above we get that $\dot{v} \leqq \bar{D}Sv$ where $\bar{D} = \mathrm{diag}[\alpha_1, \ldots, \alpha_N]$. Since multiplying an M-matrix by a diagonal matrix with positive entries does not change its status (Siljak, 1977), it follows that $\bar{D}S$ is an M-matrix, hence DA_f is stable. Alternatively, let the new scalar Lyapunov function be $v = \sum_{i=1}^{N} \beta_i v_i$, where $\beta_i > 0$. Suppose there exist numbers β_i such that the time derivative of v is negative definite, than $\mathrm{Re}\lambda\{A_f\} < 0$. Now multiply A_f by D.

Choosing $v=\sum_{i=1}^{N}(\beta_i/\alpha_i)v_i$ as a Lyapunov function, we find that the time derivative is the same as before. Hence, $\mathrm{Re}\lambda\{DA_f\}<0$ for any $\alpha_i>0$. Subsequently A_f is block D-stable.

When the bounds m_i, M_i in (12) are given finite numbers, the class of matrices A_f satisfying (14) is less conservative than the class of block D-stable matrices. However, the search for sufficient conditions may be more difficult. Our problem is to find sufficient conditions for aymptotic stability of a family of linear systems $\dot{y}=DA_f y$ where D is function of the parameter vector $\alpha=(\alpha_1,\ldots,\alpha_N)$. The set Λ of allowable α is a convex polyhedron

$$\Lambda=\{\alpha:\underline{\alpha}\leq\alpha\leq\bar{\alpha},\alpha\in R^N\} \tag{29}$$

where $\underline{\alpha}$, $\bar{\alpha}$ are constant vectors depending on m_i, M_i, $i=1,\ldots,N$, and the inequality is understood to be component-wise. We denote by $\alpha^{(1)}$, $\alpha^{(2)}$, ..., the 2^N vertices of Λ, and the corresponding D matrix by $D^{(i)}=\text{block-diag }(\alpha_1^{(i)}I_1,\ldots,\alpha_N^{(i)}I_N)$. We notice that the elements of DA_f are linear functions of α. For this stability problem it has been established in (Horisberger and Belanger, 1976) that a sufficient condition for DA_f to be stable for all $\alpha\in\Lambda$ is the existence of a symmetric positive definite matrix P such that

$$PD^{(i)}A_f+A_f'D^{(i)}P<0,\quad i=1,2,\ldots,2^N. \tag{30}$$

This condition requires that the same P satisfies (30) for all vertices and in its present form is not finitely verifiable, that is there is no algorithm with a finite number of steps to verify the existence of P. However, the situation is greatly helped by the following result (Horisberger and Belanger, 1976), which reduces the test of (30) to a convex minimax problem with linear constraints.

Lemma. Assume that at least one of the $D^{(i)}A_f$ is known to be stable, then there exists a $P=P'>0$ such that (30) is satisfied if and only if

$$\min_{\|P\|<1}\{\max_{i=1,2,\ldots,2^N}\eta_i(P)\}<0 \tag{31}$$

where $\eta_i(P)=\lambda_{\max}(PD^{(i)}A_f+A_f'D^{(i)}P)$ for all real symmetric matrices.

5. ELECTRICAL NETWORK EXAMPLES

All the above conditions attempt to delineate classes of systems for which no physically meaningful combination of parameter perturbations will exist causing instability. As a well known example of such systems consider passive electrical networks which remain stable for all positive values of R, L, C-parameters and therefore cause no difficulties with multiparameter singular perturbations. Are all the passive networks D-stable? If so do they satisfy the known sufficiency conditions?

We consider first the case when A_f is the matrix of a passive RLC network without coupling between inductors. We are going to show that if the network is asymptotically stable, that is if $\mathrm{Re}\lambda(A_f)<0$ then A_f is D-stable. Let us assume that the network has n_l inductors and $n_C=\Sigma n_i-n_l$ capacitors. A standard choice of the state variables is

x_i = current through i-th inductor, when $i=1,\ldots,n_l$

x_j = voltage through j-th capacitor, when $j=n_l+1,\ldots,\Sigma n_i$.

Then the state space equation takes the form (Kalman, 1963)

$$T\dot{x}=Ax \tag{32}$$

where

$$T=\begin{bmatrix} L & O \\ O & C\end{bmatrix},\quad A=\begin{bmatrix} A_1 & -B' \\ B & A_2\end{bmatrix}.$$

Here L is a diagonal matrix whose positive diagonal element L_{ii} is the value of the i-th inductor, and C is a diagonal matrix with C_{jj} being the value of the j-th capacitor. The matrix A does not depend on the values of the inductors or the capacitors. It depends only on the network configuration and the values of the passive resistors. The matrices A_1 and A_2 are symmetric negative semidefinite.

Let us assume that this network is asymptotically stable and recall that this property does not depend on particular values of L and C, but solely on the presence of the resistors as dissipative elements. In other words if a network is asymptotically stable for one set of L, C values, it will remain asymptotically stable for any set of physically meaningful values of L and C. From the form of the system (32), we observe that changing values of L and C is equivalent to multiplying the system matrix from the left by a diagonal matrix. The preservation of asymptotic stability under this operation is the D-stability. Thus the network matrix $A_f=T^{-1}A$ is D-stable and therefore block D-stable. However it is easy to show that such a network does not have to satisfy any of the sufficient conditions of D-stability given in Johnson (1974) or Khalil and Kokotovic (1979). An example is the network in Fig. 1 whose A_1 and A_2 are negative semidefinite and whose matrix A_f is D-stable, but for which the known sufficiency tests fail.

Another interesting observation is that if the network contains mutual coupling between in-

ductors, then A_f is not necessarily D-stable. For example the network in Fig. 2 has

$$A_f = \tfrac{4}{7}\begin{bmatrix} -6 & -3 & -2 \\ \tfrac{9}{4} & \tfrac{1}{4} & \tfrac{3}{4} \\ -2 & -1 & -\tfrac{5}{4} \end{bmatrix} \tag{33}$$

which is not D-stable. The reason is that for the case of mutual coupling the matrix L becomes a positive definite symmetric matrix instead of being diagonal as in the case without mutual coupling. Multiplying A_f by a diagonal matrix, which is equivalent to multiplying L by a diagonal matrix, results now in a nonsymmetric matrix and hence cannot be interpreted as changing the values of inductors and capacitors. A special case is when mutual inductive coupling exists within each subnetwork, but there is no inductive coupling between different subnetworks.

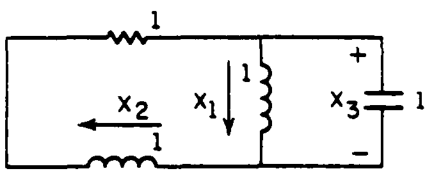

FIG. 1.

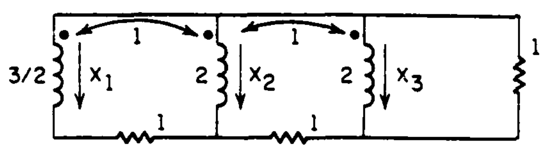

FIG. 2.

The matrix T of such a network takes the form

$$T = \text{block-diag}[T_1, \ldots, T_N], \tag{34}$$

where

$$T_i = \begin{bmatrix} L_i & 0 \\ 0 & C_i \end{bmatrix}. \tag{35}$$

Multiplying A_f by a diagonal matrix of the form $D = \text{block-diag}[\alpha_1 I_1, \ldots, \alpha_N I_N]$ is equivalent to multiplying L_i, C_i by α_i, which is merely scaling of the L and C elements of the i-th subsystem by α_i. Thus the matrix representation of such a network is block D-stable.

Finally, if the network is not passive then A_f is no longer block D-stable if $\mathrm{Re}\lambda(A_f) < 0$ for some values of L and C. However, A_f may still satisfy the boundary layer stability condition for some finite bounds. Consider for example the active network in Fig. 3, whose matrix

$$A_f = \begin{bmatrix} 1/L & 0 \\ 0 & 1/C \end{bmatrix}\begin{bmatrix} -1 & -1 \\ 1 & \tfrac{1}{2} \end{bmatrix}$$

has $\mathrm{Re}\lambda(A_f) < 0$ for $1/2 < L/C < \infty$.

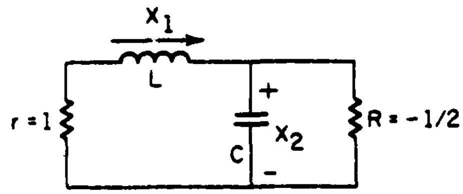

FIG. 3.

6. APPLICATION TO CONTROL AND GAME PROBLEMS

The problem to optimally control the system

$$\dot{x} = A_0 x + \sum_{j=1}^{N} A_{0j} z_j + B_0 u, \quad x(0) = x_0 \tag{36}$$

$$\varepsilon_i \dot{z}_i = A_{i0} x + \sum_{j=1}^{N} A_{ij} z_j + B_i u, \quad z_i(0) = z_{i0} \tag{37}$$

with respect to the performance index

$$J = \tfrac{1}{2}\int_0^{\infty} (y'y + u'Ru)\,\mathrm{d}t, \quad R > 0 \tag{38}$$

where

$$y = C_0 x + \sum_{j=1}^{N} C_j z_j. \tag{39}$$

is interpreted as a perturbation of a reduced problem

$$\dot{x} = A_0 x + A_{0f} z + B_0 u \tag{40}$$

$$0 = A_{f0} x + A_f z + B_f u \tag{41}$$

$$y = C_0 x + C_f z \tag{42}$$

in which $\varepsilon = 0$ and the matrices B_f, C_f are formed of the submatrices B_i, C_i, $i = 1, \ldots, N$, respectively. Assuming that A_f is nonsingular z is solved from (41) and eliminated from (40), (42) and (38). Then the reduced problem is to optimally control the system

$$\dot{x} = A_r x + B_r u \tag{43}$$

with respect to

$$J_r = \tfrac{1}{2}\int_0^{\infty} (x'C_r'C_r x + 2u'E_r'C_r x + u'R_r u)\,\mathrm{d}t \tag{44}$$

where

$$A_r = A_0 - A_{0f} A_f^{-1} A_{f0}, \quad B_r = B_0 - A_{0f} A_f^{-1} B_f,$$
$$C_r = C_0 - C_f A_f^{-1} B_f, \quad E_r = -C_f A_f^{-1} B_f,$$
$$R_r = R + E_r' E_r. \tag{45}$$

The reduced problem (43), (44) is considerably simpler than the original problem (36)–(38) because of the elimination of the fast variables and the reduction of the system order. One of the tasks of the singular perturbation analysis is to establish whether the full problem is well-posed in the sense that its solution tends to the solution of the reduced problem as $\varepsilon \to 0$ in H. If so, then the next task is to use the solution of the reduced problem as a basis for a simplified design procedure.

If the triple (A_r, B_r, C_r) is stabilizable-detectable, then the optimal control for the reduced problem is (Chow and Kokotovic, 1976)

$$u = -R_r^{-1}(E_r'C_r + B_r'K_r)x \tag{46}$$

where K_r is the unique positive semidefinite solution of the Riccati equation

$$K_r(A_r - B_rR_r^{-1}E_r'C_r) + (A_r - B_rR_r^{-1}E_r'C_r)'K_r - K_rB_rR_r^{-1}B_r'K_r + C_r'(I - E_rR_r^{-1}E_r')C_r = 0. \tag{47}$$

On the other hand, the optimal control for the original problem is

$$u = -R^{-1}B'K\begin{bmatrix} x \\ z \end{bmatrix} \tag{48}$$

where K is the stabilizing solution of the Riccati equation

$$KA + A'K + C'C - KSK = 0 \tag{49}$$

and

$$A = \begin{bmatrix} A_0 & A_{0f} \\ \frac{1}{\mu(\varepsilon)}D(\varepsilon)A_{f0} & \frac{1}{\mu(\varepsilon)}D(\varepsilon)A_f \end{bmatrix},$$

$$B = \begin{bmatrix} B_0 \\ \frac{1}{\mu(\varepsilon)}D(\varepsilon)B_f \end{bmatrix},$$

$$C = [C_0 \quad C_f], \quad S = BR^{-1}B'. \tag{50}$$

To avoid unboundedness, the solution of (49) is sought in the form

$$K = K(\varepsilon) = \begin{bmatrix} K_1(\varepsilon) & \mu(\varepsilon)K_2(\varepsilon) \\ \mu(\varepsilon)K_2'(\varepsilon) & \mu(\varepsilon)K_3(\varepsilon) \end{bmatrix}. \tag{51}$$

The asymptotic behavior of K as $\varepsilon \to 0$ is given in the following theorem.

Theorem 3. If A_f satisfies (14) and (A_r, B_r, C_r) is stabilizable-detectable, then for all sufficiently small $\varepsilon \in H$

$$K_1(\varepsilon) = K_r + O(\|\varepsilon\|) \tag{52}$$

$$K_2(\varepsilon) = K_m(\varepsilon) + O(\|\varepsilon\|) \tag{53}$$

$$K_3(\varepsilon) = K_f(\varepsilon) + O(\|\varepsilon\|) \tag{54}$$

where K_r is the positive semidefinite solution of (47),

$$K_m = [K_r(B_0R^{-1}B_f'D(\varepsilon)K_f(\varepsilon) - A_{0f}) - (A_{f0}'D(\varepsilon)K_f(\varepsilon) + C_0'C_f)] \cdot [D(\varepsilon)A_f - D(\varepsilon)B_fR^{-1}B_f'D(\varepsilon)K_f(\varepsilon)]^{-1} \tag{55}$$

and $K_f(\varepsilon)$ is the positive semidefinite solution of the Riccati equation

$$K_f(\varepsilon)D(\varepsilon)A_f + A_f'D(\varepsilon)K_f(\varepsilon) + C_f'C_f - K_f(\varepsilon)D(\varepsilon)B_fR^{-1}B_f'D(\varepsilon)K_f(\varepsilon) = 0. \tag{56}$$

The proof is given in Appendix A. We notice that since for every $\varepsilon \in H$ the matrix $D(\varepsilon)A_f$ is stable, the Riccati equation (56) has a unique positive semidefinite stabilizing solution $K_f(\varepsilon)$. Thus the matrix $D(\varepsilon)(A_f - B_fR^{-1}B_f'D(\varepsilon)K_f(\varepsilon))$ is stable and its inverse in (55) is well-defined.

When ε is scalar the limiting values of K_2, K_3 are not dependent on ε (Chow and Kokotovic, 1976). In contrast, we see from (53), (54) that in the multiparameter problem the limiting behavior of K_2 and K_3 depends on ε, and hence on the particular path along which $\varepsilon \to 0$. However, it is seen from (51) that both K_2 and K_3 are multiplied by $\mu(\varepsilon)$ which tends to zero as $\varepsilon \to 0$. Therefore, the part of K that is crucial as $\varepsilon \to 0$ is only $K_1(\varepsilon)$ which tends to the solution of the reduced problem K_r as $\varepsilon \to 0$ along any arbitrary path in H. In other words, the optimal value function of the full problem $J_{opt} = \frac{1}{2}[x_0/z_0]'K[x_0/z_0]$ tends to the optimal value function of the reduced problem $J_{r_{opt}} = \frac{1}{2}x_0'K_rx_0$ as $\varepsilon \to 0$ in H. Thus, the optimal solution of the full problem is well-posed. The next theorem utilizes this well-posedness property.

Theorem 4. If A_f satisfies (14) and (A_r, B_r, C_r) is stabilizable-detectable, then the use of the reduced control

$$u_r = -R_r^{-1}(E_r'C_r + B_r'K_r)x \tag{57}$$

results in J satisfying

$$J = J_{opt} + O(\|\varepsilon\|), \quad \forall \varepsilon \in H. \tag{58}$$

Theorem 4 is proved in Appendix B.

Remark 4. The reduced control (57) does not require knowing the values of the parameters ε_1, .

. ., ε_N. Thus it is suitable for cases when $\varepsilon_1, \ldots, \varepsilon_N$ are small uncertain parameters taking values in H.

Remark 5. In the single parameter optimal control problem (Chow and Kokotovic, 1976) it is possible to achieve $O(\varepsilon^2)$ approximation of the optimal value of the performance index without knowing the value of ε. This is done by adding to the reduced control (57) another term which accounts for the optimization of the fast modes. An analogous possibility does not exist in the multiparameter problem because such a term would involve $K_f(\varepsilon)$, which requires the knowledge of ε. Even if the value of ε, say ε_0, is known and K_f is evaluated at ε_0, the matrix $[D(\varepsilon)A_f - D(\varepsilon)B_f R^{-1}B_f' D(\varepsilon)K_f(\varepsilon_0)]$ is not necessarily stable for all $\varepsilon \in H$. This means that including K_f in the control, limits its validity to small variations of ε around ε_0.

Remark 6. The intuitive result of Theorem 4 should not be surprising to people familiar with the regulator problem. However, it should be noticed that this result is a consequence of the well-posedness of the optimal control problem and it is no longer true for ill-posed problems. We illustrate this by discussing Nash games.

In the design of multi-input control problems, the objectives in the optimal policy may be met by formulating the control problem as a differential game. In a game where cooperation among the players cannot be guaranteed, a solution is the Nash equilibrium strategy. For the multiparameter singularly perturbed system

$$\dot{x} = A_0 x + A_{0f} z + B_{01} u_1 + B_{02} u_2, \quad x(0) = x_0 \tag{59}$$

$$\mu(\varepsilon)\dot{z} = D(\varepsilon)(A_{f0} x + A_f z + B_{f1} u_1 + B_{f2} u_2), \; z(0) = z_0 \tag{60}$$

the i-th player chooses his strategy u_i to minimize his performance index

$$J_i = \tfrac{1}{2} \int_0^\infty \left(\begin{bmatrix} x \\ z \end{bmatrix}' Q_i \begin{bmatrix} x \\ z \end{bmatrix} + u_i' R_{ii} u_i + u_j' R_{ij} u_j \right) dt,$$
$$R_{ii} > 0, \quad Q_i \geqq 0, \quad i = 1, 2. \tag{61}$$

A Nash equilibrium solution of this game is a pair (u_1^*, u_2^*) such that

$$J_i(u_i^*, u_j^*) \leqq J_i(u_i, u_j^*), \quad i \neq j, \quad i = 1, 2 \tag{62}$$

We discuss here the closed-loop linear solution of the two player game (59)–(61) which is given by (Starr and Ho, 1969)

$$u_i = -R_{ii}^{-1} B_i' K_i \begin{bmatrix} x \\ z \end{bmatrix}, \quad i = 1, 2 \tag{63}$$

where (K_1, K_2) is a solution of the coupled algebraic Riccati equations

$$0 = K_i A + A' K_i + Q_i - K_i B_i R_{ii}^{-1} B_i' K_i$$
$$- K_i B_j R_{jj}^{-1} B_j' K_j - K_j B_j R_{jj}^{-1} B_j' K_i$$
$$+ K_j B_j R_{jj}^{-1} R_{ij} R_{jj}^{-1} B_j' K_j,$$
$$i \neq j, \quad i, j = 1, 2. \tag{64}$$

such that $\mathrm{Re}\lambda(A - B_1 R_{11}^{-1} B_1' K_1 - B_2 R_{22}^{-1} B_2' K_2) < 0$. The matrix A is as in (50), $B_i' = (B_{0i}' \; (1/\mu(\varepsilon)) B_{fi}' D(\varepsilon))$ and $K_i = K_i(\varepsilon)$ is sought in the form (51).

Closed-loop solutions of singularly perturbed Nash games when ε is scalar have been investigated in (Gardner and Cruz, 1978). It turns out that setting $\varepsilon = 0$ at two different stages leads to different solutions. This means that setting $\varepsilon = 0$ in the original problem to obtain a reduced problem and solve it is in general different from solving the original problem first and then setting $\varepsilon = 0$. The reason of this phenomenon has been discussed in Khalil (1978) and we briefly summarize it here. The necessary conditions for a Nash solution (Starr and Ho, 1969) depend on the partial derivatives $\partial u_i/\partial x$, $\partial u_i/\partial z$. Therefore the Nash game unlike the optimal control problem has different open-loop and closed-loop solutions. When the Nash game is singularly perturbed the difference in the solutions obtained by setting $\varepsilon = 0$ at two different stages is due to the partial derivative $\partial u_i/\partial z$. Setting $\varepsilon = 0$ to obtain a reduced problem, one eliminates z. Hence automatically $\partial u_i/\partial z$ is set equal to zero. When the original problem is solved first and then ε is set equal to zero it is not necessarily true that $\partial u_i/\partial z$ is zero. Thus one should not expect the closed-loop solution of the full problem to tend to the closed-loop solution of the reduced problem as $\varepsilon \to 0$.

In the multiparameter Nash problem (when ε is a vector) the situation is richer. We first give some examples and then discuss the asymptotic behavior of K_1, K_2 as $\varepsilon \to 0$ in H.

Example 1. Consider the system

$$\dot{x} = -x + z_1 + z_2 \tag{65}$$

$$\varepsilon_1 \dot{z}_1 = -z_1 + u_1 + 2u_2 \tag{66}$$

$$\varepsilon_2 \dot{z}_2 = -z_2 + u_1 + u_2 \tag{67}$$

and the performance indices

$$J_i = \tfrac{1}{2} \int_0^\infty (x^2 + z_1^2 + z_2^2 + u_i^2) dt. \tag{68}$$

We solve the associated coupled algebraic Riccati equation (64) where K_i takes the form

$$K_i=\begin{bmatrix} K_{i1} & \mu K_{i2} \\ \mu K_{i2} & \mu K_{i3} \end{bmatrix}.$$

We let $\varepsilon \to 0$ along different paths. For simplicity we take the paths to be the rays $\varepsilon_2/\varepsilon_1 = 0.25$, $\varepsilon_2/\varepsilon_1 = 4$. The limiting values of K_{11}, K_{21} are shown in Table 1, where it is obvious that the limiting behaviour of K_{11}, K_{21} depends on the particular path along which $\varepsilon \to 0$.

$\varepsilon_2/\varepsilon_1$	0.25	1	4
K_{11}	0.0785	0.0827	0.084
K_{21}	0.3873	0.3725	0.3595

Example 2. Consider the system

$$\dot{x} = -x + z_1 + z_2 \tag{69}$$

$$\varepsilon_i \dot{z}_i = -z_i + u_1 + u_2, \quad i = 1, 2 \tag{70}$$

with the performance indices as in (68).

Solving (64) we find that $K_1 = K_2$ and $\lim_{\varepsilon \to 0} K_{i1} = 0.3777$ as $\varepsilon \to 0$ along any arbitrary path. However if we set $\varepsilon_1 = \varepsilon_2 = 0$ in (70), eliminate z and solve the resulting reduced game we get that $K_{1r} = K_{2r} = 0.375$ which is different from $\lim_{\varepsilon \to 0} K_{i1}$.

Example 3. Consider the system

$$\dot{x} = z_1 - z_2 \tag{71}$$

$$\varepsilon_i \dot{z}_i = -z_i + u_i, \quad i = 1, 2. \tag{72}$$

with

$$J_i = \tfrac{1}{2}\int_0^\infty (x^2 + z_i^2 + u_i^2). \tag{73}$$

Solving the full game we find that as $\varepsilon \to 0$ along any arbitrary path $K_{i1} \to K_{ir} = \sqrt{2/3}$, $i = 1, 2$, where K_{1r}, K_{2r} are the solution of the reduced game.

Example 1 shows that, in general, the limit of the closed-loop solution of the full Nash game depends on the path along which $\varepsilon \to 0$. Example 2 shows a special case when the limit does not depend on the path but it is different from the closed-loop solution of the reduced game. For this case one may try to define a reduced problem whose solution is the limit of the full solution, as in the single parameter case (Gardner and Cruz, 1978). Finally, Example 3 shows a well-posed special case where the closed-loop reduced solution may be used to approximate the closed-loop full solution as in the optimal control problem. Notice that in the three examples A_f is D-stable. Hence control problems for these systems would be well-posed.

6. CONCLUSIONS

The basic difficulty in extending the single parameter singular perturbation techniques to the multi-parameter case is in testing the stability of the boundary layer system. Although block D-stability is a sufficient condition for the boundary layer stability, there are two limitations on using it. First, we still do not have a complete characterization of the class of block D-stable matrices. Second, for most problems of interest block D-stability is a conservative condition and more work is needed to develop conditions different from block D-stability to check the boundary layer stability.

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APPENDIX A

Proof of Theorem 3. The substitution of (51) into (49) yields the equations

$$K_1A_0+A_0'K_1+K_2DA_{f0}+A_{f0}'DK_2'+C_0'C_0-K_1S_1K_1 -K_1S_2DK_2'-K_2DS_2'K_1-K_2DS_3DK_2'=0 \quad (A1)$$

$$K_1A_{0f}+K_2DA_f+\mu A_0'K_2+A_{f0}'DK_3+C_0'C_f-\mu K_1S_1K_2 -K_1S_2DK_3-\mu K_2DS_2'K_2-K_2DS_3DK_3=0 \quad (A2)$$

$$\mu K_2'A_{0f}+\mu A_{0f}'K_2+K_3DA_f+A_f'DK_3+C_f'C_f -\mu^2K_2'S_1K_2-\mu K_2'S_2DK_3-\mu K_3DS_2'K_2-K_3DS_3DK_3=0 \quad (A3)$$

where

$$S_1=B_0R^{-1}B_0', \quad S_2=B_0R^{-1}B_f' \quad \text{and} \quad S_3=B_fR^{-1}B_f'.$$

Let $\bar K_1$, $\bar K_2$, and $\bar K_3$ satisfy the equations

$$\bar K_1A_0+A_0'\bar K_1+\bar K_2DA_{f0}+A_{f0}'D\bar K_2'+C_0'C_0 -\bar K_1S_1\bar K_1-\bar K_1S_2D\bar K_2'-\bar K_2DS_2'\bar K_1-\bar K_2DS_3D\bar K_2'=0 \quad (A4)$$

$$\bar K_1A_{0f}+\bar K_2DA_f+A_{f0}'D\bar K_3+C_0'C_f-\bar K_1S_2D\bar K_3-\bar K_2DS_3D\bar K_3=0 \quad (A5)$$

$$\bar K_3DA_f+A_f'D\bar K_3+C_f'C_f-\bar K_3DS_3D\bar K_3=0. \quad (A6)$$

We first consider the solution of (A4)–(A6), then we establish the relation between K_i and $\bar K_i$. Under assumption (14), equations (56) and (A6) imply that $\bar K_3(\varepsilon)=K_f(\varepsilon)$. Hence solving (A5) for $\bar K_2$ we get

$$\bar K_2=[\bar K_1(S_2DK_f-A_{0f})-(C_0'C_f+A_{f0}'DK_f)][DA_f-DS_3DK_f]^{-1} \triangleq \bar K_1E_1-E_2. \quad (A7)$$

Substituting (A7) into (A4) results in the Riccati equation

$$\bar K_1\hat A+\hat A'\bar K_1+\hat Q-\bar K_1\hat BR^{-1}\hat B'\bar K_1=0 \quad (A8)$$

where

$$\hat A=A_0+E_1DA_{f0}+S_2DE_2'+E_1DS_3DE_2' \quad (A9)$$

$$\hat B=B_0+E_1DB_f \quad (A10)$$

$$\hat Q=C_0'C_0-E_2DA_{f0}-A_{f0}'DE_2'-E_2DS_3DE_2'. \quad (A11)$$

Algebraic manipulation using (56) yields

$$\hat A=A_r-B_rR_r^{-1}E_r'C_r \quad (A12)$$

$$\hat BR^{-1}\hat B'=B_rR_r^{-1}B_r' \quad (A13)$$

$$\hat Q=C_r'(I-E_rR_r^{-1}E_r')C_r. \quad (A14)$$

Under the assumption that (A_r, B_r, C_r) is stabilizable-detectable, (47) and (A8) imply that $\bar K_1=K_r$. Hence $\bar K_1$ is not dependent on ε. From (55) and (A7), $\bar K_2=K_m$.

Subtracting (A4), (A5), and (A6) from (A1), (A2) and (A3), respectively, and using that the elements of D are bounded for $\varepsilon\in H$, we find that $\Delta K_1=K_1-\bar K_1$, $\Delta K_2=K_2-\bar K_2$, $\Delta K_3=K_3-\bar K_3$ satisfy the equations

$$\Delta K_1A_0+A_0'\Delta K_1+\Delta K_2DA_{f0}+A_{f0}'D\Delta K_2'-\Delta K_1S_1\Delta K_1 -\Delta K_1S_2D\Delta K_2'-\Delta K_2DS_2'\Delta K_1-\Delta K_2DS_3D\Delta K_2' -\Delta K_1S_1K_r-K_rS_1\Delta K_1-\Delta K_1S_2DK_m' -K_mDS_2'\Delta K_1-K_rS_2D\Delta K_2'-\Delta K_2DS_2'K_r -\Delta K_2DS_3DK_m'-K_mDS_3D\Delta K_2'=0 \quad (A15)$$

$$\Delta K_1A_{0f}+\Delta K_2DA_f+A_{f0}'D\Delta K_3-\Delta K_1S_2D\Delta K_3-K_rS_2D\Delta K_3 -\Delta K_1S_2DK_f-\Delta K_2DS_3D\Delta K_3-\Delta K_2DS_3DK_f -K_mDS_3D\Delta K_3+O(\|\varepsilon\|)=0 \quad (A16)$$

$$\Delta K_3[DA_f-DS_3DK_f]+[DA_f-DS_3DK_f]'\Delta K_3 -\Delta K_3DS_3D\Delta K_3+O(\|\varepsilon\|)=0. \quad (A17)$$

From (A17) we get that $\Delta K_3=O(\|\varepsilon\|)$. Then from (A16) we have

$$\Delta K_2=\Delta K_1E_1+O(\|\varepsilon\|). \quad (A18)$$

Substituting (A18) into (A15), it can be shown that ΔK_1 satisfies the equation

$$\Delta K_1(A_r-B_rR_r^{-1}E_r'C_r-B_rR_r^{-1}B_r'K_r)+(A_r-B_rR_r^{-1}E_r'C_r -B_rR_r^{-1}B_r'K_r)'\Delta K_1-\Delta K_1B_rR_r^{-1}B_r'\Delta K_1+O(\|\varepsilon\|)=0. \quad (A19)$$

Since $(A_r-B_rR_r^{-1}E_r'C_r-B_rR_r^{-1}B_r'K_r)$ is stable it follows that $\Delta K_1=O(\|\varepsilon\|)$, hence $\Delta K_2=O(\|\varepsilon\|)$.

APPENDIX B

Proof of Theorem 4. Let

$$H=I+R^{-1}B_f'DK_f(A_f-S_3DK_f)^{-1}B_f. \quad (B1)$$

It can be easily shown that

$$H^{-1}=I-R^{-1}B_f'DK_fA_f^{-1}B_f \quad (B2)$$

$$R_r^{-1}=HR^{-1}H' \quad (B3)$$

$$B_rH=\hat B=B_0+E_1DB_f \quad (B4)$$

$$A_f^{-1}B_fH=(A_f-S_3DK_f)^{-1}B_f \quad (B5)$$

$$HR^{-1}B_f'D=R^{-1}B_f'D[I+K_f(A_f-S_3DK_f)^{-1}S_3D]. \quad (B6)$$

Using (B1)–(B6) we rewrite (57) in a more convenient form

$$\begin{aligned}u_r&=[HR^{-1}H'B_f'A_f'^{-1}C_f'(C_0-C_fA_f^{-1}A_{f0})-HR^{-1}H'B_r'K_r]x\\&=[HR^{-1}B_f'(A_f-S_3DK_f)'^{-1}(C_f'C_0+(K_fDA_f+A_f'DK_f-K_fDS_3DK_f)A_f^{-1}A_{f0})\\&\quad-HR^{-1}(B_0+E_1DB_f)'K_r]x\\&=[HR^{-1}B_f'(A_f-S_3DK_f)'^{-1}(C_f'C_0+K_fDA_{f0})+HR^{-1}B_f'DK_fA_f^{-1}A_{f0}\\&\quad-R^{-1}B_0'K_r-R^{-1}B_f'DK_f(A_f-S_3DK_f)^{-1}S_2'K_r-HR^{-1}B_f'DE_1'K_r]x\\&=-R^{-1}B_0'K_rx-HR^{-1}B_f'D[K_m'+K_fA_f^{-1}S_2'K_r-K_fA_f^{-1}A_{f0}]x\\&=-R^{-1}B_0'K_rx-R^{-1}B_f'D[I+K_f(A_f-S_3DK_f)^{-1}S_3D][K_m'+K_fA_f^{-1}(S_2'K_r-A_{f0})]x\\&=-R^{-1}B'\begin{bmatrix}K_r & 0\\ \mu K_p' & 0\end{bmatrix}\begin{bmatrix}x\\ z\end{bmatrix}=-R^{-1}B'L\begin{bmatrix}x\\ z\end{bmatrix}\end{aligned} \quad (B7)$$

where

$$K_p=[K_m+(K_rS_2-A_{f0}')A_f'^{-1}K_f][I+DS_3(A_f-S_3DK_f)'^{-1}K_f]. \quad (B8)$$

When u_r is used, the value of the performance index is

$$J=\tfrac{1}{2}\begin{bmatrix}x_0\\ z_0\end{bmatrix}'W\begin{bmatrix}x_0\\ z_0\end{bmatrix} \quad (B9)$$

where W is the positive semidefinite solution of the Lyapunov equation

$$W(A-SL)+(A-SL)'W+L'RL+Q=0. \qquad (B10)$$

On the other hand the optimal value is

$$J_{opt}=\tfrac{1}{2}\begin{bmatrix}x_0\\ z_0\end{bmatrix}'K\begin{bmatrix}x_0\\ z_0\end{bmatrix} \qquad (B11)$$

where K is the stabilizing solution of (49). Subtracting (49) from (B10) we find that $V=W-K$ satisfies the Lyapunov equation

$$V(A-SL)+(A-SL)'V+(K-L)'S(K-L)=0. \qquad (B12)$$

We seek V in the form (51), that is

$$V=\begin{bmatrix}V_1(\varepsilon) & \mu(\varepsilon)V_2(\varepsilon)\\ \mu(\varepsilon)V_2'(\varepsilon) & \mu(\varepsilon)V_3(\varepsilon)\end{bmatrix}. \qquad (B13)$$

Substituting (B13) into (B12) yields the equations

$$\begin{aligned}&V_1(A_0-S_1K_r-S_2DK_p')+(A_0-S_1K_r-S_2DK_p')'V_1+V_2D(A_{f0}\\ &-S_2'K_r-S_3DK_p')+(A_{f0}-S_2'K_r-S_3DK_p')'DV_2'\\ &+(K_1-K_r)S_1(K_1-K_r)+(K_1-K_r)S_2D(K_2-K_p)'\\ &+(K_2-K_p)DS_2'(K_1-K_r)+(K_2-K_p)DS_3D(K_2-K_p)'=0\end{aligned} \qquad (B14)$$

$$\begin{aligned}&V_1A_{0f}+V_2DA_f+\mu(A_0-S_1K_r-S_2DK_p')'V_2\\ &+(A_{f0}-S_2'K_r-S_3DK_p')'DV_3+\mu(K_1-K_r)S_1K_2+(K_1-K_r)S_2DK_3\\ &+\mu(K_2-K_p)DS_2'K_2+(K_2-K_p)DS_3DK_3=0\end{aligned} \qquad (B15)$$

$$\begin{aligned}&\mu V_2'A_{0f}+\mu A_{0f}'V_2+V_3DA_f+A_f'DV_3+\mu^2K_2'S_1K_2\\ &+\mu K_2'S_2DK_3+\mu K_3DS_2'K_2+K_3DS_3DK_3=0.\end{aligned} \qquad (B16)$$

Let $\bar{V}_1$, $\bar{V}_2$, and $\bar{V}_3$ satisfy the equations

$$\begin{aligned}&\bar{V}_1(A_0-S_1K_r-S_2DK_p')+(A_0-S_1K_r-S_2DK_p')'\bar{V}_1+\bar{V}_2D(A_{f0}\\ &-S_2'K_r-S_3DK_p')+(A_{f0}-S_2'K_r-S_3DK_p')'D\bar{V}_2'\\ &+(K_m-K_p)DS_3D(K_m-K_p)'=0\end{aligned} \qquad (B17)$$

$$\begin{aligned}&\bar{V}_1A_{0f}+\bar{V}_2DA_f+(A_{f0}-S_2'K_r-S_3DK_p')'D\bar{V}_3+(K_m\\ &-K_p)DS_3DK_f=0\end{aligned} \qquad (B18)$$

$$\bar{V}_3DA_f+A_f'D\bar{V}_3+K_fDS_3DK_f=0. \qquad (B19)$$

We first evaluate $\bar{V}_1$, $\bar{V}_2$, and $\bar{V}_3$. Since A_f satisfies (14), there exists a unique positive semidefinite solution $\bar{V}_3$ of (B19). Expressing $\bar{V}_2$ in terms of $\bar{V}_1$ and $\bar{V}_3$, substituting in (B17) and using (B19) to eliminate $\bar{V}_1$ we obtain

$$\begin{aligned}&\bar{V}_1(\bar{A}_0-A_{0f}A_f^{-1}\bar{A}_{f0})+(\bar{A}_0-A_{0f}A_f^{-1}\bar{A}_{f0})'\bar{V}_1\\ &+(K_m-K_p-\bar{A}_{f0}'A_f'^{-1}K_f)DS_3D(K_m-K_p-\bar{A}_{f0}'A_f'^{-1}K_f)'=0\end{aligned} \qquad (B20)$$

where

$$\bar{A}_0=A_0-S_1K_r-S_2DK_p', \quad \bar{A}_{f0}=A_{f0}-S_2'K_r-S_3DK_p'. \qquad (B21)$$

From (B8) we have

$$\begin{aligned}&K_pDS_3=[K_m+(K_rS_2-A_{f0}')A_f'^{-1}K_f]DS_3(A_f-S_3DK_f)'^{-1}A_f'\\ &K_pDS_3A_f'^{-1}=[K_m+(K_rS_2-A_{f0}')A_f'^{-1}K_f]DS_3(A_f-S_3DK_f)'^{-1}.\end{aligned} \qquad (B22)$$

Thus

$$\begin{aligned}&K_p=K_m+(A_{f0}-S_2'K_r)'A_f'^{-1}K_f+K_pDS_3A_f'^{-1}K_f\\ &0=K_m-K_p-\bar{A}_{f0}'A_f'^{-1}K_f.\end{aligned} \qquad (B23)$$

Furthermore, $\bar{A}_0-A_{0f}A_f^{-1}\bar{A}_{f0}=A_r-B_rR_r^{-1}(E_r'C_r+B_r'K_r)$, which is stable. Thus the solution of (B20) is $\bar{V}_1=0$. Substracting (B20) from (B14) and using Theorem 3 we obtain that $V_1=\bar{V}_1+O(\|\varepsilon\|)$. Hence

$$V_1=O(\|\varepsilon\|), \quad \varepsilon\in H \qquad (B24)$$

implying (58).

SECTION 9

WELL-POSEDNESS OF DIFFERENTIAL GAMES

Well-Posedness of Singularly Perturbed Nash Games†

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ABSTRACT: *This paper discusses linear-quadratic infinite-time nonzero-sum closed-loop Nash games for systems with fast and slow modes. It is shown via example that the usual order reduction processes utilizing control ideas of singular perturbation analysis leads to an ill-posed reduced order problem. A modification of the performance indices is presented which leads to a well-posed problem, when the usual order reduction method is used. Finally, a hierarchical reduction procedure is proposed which leads to well-posed fast and slow game problems even when the performance indices are not modified.*

I. Introduction

In a general multi-input system there may be many decision makers or players each trying to minimize his own performance index. The system is described by a vector differential equation and the performance indices are functions of control input vectors and state vectors over some period of time. We consider the case where the system equation is linear and the performance indices are quadratic functions of state and control vectors. A particular strategy, or rationale for choosing controls, is the Nash strategy which is appropriate when cooperation among the players cannot be guaranteed. It has the advantage that if one player deviates unilaterally from the Nash strategy his performance index will not improve. When the sum of the performance indices is zero the game is called zero-sum, otherwise the game is called nonzero-sum. An early paper on Nash strategy is given by (1) and necessary conditions for open- and closed-loop Nash strategies have been presented in (2) and (3) respectively.

When the system has slow and fast modes, the control problem is ill-conditioned, that is it is numerically "stiff". To alleviate this ill-conditioning and to reduce the amount of computation, singular perturbation techniques have been developed, some of which are presented in (4, 5).

In this paper we investigate the well-posedness of closed-loop Nash strategies with respect to singular perturbation. There are two principal reasons for singular perturbation. First the model can only be an approximation of the actual system and we must insure that the control is robust with respect to neglected fast dynamics. A second major reason is computational simplification

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when an original full order game is approximately decomposed into fast and slow subgames.

We give an example of a nonzero-sum Nash game whereby the natural singular perturbation leads to a strategy which results in limiting values of performance indices different from the limiting values of those corresponding to the full order Nash strategy. In contrast we have shown elsewhere (6) that the corresponding performance indices have the same limiting values when the game is zero-sum.

We then show that a physically justified modification of the performance indices consistent with inadequate modeling of fast dynamics results in a well-posed singularly perturbed nonzero-sum Nash game problem when the natural perturbation method is applied. With this modification, computational savings can be gained and a close approximation to the optimal performance indices obtained by order reduction of the Riccati equations.

Finally, we present a hierarchical reduction procedure which leads to a well-posed singularly perturbed modified slow game. This reduced order slow game differs from the natural one in that it contains information about the low order fast game. The problem is well-posed with respect to the original performance index for the full order game. Computational savings and a close approximation of the performance indices are achieved.

II. Ill-Posedness of Nonzero-Sum Nash Games with Respect to Singular Perturbation

Consider a singularly perturbed time-invariant system

$$\dot{x}_1 = A_{11}x_1 + A_{12}x_2 + B_{11}u_1 + B_{12}u_2; \quad x_1(t_0) = x_{10} \tag{1a}$$

$$\mu\dot{x}_2 = A_{21}x_1 + A_{22}x_2 + B_{21}u_1 + B_{22}u_2; \quad x_2(t_0) = x_{20} \tag{1b}$$

and performance criteria

$$J_i = \frac{1}{2}\int_{t_0}^{\infty} \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' \begin{bmatrix} Q_{i1} & Q_{i2} \\ Q_{i2}' & Q_{i3} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + u_i' R_{ii} u_i + u_j' R_{ij} u_j \right\} dt; \quad i, j = 1, 2, i \neq j \tag{2}$$

where μ is a small positive scalar, x_1 and x_2 are n_1 and n_2 dimensional components of the state vector, u_1 and u_2 are m_1 and m_2 dimensional control vectors to be chosen by Players 1 and 2 respectively in accordance with the Nash solution concept, and the control strategies are restricted to be linear feedback functions of the state. Denote

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21}/\mu & A_{22}/\mu \end{bmatrix}, \quad B_i = \begin{bmatrix} B_{1i} \\ B_{2i}/\mu \end{bmatrix}, \quad \text{and} \quad Q_i = \begin{bmatrix} Q_{i1} & Q_{i2} \\ Q_{i2}' & Q_{i3} \end{bmatrix}.$$

The usual definiteness assumptions are made on Q_i and R_{ij}, $i, j = 1, 2$.

The optimal closed-loop Nash strategy for Player i for (2) subject to (1) is well known (3) and given by

$$u_i = -R_{ii}^{-1} B_i' K_i x \tag{3}$$

where K_i is a stabilizing solution of the coupled Riccati equations given by

$$0=-(Q_i+K_iA+A'K_i)+K_iB_iR_{ii}^{-1}B_i'K_i+K_iB_jR_{jj}^{-1}B_j'K_j+K_jB_jR_{jj}^{-1}B_j'K_i$$
$$-K_jB_jR_{jj}^{-1}R_{ij}R_{jj}^{-1}B_j'K_j, \quad \text{for} \quad i,j=1,2; \quad i\neq j. \quad (4)$$

Notice that since A and B_i are functions of the small parameter μ, K_i is also a function of μ. In general even for low order problems the presence of μ causes numerical "stiffness" in (4). For this reason and for computational reduction the problem (1), (2) in the one Player, i.e. control, case is generally approximated by a lower order problem by formally setting $\mu=0$. This produces a control which when applied to the full order plant gives a close approximation to the optimal cost. In this section we examine the standard reduction method.

The standard approach to obtaining a reduced order model is to set $\mu=0$ in (1b), solve for x_2, assuming A_{22} is non-singular, and substitute in (1a) to obtain

$$\dot{\bar{x}}_1=(A_{11}-A_{12}A_{22}^{-1}A_{21})\bar{x}_1+(B_{11}-A_{12}A_{22}^{-1}B_{21})\bar{u}_1$$
$$+(B_{12}-A_{12}A_{22}^{-1}B_{22})\bar{u}_2, \quad \bar{x}_1(t_0)=x_{10} \quad (5)$$

where the bar indicates that $\mu=0$. Rewriting (5) we get the "slow" (since setting $\mu=0$ is equivalent to saying that the fast states are infinitely fast) subsystem

$$\dot{x}_s=A_0x_s+B_{01}u_{1s}+B_{02}u_{2s}, \quad x_s(t_0)=x_{10} \quad (6)$$

where

$$A_0=A_{11}-A_{12}A_{22}^{-1}A_{21},$$
$$B_{0i}=B_{1i}-A_{12}A_{22}^{-1}B_{2i}, \quad i=1,2.$$

The corresponding "slow" performance criteria found by substituting x_2 when $\mu=0$ into (2) is

$$J_{is}=\frac{1}{2}\int_{t_0}^{\infty}\{x_s'\hat{Q}_{i1}x_s+2x_s'\hat{Q}_{i2}(B_{2i}u_{is}+B_{2j}u_{js})+u_{is}'\hat{R}_{ii}u_{is}+u_{js}'\hat{R}_{ij}u_{js}$$
$$+2u_{is}'\hat{Q}_{i3}u_{js}\}\,dt \quad i,j=1,2, \quad i\neq j \quad (7)$$

where

$$\hat{Q}_{i1}=Q_{i1}-Q_{i2}A_{22}^{-1}A_{21}-(A_{22}^{-1}A_{21})'Q_{i2}'+(A_{22}^{-1}A_{21})'Q_{i3}A_{22}^{-1}A_{21},$$
$$\hat{Q}_{i2}=(A_{22}^{-1}A_{21})'Q_{i3}A_{22}^{-1}-Q_{i2}A_{22}^{-1},$$
$$\hat{R}_{ii}=R_{ii}+B_{2i}'(A_{22}^{-1})'Q_{i3}A_{22}^{-1}B_{2i},$$
$$\hat{R}_{ij}=R_{ij}+B_{2j}'(A_{22}^{-1})'Q_{i3}A_{22}^{-1}B_{2j},$$
$$\hat{Q}_{i3}=B_{2i}'(A_{22}^{-1})'Q_{i3}A_{22}^{-1}B_{2j}.$$

Solving for the reduced order closed-loop Nash strategies, we have

$$u_{is}=-\hat{R}_{ii}^{-1}[B_{0i}'K_{is}x_s+B_{2i}'\hat{Q}_{i2}'x_s+\hat{Q}_{i3}u_{js}] \quad (8a)$$
$$=-M_{is}x_s \quad (8b)$$

where K_{is} is a stabilizing solution of

$$0 = -(\hat{Q}_{i1} + A_0'K_{is} + K_{is}A_0) + M_{is}'\hat{R}_{ii}M_{is} - M_{is}'\hat{R}_{ij}M_{js} + [K_{is}B_{0j} + \hat{Q}_{i2}B_{2j}]M_{js} + M_{js}'[B_{0j}'K_{is} + B_{2j}'\hat{Q}_{i2}'], \quad \text{for} \quad i, j = 1, 2, \quad i \neq j. \tag{9}$$

Using the gain matrix M_{is} from (8b), we implement the control

$$u_i = -M_{is}x_1 \tag{10}$$

and apply it to the system in (1). The resulting value of the suboptimal performance criteria in (2) can be expressed as

$$J_{i_{sub}} = \tfrac{1}{2}x'(t_0)V_{i_{sub}}x(t_0) \tag{11}$$

where $V_{i_{sub}}$ satisfies the Lyapunov equation

$$V_{i_{sub}}\{A - B_i[M_{is} \,\vdots\, 0] - B_j[M_{js} \,\vdots\, 0]\} + \{A - B_i[M_{is} \,\vdots\, 0] - B_j[M_{js} \,\vdots\, 0]\}'V_{i_{sub}} + \left[\begin{array}{c:c} Q_{i1} + M_{is}'R_{ii}M_{is} + M_{js}'R_{ij}M_{js} & Q_{i2} \\ \hdashline Q_{i2}' & Q_{i3} \end{array}\right] = 0. \tag{12}$$

The matrix $V_{i_{sub}}$ depends on μ since A and B_i contain μ. Hence the reduced cost is dependent on μ.

If the optimal Nash controls given by (3) are applied to (1), the values of the optimal performance criteria are given by

$$J_i = \tfrac{1}{2}x'(t_0)K_ix(t_0) \tag{13}$$

where K_i satisfies (4). We wish to examine the nature of the optimal criteria J_i as $\mu \to 0$. In particular we wish to verify if J_i approaches $J_{i_{sub}}$ as μ approaches zero. We will say that the reduced order game is well-posed if J_i approaches $J_{i_{sub}}$ as $\mu \to 0$. Otherwise, we say that it is ill-posed. We perform this comparison on a specific numerical example.

Consider the second order system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -1/\mu & -2/\mu \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 2/\mu \end{bmatrix}u_1 + \begin{bmatrix} 1 \\ 2/\mu \end{bmatrix}u_2 \tag{14}$$

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

with performance criteria

$$J_1 = \frac{1}{2}\int_{t_0}^{\infty}\left\{x'\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}x + u_1^2 + 2u_2^2\right\}dt \tag{15a}$$

$$J_2 = \frac{1}{2}\int_{t_0}^{\infty}\left\{x'\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}x + 2u_1^2 + u_2^2\right\}dt. \tag{15b}$$

For this example, the resulting K_{is} and M_{is} from (9) and (8) are

$$K_{1s} = K_{2s} = \sqrt{\frac{25}{54}} = 0.6804 \tag{16}$$

and

$$M_{1s} = M_{2s} = 0.4082. \tag{17}$$

Calculation of the resulting values of J_1 for several values of μ are given in Table I. Because of symmetry, $J_1 = J_2 = J$.

TABLE I

μ	0.5	0.2	0.1	0.01	0.005	0.001	0
J	1.3012	0.73245	0.5425	0.3724	0.3630	0.3630	0.3536
J_{sub}	1.84127	0.86558	0.59083	0.36420	0.35217	0.34259	0.3402

It is seen that the limit of J_i as $\mu \to 0$ is different from the corresponding limit of $J_{i_{sub}}$. This discrepancy between the J's in the neighborhood of $\mu = 0$ indicates that the reduced order Nash strategy obtained by the standard method is ill-posed.

III. Regularization of the Cost Functional Consistent with Inadequate Modeling of Fast Dynamics

The manner in which the singular perturbation approach could be modified so that we have a well-posed problem depends on the reason for the appearance of the singular perturbation parameter in the system model. In this section we discuss the first of two reasons considered in this paper. Let us suppose that we have a Nash strategy using a model represented by (1a) and

$$0 = A_{21}x_1 + A_{22}x_2 + B_{21}u_1 + B_{22}u_2. \tag{18}$$

We wish to examine the robustness of the Nash strategies when the actual system is represented by (1b) instead of (18). The performance index in (2) leads to an ill-posed problem as we demonstrated.

If indeed the original model used for design is based on (1a) and (18), then for *consistency* it is appropriate to assume that the vector x_2 that appears in (2) is constrained by (18). That is, from (18) we have

$$x_2 = -A_{22}^{-1}[A_{21}x_1 + B_{21}u_1 + B_{22}u_2]. \tag{19}$$

Substituting (19) into (1a) we obtain (5) and substituting (19) into (2) we obtain

$$J_i = \frac{1}{2}\int_{t_0}^{\infty} \{x_1'\hat{Q}_{i1}x_1 + 2x_1'\hat{Q}_{i2}(B_{2i}u_i - B_{2j}u_j) + u_i'\hat{R}_{ii}u_i + u_j'\hat{R}_{ij}u_j + 2u_i'\hat{Q}_{i3}u_j\}\,dt. \tag{20}$$

The modified performance index in (20) for $i, j = 1, 2$, $i \neq j$, reflects the model constraint of (18). In this case, the variable x_2 in (2) is not a component vector of the state x, but it is simply a function of x_1, u_1 and u_2 as given in (19). For example, in a d.c. motor model, we may be interested in penalizing the

armature current. However if our model neglects armature inductance then the armature current is expressed as a function of the speed and the voltage. On the other hand, in our earlier ill-posed example, x_2 in (2) is not constrained to satisfy (19) but instead, it is part of the state as given in (1b). Thus in this reformulated problem, we are interested in comparing the Nash strategy that is obtained from (1a), (18) and (20), which is the same as (6) and (7), with the Nash strategy that is obtained from (1a), (1b) and (20) as $\mu \to 0$. We show that this is a well-posed problem with respect to singular perturbation so that the Nash strategy is robust against inaccuracies caused by neglecting fast dynamics, provided that these are stable (i.e. A_{22} is stable).

For the full order problem (1) and (20), the optimal closed-loop Nash solution is given by

$$u_i = -\hat{R}_{ii}^{-1}\{B'_{2i}[\hat{Q}'_{i2} : 0]x + [B'_{1i} : B'_{2i}/\mu]\bar{K}_i x + \hat{Q}_{i3}u_j\} \tag{21a}$$

$$= -\bar{M}_i x \tag{21b}$$

where $\bar{K}_i$ is a stabilizing solution of

$$0 = \begin{bmatrix} \hat{Q}_{i1} & 0 \\ 0 & 0 \end{bmatrix} + \bar{K}_i A + A'\bar{K}_i - \bar{K}_i B_j \bar{M}_j - \bar{M}'_j B'_j \bar{K}_i$$

$$- \begin{bmatrix} \hat{Q}_{i2} \\ 0 \end{bmatrix} B_{2i}\bar{M}_i - \bar{M}'_i B'_{2i}[\hat{Q}'_{i2} : 0] + \bar{M}'_j \hat{R}_{ij}\bar{M}_j - \bar{M}'_i \hat{R}_{ii}\bar{M}_i. \tag{22}$$

The optimal cost for (20) subject to (1) is given by

$$J_{i_{\text{opt}}} = \tfrac{1}{2}x'(t_0)\bar{K}_i x(t_0). \tag{23}$$

If the control

$$u_{ir} = -M_{is}x_1 \tag{24}$$

where M_{is} is from (8) is applied to (1) for performance indices (20) a suboptimal performance cost results which can be written as

$$V_{ir} = \tfrac{1}{2}x'(t_0)P_{ir}x(t_0) \tag{25}$$

where P_{ir} is the positive semidefinite solution of the Lyapunov equation

$$P_{ir}\{A - B_i[M_{is} : 0] - B_j[M_{js} : 0]\} + \{A - B_i[M_{is} : 0] - B_j[M_{js} : 0]\}'P_{ir} + \begin{bmatrix} \xi_i & 0 \\ 0 & 0 \end{bmatrix} = 0 \tag{26}$$

and

$$\xi_i = \hat{Q}_{i1} - \hat{Q}_{i2}[B_{2i}M_{is} + B_{2j}M_{js}] - [B_{2i}M_{is} + B_{2j}M_{js}]'\hat{Q}'_{i2} + M'_{is}\hat{R}_{ii}M_{is}$$
$$+ M'_{js}\hat{R}_{ij}M_{js} + M'_{is}\hat{Q}_{i3}M_{js} + M'_{js}\hat{Q}'_{i3}M_{is}.$$

We shall compare the optimal Nash performance cost (23) to the suboptimal performance cost (25). In order to perform this comparison we need some relationship between the optimal Riccati gain $\bar{K}_i$ and P_{ir}. A relationship is found by first giving conditions under which $\bar{K}_i$ possesses a power series expansion at $\mu = 0$ and then giving conditions under which P_{ir} possesses a power

series expansion. Finally, we form a new Lyapunov equation by substracting (26) from (22) and show that there is in fact a relationship between the optimal performance cost and the suboptimal performance cost.

Represent $\bar{K}_i$, the solution of (22), as

$$\bar{K}_i(\mu) = \begin{bmatrix} \bar{K}_{i1}(\mu) & \mu \bar{K}_{i2}(\mu) \\ \mu \bar{K}'_{i2}(\mu) & \mu \bar{K}_{i3}(\mu) \end{bmatrix}, \quad i = 1, 2. \tag{27}$$

$$\begin{aligned} 0 = {} & \hat{Q}_{i1} + \bar{K}_{i1}^{(0)} A_{11} + A'_{11} \bar{K}_{i1}^{(0)} + \bar{K}_{i2}^{(0)} A_{21} + A'_{21} \bar{K}'^{(0)}_{i2} \\ & -[\bar{K}_{i1}^{(0)} B_{1j} + \bar{K}_{i2}^{(0)} B_{2j} + \hat{Q}_{i2} B_{2j}][\hat{R}_{jj} - \hat{Q}_{j3} \hat{R}_{ii}^{-1} \hat{Q}_{i3}]^{-1} S_{j1} \\ & - S'_{j1}[\hat{R}_{jj} - \hat{Q}_{j3} \hat{R}_{ii}^{-1} \hat{Q}_{i3}]^{-1}[B'_{1j} \bar{K}_{i1}^{(0)} + B'_{2j} \bar{K}'^{(0)}_{i2} + B'_{2j} \hat{Q}'_{i2}] \\ & + S'_{j1}[\hat{R}_{jj} - \hat{Q}_{j3} \hat{R}_{ii}^{-1} \hat{Q}_{i3}]^{-1} \hat{R}_{ij}[\hat{R}_{jj} - \hat{Q}_{j3} \hat{R}_{ii}^{-1} \hat{Q}_{i3}]^{-1} S_{j1} \\ & - S'_{i1}[\hat{R}_{ii} - \hat{Q}_{i3} \hat{R}_{jj}^{-1} \hat{Q}_{j3}]^{-1} \hat{R}_{ii}[\hat{R}_{ii} - \hat{Q}_{i3} \hat{R}_{jj}^{-1} \hat{Q}_{j3}]^{-1} S_{i1} \end{aligned} \tag{28}$$

$$\begin{aligned} 0 = {} & \bar{K}_{i1}^{(0)} A_{12} + \bar{K}_{i2}^{(0)} A_{22} + A'_{21} \bar{K}_{i3}^{(0)} - [\bar{K}_{i1}^{(0)} B_{1j} + \bar{K}_{i2}^{(0)} B_{2j} + \hat{Q}_{i2} B_{2j}] \\ & \times [\hat{R}_{jj} - \hat{Q}_{j3} \hat{R}_{ii}^{-1} \hat{Q}_{i3}]^{-1}[B'_{2j} \bar{K}_{j3}^{(0)} - \hat{Q}_{j3} \hat{R}_{ii}^{-1} B'_{2i} \bar{K}_{i3}^{(0)}] \\ & - S'_{j1}[\hat{R}_{jj} - \hat{Q}_{j3} \hat{R}_{ii}^{-1} \hat{Q}_{i3}]^{-1} B'_{2j} \bar{K}_{i3}^{(0)} + S'_{j1}[\hat{R}_{jj} - \hat{Q}_{j3} \hat{R}_{ii}^{-1} \hat{Q}_{i3}]^{-1} \\ & \times \hat{R}_{ij}[\hat{R}_{jj} - \hat{Q}_{j3} \hat{R}_{ii}^{-1} \hat{Q}_{i3}]^{-1}[B'_{2j} \bar{K}_{j3}^{(0)} - \hat{Q}_{j3} \hat{R}_{ii}^{-1} B'_{2i} \bar{K}_{i3}^{(0)}] \\ & - S'_{i1}[\hat{R}_{ii} - \hat{Q}_{i3} \hat{R}_{jj}^{-1} \hat{Q}_{j3}]^{-1} \hat{R}_{ii}[\hat{R}_{ii} - \hat{Q}_{i3} \hat{R}_{jj}^{-1} \hat{Q}_{j3}]^{-1} \\ & \times [B'_{2i} \bar{K}_{i3}^{(0)} - \hat{Q}_{j3} \hat{R}_{jj}^{-1} B'_{2j} \bar{K}_{j3}^{(0)}] \end{aligned} \tag{29}$$

$$\begin{aligned} 0 = {} & \bar{K}_{i3}^{(0)} A_{22} + A'_{22} \bar{K}_{i3}^{(0)} - \bar{K}_{i3}^{(0)} B_{2j}[\hat{R}_{jj} - \hat{Q}_{j3} \hat{R}_{ii}^{-1} \hat{Q}_{i3}]^{-1} \\ & \times (B'_{2j} \bar{K}_{j3}^{(0)} - \hat{Q}_{j3} \hat{R}_{ii}^{-1} B'_{2i} \bar{K}_{i3}^{(0)}) - (\bar{K}_{j3}^{(0)} B_{2j} - \bar{K}_{i3}^{(0)} B_{2i} \hat{R}_{ii}^{-1} \hat{Q}'_{j3}) \\ & \times [\hat{R}_{jj} - \hat{Q}_{j3} \hat{R}_{ii}^{-1} \hat{Q}_{i3}]^{-1} B'_{2j} \bar{K}_{i3}^{(0)} + (\bar{K}_{j3}^{(0)} B_{2j} - \bar{K}_{i3}^{(0)} B_{2i} \hat{R}_{ii}^{-1} \hat{Q}'_{j3}) \\ & \times [\hat{R}_{jj} - \hat{Q}_{j3} \hat{R}_{ii}^{-1} \hat{Q}_{i3}]^{-1} \hat{R}_{ij}[\hat{R}_{jj} - \hat{Q}_{j3} \hat{R}_{ii}^{-1} \hat{Q}_{i3}]^{-1} \\ & \times (B'_{2j} \bar{K}_{j3}^{(0)} - \hat{Q}_{j3} \hat{R}_{ii}^{-1} B'_{2i} \bar{K}_{i3}^{(0)}) - (\bar{K}_{i3}^{(0)} B_{2i} - \bar{K}_{j3}^{(0)} B_{2j} \hat{R}_{jj}^{-1} \hat{Q}'_{i3}) \\ & \times [\hat{R}_{ii} - \hat{Q}_{i3} \hat{R}_{jj}^{-1} \hat{Q}_{j3}]^{-1} \hat{R}_{ii}[\hat{R}_{ii} - \hat{Q}_{i3} \hat{R}_{jj}^{-1} \hat{Q}_{j3}]^{-1} \cdot (B'_{2i} \bar{K}_{i3}^{(0)} - \hat{Q}_{i3} \hat{R}_{jj}^{-1} B'_{2j} \bar{K}_{j3}^{(0)}) \end{aligned} \tag{30}$$

where

$$S_{i1} = B'_{2i} \hat{Q}'_{i2} + B'_{1i} \bar{K}_{i1}^{(0)} + B'_{2i} \bar{K}'^{(0)}_{i2} - \hat{Q}_{i3} \hat{R}_{jj}^{-1}[B'_{2j} \hat{Q}'_{j2} + B'_{1j} \bar{K}_{j1}^{(0)} + B'_{2j} \bar{K}'^{(0)}_{j2}] \tag{31}$$

and

$$\bar{K}_{ik}^{(0)} = \bar{K}_{ik}(\mu)|_{\mu=0}, \quad i = 1, 2, \quad k = 1, 2, 3, \quad j = 1, 2, \quad i \neq j. \tag{32}$$

In the comparison of the optimal performance and the suboptimal performance costs we need the following condition:

Condition a

$$\begin{bmatrix} \mathcal{A}_1 & \mathcal{B}_1 \\ \mathcal{B}_2 & \mathcal{A}_2 \end{bmatrix} \tag{33}$$

is non-singular where

$$\mathcal{A}_i = I \otimes \hat{A}'_i + \hat{A}'_i \otimes I$$
$$\mathcal{B}_i = I \otimes \hat{B}'_i + \hat{B}'_i \otimes I$$

and

$$\hat{A}_i = A_0 - B_{0j}M_{js} + B_{0i}\bar{R}_{ii}^{-1}\hat{Q}'_{i3}\bar{R}_{jj}^{-1}[B'_{2j}\hat{Q}'_{i2} + B'_{0j}K_{is} - \hat{R}_{ij}M_{js}] - B_{0i}\bar{R}_{ii}^{-1}\hat{R}_{ii}M_{is}$$
$$\hat{B}_i = B_{0j}\bar{R}_{jj}^{-1}[B'_{2j}\hat{Q}'_{i2} + B'_{0j}K_{is} - \hat{R}_{ij}M_{js}] + B_{0j}\bar{R}_{jj}^{-1}\hat{Q}'_{i3}\bar{R}_{ii}^{-1}\hat{R}_{ii}M_{is}$$

where

$$\bar{R}_{ii} = \hat{R}_{ii} - \hat{Q}_{i3}\hat{R}_{jj}^{-1}\hat{Q}_{j3}, \quad i, j = 1, 2, \quad i \neq j.$$

$\otimes$ is the Kronecker product operator.

Theorem I

If

(1) A_{22} from (1) is stable;

(2) The slow game (6), (7) has a unique stabilizing closed-loop Nash strategy pair;

(3) $\bar{K}_{i3}^{(0)} = 0$ is the unique positive semidefinite solution of (30);

and

(4) Condition a is satisfied;

then the solution $\bar{K}_i = \bar{K}_i(\mu)$ of (22) possesses a power series expansion at $\mu = 0$, that is,

$$\bar{K}_i(\mu) = \sum_{j=0}^{\infty} \frac{\mu^j}{j!} \begin{bmatrix} \bar{K}_{i1}^{(j)} & \mu\bar{K}_{i2}^{(j)} \\ \mu\bar{K}_{i2}^{(j)\prime} & \mu\bar{K}_{i3}^{(j)} \end{bmatrix} \tag{34}$$

where

$$\bar{K}_{ik}^{(j)} = \left. \frac{\partial^j \bar{K}_{ik}(\mu)}{\partial \mu^j} \right|_{\mu=0}, \quad i = 1, 2; \quad k = 1, 2, 3. \tag{35}$$

Furthermore, the matrices $\bar{K}_{i1}^{(0)}$, $\bar{K}_{i2}^{(0)}$ and $\bar{K}_{i3}^{(0)}$ satisfy the identities

$$\bar{K}_{i1}^{(0)} = K_{is} \tag{36a}$$
$$\bar{K}_{i2}^{(0)} = -K_{is}A_{12}A_{22}^{-1} \tag{36b}$$
$$\bar{K}_{i3}^{(0)} = 0. \tag{36c}$$

Proof: The proof is given in Appendix A.

A relationship between the suboptimal control (24) and the optimal control (21) is found by substituting (34) into (21) and letting $\mu = 0$. Comparison of the resulting equation and (24) using the identities (36) yields

$$u_{ir} = u_i + 0(\mu), \quad i = 1, 2. \tag{37}$$

The result in (37) is analogous to the "composite" control formulation in (5). Even though there is no x_2 present in (24), the result in (37) is not unexpected since the fast part of $\bar{K}_i$ for $\mu = 0$ is zero. Thus we really have a "composite" control but the fast part of that composite control is zero.

Since the reduced control (24) is close to the optimal control (21) we expect that the reduced cost (25) is close to the optimal cost (23). We state the following results.

Theorem II

If A_{22} is stable and if there exists a unique stabilizing closed-loop Nash solution to the slow game (6), (7), then P_{ir}, $i = 1, 2$, in (25), (26) possesses a power series expansion at $\mu = 0$, that is,

$$P_{ir}(\mu) = \sum_{j=0}^{\infty} \frac{\mu^j}{j!} \begin{bmatrix} P_{i1}^{(j)} & \mu P_{i2}^{(j)} \\ \mu P_{i2}^{(j)\prime} & \mu P_{i3}^{(j)} \end{bmatrix} \tag{38}$$

where

$$P_{ik}^{(j)} = \left.\frac{\partial^j P_{ik}(\mu)}{\partial \mu^j}\right|_{\mu=0}, \quad i = 1, 2, \quad k = 1, 2, 3.$$

Proof: See Appendix B.

Applying the reduced control (24) to the system (1) and comparing the resulting cost to the optimal cost gives the following theorem.

Theorem III

The first terms of the power series expansion at $\mu = 0$ of $J_{i_{opt}}$ in (23) and V_{ir} in (25) are the same, that is,

$$V_{ir} = J_{i_{opt}} + 0(\mu), \quad i = 1, 2. \tag{39}$$

Thus the reduced order slow game in (6) and (7) is well-posed with respect to the full order game in (1) and (20).

Proof: See Appendix C.

It should be noted that in (20) x_2 does not appear explicitly and that there are cross terms of x_1 with u_1 and u_2 and also cross terms of u_1 and u_2. Using a linear transformation among x_1, u_1 and u_2, a performance criterion without cross terms could be obtained. However, in this case, the transformation would induce an additional structural constraint on the control, and the Nash solution might be different. Thus, no such transformation is used in this section. A second point to note is that although x_2 does not appear in (20), the "slow" part of x_2 as given by (19) does appear, since (19) was substituted into (2) to obtain (20).

IV. Hierarchical Reduction Scheme which Transfers Fast Game Information to a Modified Slow Game

In Section III we demonstrated that if the system model for control design contains only slow modes and we wish to determine the robustness of the nonzero-sum Nash strategy to the presence of fast modes in the actual system, then the performance indices should not include the fast modes of the system. That is, if we have a system with fast and slow modes, then in order to have a well-posed reduced problem under the usual singular perturbation reduction method, the fast modes of the system should not be penalized in the performance indices. On the other hand, if the system is assumed to be adequately

modeled and the fast modes appear in both the state equations and performance indices, and it is desired to reduce the amount of computation and alleviate the numerical stiffness of the closed-loop Nash control problem, we have seen via the example in Section II that the usual order reduction method of singularly perturbed optimal control problems does not lead to a well-posed Nash game.

In the method of Section II it is implicitly assumed that the fast modes and slow modes can be completely decoupled. However, we have shown that if we directly penalize the fast modes, the fast and slow modes cannot be completely decoupled. Taking this into account we propose to first solve a fast low order game and then implement the fast feedback control in the system and performance indices before obtaining a reduced order slow game. Thus we are proposing a block triangular or hierarchical rather than the usual block diagonal decomposition.

To derive the fast subsystem, we assume that the slow variables are constant during fast transients. Denoting the fast variables by the subscript f we have the fast subsystem and performance indices

$$\mu\dot{x}_f = A_{22}x_f + B_{21}u_{1f} + B_{22}u_{2f}; \quad x_f(t_0) = x_{20} - \bar{x}_2(t_0) \tag{40}$$

$$J_{if} = \frac{1}{2}\int_{t_0}^{\infty}[x_f'Q_{i3}x_f + u_{if}'R_{ii}u_{if} + u_{jf}'R_{ij}u_{jf}]\,dt; \quad i, j = 1, 2; \quad i \neq j, \tag{41}$$

where $x_f = x_2 - x_{2s}$. The closed-loop Nash controls for (41) subject to (40) are

$$u_{if} = -R_{ii}^{-1}B_{2i}'K_{if}x_f, \quad i = 1, 2 \tag{42}$$

where K_{if} is a stabilizing solution of

$$0 = -Q_{i3} - K_{if}A_{22} - A_{22}'K_{if} + K_{if}B_{2i}R_{ii}^{-1}B_{2i}'K_{if} + K_{if}B_{2j}R_{jj}^{-1}B_{2j}'K_{jf}$$
$$+ K_{jf}B_{2j}R_{jj}^{-1}B_{2j}'K_{if} - K_{jf}B_{2j}R_{jj}^{-1}R_{ij}R_{jj}^{-1}B_{2j}'K_{jf}, \quad i, j = 1, 2, \quad i \neq j. \tag{43}$$

Next we make use of the fast control and substitute the following for u_i in our original system (1) and performance indices (2). Let

$$u_i = -R_{ii}^{-1}B_{2i}'K_{if}x_2 + \hat{u}_i \tag{44}$$

be our modified control. This gives a new system and performance indices given by

$$\dot{x}_1 = A_{11}x_1 + \hat{A}_{12}x_2 + B_{11}\hat{u}_1 + B_{12}\hat{u}_2; \quad x_1(t_0) = x_{10} \tag{45a}$$

$$\mu\dot{x}_2 = A_{21}x_1 + \hat{A}_{22}x_2 + B_{21}\hat{u}_1 + B_{22}\hat{u}_2; \quad x_2(t_0) = x_{20} \tag{45b}$$

and

$$J_i = \frac{1}{2}\int_{t_0}^{\infty}\left\{x'\begin{bmatrix} Q_{i1} & Q_{i2} \\ Q_{i2}' & \hat{Q}_{i3} \end{bmatrix}x - 2x_2'K_{if}B_{2i}\hat{u}_i - 2x_2'K_{jf}B_{2j}R_{jj}^{-1}R_{ij}\hat{u}_j\right.$$
$$\left. + \hat{u}_i'R_{ii}\hat{u}_i + \hat{u}_j'R_{ij}\hat{u}_j\right\}dt \quad i, j = 1, 2, \quad i \neq j, \tag{46}$$

where

$$\hat{A}_{12} = A_{12} - B_{11}R_{11}^{-1}B'_{21}K_{1f} - B_{12}R_{22}^{-1}B'_{22}K_{2f}$$
$$\hat{A}_{22} = A_{22} - B_{21}R_{11}^{-1}B'_{21}K_{1f} - B_{22}R_{22}^{-1}B'_{22}K_{2f}$$
$$\hat{Q}_{i3} = Q_{i3} + K_{if}B_{2i}R_{ii}^{-1}B'_{2i}K_{if} + K_{jf}B_{2j}R_{jj}^{-1}R_{ij}R_{jj}^{-1}B'_{2j}K_{jf}.$$

To get our "modified slow" subsystem we formally set $\mu = 0$ in (45b) and solve for x_2. This gives

$$\bar{x}_2 = -\hat{A}_{22}^{-1}[A_{21}\bar{x}_1 + B_{21}\bar{u}_1 + B_{22}\bar{u}_2]. \tag{47}$$

Substitution of (47) into (45a) and (46) gives us the "modified slow" subsystem and performance indices

$$\dot{x}_{sm} = \bar{A}_0 x_{sm} + \bar{B}_{01}u_{1sm} + \bar{B}_{02}u_{2sm}; \quad x_{sm}(t_0) = x_{10} \tag{48}$$

and

$$J_{ism} = \frac{1}{2}\int_{t_0}^{\infty} [x'_{sm}\bar{Q}_{i1}x_{sm} + 2x'_{sm}\bar{Q}_{i2}u_{ism} + 2x'_{sm}\tilde{Q}_{i2}u_{jsm} + 2u'_{ism}\bar{Q}_{i3}u_{jsm}$$
$$+ u'_{ism}R_{ii}u_{ism} + u'_{jsm}\bar{R}_{ij}u_{jsm}]\,dt; \quad i, j = 1, 2, \quad i \neq j, \tag{49}$$

where

$$\bar{A}_0 = A_{11} - \hat{A}_{12}\hat{A}_{22}^{-1}A_{21}$$
$$\bar{B}_{0i} = B_{1i} - \hat{A}_{12}\hat{A}_{22}^{-1}B_{2i}$$
$$\bar{Q}_{i1} = Q_{i1} - Q_{i2}\hat{A}_{22}^{-1}A_{21} - (\hat{A}_{22}^{-1}A_{21})'Q'_{i2} + (\hat{A}_{22}^{-1}A_{21})'\hat{Q}_{i3}\hat{A}_{22}^{-1}A_{21}$$
$$\bar{Q}_{i2} = -Q_{i2}\hat{A}_{22}^{-1}B_{2i} + (\hat{A}_{22}^{-1}A_{21})'\hat{Q}_{i3}\hat{A}_{22}^{-1}B_{2i} + (\hat{A}_{22}^{-1}A_{21})'K_{if}B_{2i}$$
$$\tilde{Q}_{i2} = -Q_{i2}\hat{A}_{22}^{-1}B_{2j} + (\hat{A}_{22}^{-1}A_{21})'\hat{Q}_{i3}\hat{A}_{22}^{-1}B_{2j} + (\hat{A}_{22}^{-1}A_{21})'K_{jf}B_{2j}\hat{R}_{jj}^{-1}R_{ij}$$
$$\bar{Q}_{i3} = B'_{2i}(\hat{A}_{22}^{-1})'\hat{Q}_{i3}\hat{A}_{22}^{-1}B_{2j} + B'_{2i}K_{if}\hat{A}_{22}^{-1}B_{2j} + B'_{2i}(\hat{A}_{22}^{-1})'K_{jf}B_{2j}R_{jj}^{-1}R_{ij}$$
$$\bar{R}_{ij} = R_{ij} + B'_{2j}(\hat{A}_{22}^{-1})'\hat{Q}_{i3}\hat{A}_{22}^{-1}B_{2j} + B'_{2j}(\hat{A}_{22}^{-1})'K_{jf}B_{2j}R_{jj}^{-1}R_{ij} + R_{ij}R_{jj}^{-1}B'_{2j}K_{jf}\hat{A}_{22}^{-1}B_{2j}.$$

We will show in this section that the reduction process we have described leads to a well-posed reduced game. Note that the modified slow subsystem and performance indices are of the same form as in the slow problem considered in Section II. However, the system matrices and performance coefficients contain information about the fast low order game. Examining (44) we see that we still have a control composed of fast and slow parts. However, since we substitute the explicit form for the fast part into the state equation and performance indices before the slow state equation and performance indices are formed, the slow modes are dependent on the fast modes. In Sections II and III the fast modes and slow modes were completely separated.

The closed-loop Nash strategy for (49) subject to (48) is given by

$$u_{ism} = -R_{ii}^{-1}[\bar{Q}'_{i2}x_{sm} + \bar{B}'_{0i}K_{ism}x_{sm} + \bar{Q}_{i3}u_{jsm}] \tag{50}$$
$$= -\bar{M}_{is}x_{sm} \tag{51}$$

where K_{1sm}, K_{2sm} satisfy the coupled Riccati equations

$$0 = -\bar{Q}_{i1} - K_{ism}\bar{A}_0 - \bar{A}_0' K_{ism} + [K_{ism}\bar{B}_{0i} + \bar{Q}_{i2}]\bar{M}_{is} + \bar{M}_{is}'[\bar{B}_{0i}'K_{ism} + \bar{Q}_{i2}'] - \bar{M}_{js}'\bar{R}_{ij}\bar{M}_{js} + \bar{M}_{is}'R_{ii}\bar{M}_{is}; \quad i, j = 1, 2; \quad i \neq j. \tag{52}$$

Of course (51) and (42) are only subsystem optimal. That is, as they stand we cannot apply them to the original system (1). Following the methodology of (5) we form a "composite" control involving both fast and slow control coefficients. The form for the composite control is suggested by (44). Forming

$$u_{ic} = -\bar{M}_{is}x_1 - R_{ii}^{-1}B_{2i}'K_{if}x_2 \tag{53a}$$

$$= -R_{ii}^{-1}B_i'\begin{bmatrix} K_{ism} & 0 \\ \mu K_{im}' & \mu K_{if} \end{bmatrix}x \tag{53b}$$

$$= -R_{ii}^{-1}B_i'M_{ic}x \tag{53c}$$

where

$$K_{im} = \{-Q_{i2} - A_{21}'K_{if} - K_{ism}\hat{A}_{12} + (K_{ism}\bar{B}_{0i} + \bar{Q}_{i2} - [K_{ism}\bar{B}_{0i} + \bar{Q}_{i2}]R_{ii}^{-1}\bar{Q}_{i3}') \cdot [R_{ij} - \bar{Q}_{i3}'R_{ii}^{-1}\bar{Q}_{i3}']^{-1}[B_{2i}'K_{if} - R_{ij}R_{jj}^{-1}B_{2j}'K_{jf}]\}\hat{A}_{22}^{-1}. \tag{54}$$

We note that the coefficient of x_1 involves both fast and slow Riccati gains while the coefficient of x_2 involves only fast Riccati gains.

If the composite control (53) is applied to (1) for performance indices (2) a suboptimal performance cost results which can be written as

$$J_{ic} = \tfrac{1}{2}x_0'P_{ic}x_0 \tag{55}$$

where

$$0 = P_{ic}[A - B_iR_{ii}^{-1}B_i'M_{ic} - B_jR_{jj}^{-1}B_j'M_{jc}] + [A - B_iR_{ii}^{-1}B_i'M_{ic} - B_jR_{jj}^{-1}B_j'M_{jc}]'P_{ic} + Q_i + M_{ic}'B_iR_{ii}^{-1}B_i'M_{ic} + M_{jc}'B_jR_{jj}^{-1}R_{ij}R_{jj}^{-1}B_j'M_{jc}. \tag{56}$$

To compare the optimal performance cost (13) and the composite performance cost (55) we need the following conditions:

Conditions b

$$[\mathscr{A}_3 - \mathscr{C}_1\mathscr{A}_3^{-1}\mathscr{C}_2] \tag{57}$$

is non-singular where

$$\mathscr{A}_3 = I \otimes \hat{A}_{22}' + \hat{A}_{22}' \otimes I$$
$$\mathscr{C}_i = I \otimes C_i' + C_i' \otimes I$$

and

$$C_i = B_{2i}R_{ii}^{-1}B_{2i}'K_{if} - B_{2i}R_{ii}^{-1}R_{ii}R_{ii}^{-1}B_{2i}'K_{if}.$$

Condition c

$$\begin{bmatrix} \check{\mathscr{A}}_1 & \check{\mathscr{B}}_1 \\ \check{\mathscr{B}}_2 & \check{\mathscr{A}}_2 \end{bmatrix} \tag{58}$$

is non-singular where

$$\check{\mathcal{A}}_i = I \otimes \check{A}_i' + \check{A}_i' \otimes I$$
$$\check{\mathcal{B}}_i = -I \otimes \check{B}_i' - \check{B}_i' \otimes I$$

and

$$\begin{aligned}
\check{A}_i &= \bar{A}_0 - \bar{B}_{0i}\bar{M}_{is} - \bar{B}_{0j}\bar{M}_{js} + N_{i1}B_{2j}R_{jj}^{-1}\tau_i \\
\check{B}_i &= [B_{1j} + N_{i2}B_{2j}]R_{jj}^{-1}\tau_i \\
\tau_i &= B_{1j}'K_{ism} + B_{2j}'K_{im}' - R_{ij}\bar{M}_{js} + [B_{2j}'K_{if} - R_{ij}R_{jj}^{-1}B_{2j}'K_{jf}]\hat{A}_{22}^{-1} \\
&\quad \times[-A_{21} + B_{2i}\bar{M}_{is} + B_{2j}\bar{M}_{js}] \\
N_{i1} &= \bar{B}_{0i}[R_{ii}^{-1}B_{2i}'K_{ff} - R_{ii}^{-1}R_{ji}R_{ii}^{-1}B_{2i}'K_{if}]\hat{A}_{22}^{-1}T_i^{-1} \\
N_{i2} &= \{-\hat{A}_{12}\hat{A}_{22}^{-1} + B_{1j}[R_{jj}^{-1}B_{2j}K_{if} - R_{jj}^{-1}R_{ij}R_{jj}^{-1}B_{2j}'K_{jf}]\hat{A}_{22}^{-1}B_{2i} \\
&\quad \times[R_{ii}^{-1}B_{2i}'K_{ff} - R_{ii}^{-1}R_{ji}R_{ii}^{-1}B_{2i}'K_{if}]\hat{A}_{22}^{-1}\}T_i^{-1} \\
T_i &= I - B_{2j}[R_{jj}^{-1}B_{2j}'K_{if} - R_{jj}^{-1}R_{ij}R_{jj}^{-1}B_{2j}'K_{jf}]\hat{A}_{22}^{-1}B_{2i} \\
&\quad \times[R_{ii}^{-1}B_{2i}'K_{ff} - R_{ii}^{-1}R_{ji}R_{ii}^{-1}B_{2i}'K_{if}]\hat{A}_{22}^{-1}.
\end{aligned}$$

If these conditions hold we have the following theorem.

Theorem IV

If

(1) the fast game (40), (41) has a unique stabilizing closed-loop Nash solution;
(2) the modified slow game (48), (49) has a unique stabilizing closed-loop Nash solution;
(3) Condition b is satisfied;

and

(4) Condition c is satisfied;

then K_i, the solution of (4), possesses a power series expansion at $\mu = 0$, that is,

$$K_i(\mu) = \sum_{j=0}^{\infty} \frac{\mu^j}{j!}\begin{bmatrix} K_{i1}^{(j)} & \mu K_{i2}^{(j)} \\ \mu K_{i2}^{(j)\prime} & \mu K_{i3}^{(j)} \end{bmatrix} \tag{59}$$

where

$$K_{ik}^{(j)} = \left.\frac{\partial^j K_{ik}(\mu)}{\partial \mu^j}\right|_{\mu=0}; \quad i = 1, 2; \quad k = 1, 2, 3. \tag{60}$$

Furthermore, the matrices $K_{i1}^{(0)}$, $K_{i2}^{(0)}$ and $K_{i3}^{(0)}$ satisfy the identities

$$K_{i1}^{(0)} = K_{ism}$$
$$K_{i2}^{(0)} = K_{im}$$
$$K_{i3}^{(0)} = K_{if}.$$

Proof: The proof is similar to the proof of Theorem I and is omitted for brevity.

An immediate result of Theorem IV is that

$$u_{ic} = u_i + 0(\mu) \tag{61}$$

where u_i is the optimal Nash control for (1), (2). This can be shown easily by substituting (59) into (3) and letting $\mu = 0$. The identities found in Theorem IV yield (61). Furthermore we have the following results.

Theorem V

If the fast game (40), (41) has a unique stabilizing closed-loop Nash solution and the modified slow game (48), (49) has a unique stabilizing closed-loop Nash solution, then P_{ic} possesses a power series expansion at $\mu = 0$, that is,

$$P_{ic} = \sum_{j=0}^{\infty} \frac{\mu^j}{j!} \begin{bmatrix} P_{ic1}^{(j)} & \mu P_{ic2}^{(j)} \\ \mu P_{ic2}^{(j)\prime} & \mu P_{ic3}^{(j)} \end{bmatrix}. \tag{62}$$

Proof: The proof is similar to the proof of Theorem II and is omitted for brevity.

As a result of K_i and P_{ic} possessing power series expansions at $\mu = 0$ it is easy to show that their difference also has a power series expansion at $\mu = 0$. Comparison of the optimal Nash performance with the composite performance costs gives the following theorem.

Theorem VI

The first terms of the power series expansion at $\mu = 0$ of K_i and P_{ic} are the same, that is,

$$J_{ic} = J_i + 0(\mu). \tag{63}$$

Proof: The proof is similar to the proof of Theorem III and is omitted for brevity.*

Thus far we have not changed the structure of the controller for the full order system (1). That is, the composite controls are a function of both x_1 and x_2 as are the optimal Nash closed-loop controls. If it is desired to implement the control as a function of x_1 only to achieve an $0(\mu)$ approximation of the optimal cost we use the following procedure. Substitute the composite control (53) into (1b) and let $\mu = 0$. This gives as an approximation of x_2

$$\bar{x}_2 = -\hat{A}_{22}^{-1}[A_{21} - B_{21}\bar{M}_{1s} - B_{22}\bar{M}_{2s}]x_1. \tag{64}$$

If (64) is substituted for x_2 in the composite control we have a "lower order"

* Thus the modified slow game is well-posed with respect to the modified reduction procedure. Furthermore, it can be shown that u_{ic} satisfies an Asymptotic Nash property in the sense that $J_i(u_{ic}, u_{jc}) \leq J_i(u_i, u_{jc}) + 0(\mu)$, $i, j = 1, 2$; $i \neq j$. The reduced order control satisfies a similiar inequality.

control as a function of x_1 only. This lower order control is

$$u_{il} = -\{\bar{M}_{is} - R_{ii}^{-1}B_{2i}'K_{if}\hat{A}_{22}^{-1}[A_{21} - B_{21}\bar{M}_{1s} - B_{22}\bar{M}_{2s}]\}x_1 \tag{65a}$$

$$= -R_{ii}^{-1}B_i'M_{il}x \tag{65b}$$

where

$$M_{il} = \begin{bmatrix} K_{im} & 0 \\ \mu\tilde{K}_{im}' & 0 \end{bmatrix}$$

and

$$\tilde{K}_{im} = K_{im} - [A_{21} - B_{21}\bar{M}_{1s} - B_{22}\bar{M}_{2s}]'(\hat{A}_{22}^{-1})'K_{if}.$$

If (65) is applied to the full order system (1) for performance indices (2) a cost results which can be written as

$$J_{il} = \tfrac{1}{2}x_0'P_{il}x_0 \tag{66}$$

where P_{il} is the positive semidefinite solution of the Lyapunov equation

$$0 = P_{il}[A - B_iR_{ii}^{-1}B_i'M_{il} - B_jR_{jj}^{-1}B_j'M_{jl}] + [A - B_iR_{ii}^{-1}B_i'M_{il} - B_jR_{jj}^{-1}B_j'M_{jl}]'P_{il}$$
$$+ Q_i + M_{il}'B_iR_{ii}^{-1}B_i'M_{il} + M_{jl}'B_jR_{jj}^{-1}R_{ij}R_{jj}^{-1}B_j'M_{jl}. \tag{67}$$

Following the method used earlier in this paper we have the following theorem.

Theorem VII

If A_{22} is stable and the modified slow game (48), (49) has a unique stabilizing closed-loop Nash solution, then P_{il} possesses a power series expansion at $\mu = 0$, that is,

$$P_{il} = \sum_{j=0}^{\infty} \frac{\mu^j}{j!}\begin{bmatrix} \mu P_{il1}^{(j)} & \mu P_{il2}^{(j)} \\ \mu P_{il2}^{(j)\prime} & \mu P_{il3}^{(j)} \end{bmatrix}. \tag{68}$$

Proof: The proof is similar to the proof of Theorem II and is omitted for brevity.

Since P_{il} and P_{ic} possess power series expansions at $\mu = 0$ it can be shown that their difference also has a power series expansion at $\mu = 0$. Comparison of the composite performance costs and the lower order performance costs gives the following theorem.

Theorem VIII

The first terms of the power series expansion at $\mu = 0$ of P_{il} and P_{ic} are the same, that is

$$J_{il} = J_{ic} + 0(\mu). \tag{69}$$

Proof: The proof is similar to the proof of Theorem III and is omitted for brevity.

As a result of Theorem VIII, it can be seen that the costs at $\mu = 0$ for the full order optimal Nash game, the full order game with composite control applied,

and the full order game with the lower order control applied are the same. Thus we have shown that the modified slow game which leads to the composite control u_{i_c} in (53) and to the reduced control $u_{i\ell}$ in (65) is a well-posed reduced order closed loop Nash game, without having to modify the original quadratic cost functions in (2).

V. Conclusions

We have shown via example that the usual order reduction procedure for singularly perturbed optimal control systems does not lead to a well-posed problem when extended directly to the linear-quadratic nonzero-sum closed-loop Nash game. If the fast dynamics are not known exactly then only the slow part of the fast states should be incorporated into the performance indices. We have shown that in this case the usual order reduction procedure for singularly perturbed optimal control systems leads to a well-posed problem.

On the other hand, if it is assumed that the fast dynamics are known and are incorporated in both the state equation and performance indices, we have shown that by using a hierarchical reduction procedure developed in Section IV the resulting modified slow game is well-posed. This hierarchical reduction procedure differs from the normal singular perturbation order reduction procedure in that it is a block triangular or sequential process rather than a parallel decomposition. In this sense it is analogous to the reduction method of Kokotovic and Yackel (7) for singularly perturbed optimal control problems where they had the slow Riccati equation dependent on the fast Riccati gain. In our hierarchical decomposition the fast subsystem may be found independently of the slow subsystem but the converse is not true. Also, a choice is provided for implementing the approximate control as either a function of fast and slow states or as a function of slow states only. As in the optimal control case, knowledge of the value of the small parameter, μ, is not necessary to obtain an $0(\mu)$ feedback control design.

In contrast, for zero-sum Nash games (6), although the performance indices contain fast modes, the natural order reduction used in optimal control formulations leads to well-posed problems. That is, in zero-sum games it does not matter whether the order reduction is due to ignorance of inadequately modeled fast dynamics or due to computational simplification only.

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Appendices

Appendix A: Proof of Theorem I

The approach to the proof of Theorem I is to represent $\bar{K}_i$ as given by (27). When this form is substituted into (22) we will show that, under the conditions of Theorem I, each term in the series expansion of $\bar{K}_i$ about $\mu = 0$ exists and is unique. Then, clearly, there is a $\mu^* > 0$ small enough to guarantee convergence of the series for all $0 \leq \mu < \mu^*$.

The substitution of (27) into (22) at $\mu = 0$ yields Eqns. (28)–(31); $i, j = 1, 2$; $i \neq j$. If

$$\bar{K}_{i3}^{(0)} = 0, \quad i = 1, 2 \tag{A.1}$$

is the unique positive semidefinite solution to (30), then (A.1) may be substituted into (29) to uniquely yield

$$\bar{K}_{i2}^{(0)} = -\bar{K}_{i1}^{(0)} A_{12} A_{22}^{-1}. \tag{A.2}$$

Substitution of (A.2) and (A.1) into (28) and manipulating gives

$$0 = \hat{Q}_{i1} + \bar{K}_{i1}^{(0)} A_0 + A_0' \bar{K}_{i1}^{(0)} - [\bar{K}_{i1}^{(0)} B_{0i} + \hat{Q}_{i2} B_{2i}] \bar{M}_i - \bar{M}_i' [B_{0i}' \bar{K}_{i1}^{(0)} + B_{2i}' \hat{Q}_{i2}']$$
$$+ \bar{M}_j' \hat{R}_{ij} \bar{M}_j - \bar{M}_i' \hat{R}_{ii} \bar{M}_i \tag{A.3}$$

where

$$\bar{M}_i = [\hat{R}_{ii} - \hat{Q}_{i3} \hat{R}_{jj}^{-1} \hat{Q}_{i3}']^{-1} [B_{2i}' \hat{Q}_{i2}' + B_{0i}' \bar{K}_{i1}^{(0)} + \hat{Q}_{i3} \hat{R}_{jj}^{-1} (B_{2i}' \hat{Q}_{i2}' - B_{0i}' \bar{K}_{i1}^{(0)})]$$
$$i, j = 1, 2; \quad i \neq j. \tag{A.4}$$

Comparison of (A.3) and (9) show that the two equations are identical with K_{is} appearing in (9) where $\bar{K}_{i1}^{(0)}$ appears in (A.3). Thus, if K_{is}, $i = 1, 2$, is the unique stabilizing solution to (9)

$$\bar{K}_{i1}^{(0)} = K_{is}, \quad i = 1, 2. \tag{A.5}$$

Substitution of (A.5) into (A.2) gives

$$\bar{K}_{i2}^{(0)} = -K_{is} A_{12} A_{22}^{-1}, \quad i = 1, 2. \tag{A.6}$$

Thus, we have shown that the first term of the series exists and is unique.

To see if the second term of the series exists we substitute (27) into (22) and take the first partial with respect to μ at $\mu = 0$. This gives, with some manipulation,

$$0 = \bar{K}_{i3}^{(1)} A_{22} + A_{22}' \bar{K}_{i3}^{(1)} - [(A_{12} A_{22}^{-1})' \bar{K}_{i1}^{(0)} A_{12} + A_{12}' \bar{K}_{i1}^{(0)} A_{12} A_{22}^{-1}], \quad i = 1, 2. \tag{A.7}$$

If A_{22} is stable (A.7) possesses a unique solution. If we now assume that $\bar{K}_{i3}^{(1)}$ is known

from (A.7), $\bar{K}_{i2}^{(1)}$ can be found to be

$$\bar{K}_{i2}^{(1)} = -\bar{K}_{i1}^{(1)} A_{12} A_{22}^{-1} + \mathcal{R}_i A_{22}^{-1}, \quad i = 1, 2 \tag{A.8}$$

where $\mathcal{R}_i$ is some known matrix.

Substitution of (A.8) into the equation for $\bar{K}_{i1}^{(1)}$ gives

$$\bar{K}_{i1}^{(1)} \hat{A}_i + \hat{A}_i' \bar{K}_{i1}^{(1)} + \bar{K}_{j1}^{(1)} \hat{B}_i + \hat{B}_i' \bar{K}_{j1}^{(1)} = \hat{\mathcal{R}}_i, \quad i, j = 1, 2, \quad i \neq j \tag{A.9}$$

where

$$\hat{A}_i = A_0 - B_{0i} M_{is} + B_{0i} \bar{R}_{ii}^{-1} \hat{Q}_{i3}' \bar{R}_{ii}^{-1} [B_{2i}' \hat{Q}_{i2}' + B_{0i}' K_{is} - \hat{R}_{ii} M_{is}] - B_{0i} \bar{R}_{ii}^{-1} \hat{R}_{ii} M_{is} \tag{A.10}$$

$$\hat{B}_i = -B_{0i} \bar{R}_{ii}^{-1} [B_{2i}' \hat{Q}_{i2}' + B_{0i}' K_{is} - \hat{R}_{ii} M_{is}] + B_{0i} \bar{R}_{ii}^{-1} \hat{Q}_{i3}' \bar{R}_{ii}^{-1} \hat{R}_{ii} M_{is}. \tag{A.11}$$

$$\bar{R}_{ii} = \hat{R}_{ii} - \hat{Q}_{i3} \bar{R}_{ii}^{-1} \hat{Q}_{is}. \tag{A.12}$$

To find conditions for $\bar{K}_{11}^{(1)}$ and $\bar{K}_{21}^{(1)}$ to exist and be unique we apply the Kronecker product operator to (A.9) to give the vector form

$$\mathcal{A}_1 k_{11} + \mathcal{B}_1 k_{21} = r_1 \tag{A.13}$$

$$\mathcal{A}_2 k_{21} + \mathcal{B}_2 k_{11} = r_2. \tag{A.14}$$

Then if

$$\begin{bmatrix} \mathcal{A}_1 & \mathcal{B}_1 \\ \mathcal{B}_2 & \mathcal{A}_2 \end{bmatrix} \tag{A.15}$$

is non-singular $\bar{K}_{11}^{(1)}$ and $\bar{K}_{21}^{(1)}$ exist and are unique.

The existence of higher derivatives follows in an analogous manner and existence and uniqueness are guaranteed by A_{22} and (A.15) non-singular. Instead of giving a specific manner in which a unique solution exists to (A.9) we could just specify that if there exists a unique solution to (A.9) then the power series exists and is unique.

Appendix B: Proof of Theorem II

Represent $P_{i\nu}$ as

$$P_{i\nu} = \begin{bmatrix} P_{i1}(\mu) & \mu P_{i2}(\mu) \\ \mu P_{i2}'(\mu) & \mu P_{i3}(\mu) \end{bmatrix}. \tag{B.1}$$

Then denote

$$P_{ik}^{(j)} = \frac{\partial^j P_{ik}(\mu)}{\partial \mu^j}\bigg|_{\mu=0}, \quad i = 1, 2; \quad j = 1, 2, \ldots; \quad k = 1, 2, 3.$$

Substituting of (B.1) into (26) at $\mu = 0$ gives

$$\begin{aligned} 0 = {} & P_{i1}^{(0)}[A_{11} - B_{1i} M_{is} - B_{1j} M_{js}] + P_{i2}^{(0)}[A_{21} - B_{2i} M_{is} - B_{2j} M_{js}] \\ & + [A_{11} - B_{1i} M_{is} - B_{1j} M_{js}]' P_{i1}^{(0)} + [A_{21} - B_{2i} M_{is} - B_{2j} M_{js}]' P_{i2}^{(0)\prime} + \xi \end{aligned} \tag{B.2a}$$

$$0 = P_{i1}^{(0)} A_{12} + P_{i2}^{(0)} A_{22} + [A_{21} - B_{2i} M_{is} - B_{2j} M_{js}]' P_{i3}^{(0)} \tag{B.2b}$$

$$0 = P_{i3}^{(0)} A_{22} + A_{22}' P_{i3}^{(0)}. \tag{B.2c}$$

If A_{22} is stable then

$$P_{i3}^{(0)} = 0 \tag{B.3}$$

is the unique solution to (B.2c). Substitution of this into (B.2b) gives

$$P_{i2}^{(0)} = -P_{i1}^{(0)} A_{12} A_{22}^{-1}. \tag{B.4}$$

Finally, substitution of (B.4) into (B.2a) gives

$$0 = P_{i1}^{(0)}[A_0 - B_{0i}M_{is} - B_{0j}M_{js}] + [A_0 - B_{0i}M_{is} - B_{0j}M_{js}]' P_{i1}^{(0)} + \xi_{si}, \quad i, j = 1, 2; \quad i \neq j. \tag{B.5}$$

Since $[A_0 - B_{0i}M_{is} - B_{0j}M_{js}]$ is stable if the slow game possesses a unique stabilizing pair K_{1s}, K_{2s}, (B.5) has a unique solution. Hence the first term in (B.1) exists and is unique.

We next examine the existence and uniqueness of the second term in (B.1). Substitute (B.1) into (26) and take the partial with respect to μ at $\mu = 0$. This gives

$$\begin{aligned} 0 = {} & P_{i1}^{(1)}[A_{11} + B_{1i}M_{is} - B_{1j}M_{js}] + P_{i2}^{(1)}[A_{21} - B_{2i}M_{is} - B_{2j}M_{js}] \\ & + [A_{11} - B_{1i}M_{is} - B_{1j}M_{js}]' P_{i1}^{(1)} + [A_{21} - B_{2i}M_{is} - B_{2j}M_{js}]' P_{i2}^{(0)\prime} \end{aligned} \tag{B.6a}$$

$$\begin{aligned} 0 = {} & P_{i1}^{(1)} A_{12} + P_{i2}^{(1)} A_{22} + [A_{21} - B_{2i}M_{is} - B_{2j}M_{js}]' P_{i3}^{(1)} \\ & + [A_{11} - B_{1i}M_{is} - B_{1j}M_{js}]' P_{i2}^{(0)} \end{aligned} \tag{B.6b}$$

$$0 = P_{i3}^{(1)} A_{22} + A_{22}' P_{i3}^{(1)} + [P_{i2}^{(0)\prime} A_{12} + A_{12}' P_{i2}^{(0)}], \quad i, j = 1, 2; \quad i \neq j. \tag{B.6c}$$

Since $P_{i2}^{(0)}$ are known from the calculations for the first term in the expansion, if A_{22} is stable (B.6c) possesses a unique solution. Then $P_{i2}^{(1)}$ may be found as

$$P_{i2}^{(1)} = -P_{i1}^{(1)} A_{12} A_{22}^{-1} - \mathcal{S}_i A_{22}^{-1}, \quad i = 1, 2 \tag{B.7}$$

where $\mathcal{S}_i$ is a known matrix. Substitution of (B.7) into (B.6a) gives

$$0 = P_{i1}^{(1)}[A_0 - B_{0i}M_{is} - B_{0j}M_{js}] + [A_0 - B_{0i}M_{is} - B_{0j}M_{js}]' P_{i1}^{(1)} + \mathcal{U}_i \tag{B.8}$$

where $\mathcal{U}_i$ is a known matrix. If $[A_0 - B_{0i}M_{is} - B_{0j}M_{js}]$ is stable then (B.8) possesses a unique solution. Hence the second term in the series exists and is unique. Higher order terms follow easily and have the same requirements for existence and uniqueness. Since each term of the series exists, clearly, there is a $\bar{\mu}^* > 0$ small enough to guarantee convergence of the series for all $0 \leq \mu < \bar{\mu}^*$

Appendix C: Proof of Theorem III

Subtracting (22) from (26), we obtain a Lyapunov equation for $\bar{W}_i = P_{i\varepsilon} - \bar{K}_i$,

$$\begin{aligned} & \bar{W}_i \left[\begin{array}{c:c} A_{11} - B_{1i}M_{is} - B_{1j}M_{js} & A_{12} \\ \hdashline \frac{1}{\mu}[A_{21} - B_{2i}M_{is} - B_{2j}M_{js}] & \frac{A_{22}}{\mu} \end{array}\right] + \left[\begin{array}{c:c} B_{1i}M_{is} + B_{1j}M_{js} & 0 \\ \hdashline \frac{1}{\mu}[B_{2i}M_{is} + B_{2j}M_{js}] & 0 \end{array}\right]' \bar{K}_i \\ & + \left[\begin{array}{c:c} \xi_i - \hat{Q}_{i1} & 0 \\ \hdashline 0 & 0 \end{array}\right] - \bar{K}_i \left[\begin{array}{c:c} B_{1i}M_{is} + B_{1j}M_{js} & 0 \\ \hdashline \frac{1}{\mu}[B_{2i}M_{is} + B_{2j}M_{js}] & 0 \end{array}\right] - \left[\begin{array}{c:c} B_{1i}M_{is} + B_{1j}M_{js} & 0 \\ \hdashline \frac{1}{\mu}[B_{2i}M_{is} + B_{2j}M_{js}] & 0 \end{array}\right]' \bar{K}_i \\ & + \left\{\bar{K}_i B_i + \left[\begin{array}{c} \hat{Q}_{i2} B_{2i} \\ \hdashline 0 \end{array}\right]\right\} \bar{M}_i + \bar{M}_i' \left\{\bar{K}_i B_i + \left[\begin{array}{c} \hat{Q}_{i2} B_{2i} \\ \hdashline 0 \end{array}\right]\right\}' - \bar{M}_i' \hat{R}_{i\mu} \bar{M}_i + \bar{M}_i' \hat{R}_{i\mu} \bar{M}_i = 0. \end{aligned} \tag{C.1}$$

$\bar{W}_i$ possesses a power series expansion about $\mu = 0$ since $P_{i\varepsilon}$ and $\bar{K}_i$ possess a power

series at $\mu = 0$. $\bar{W}_i$ can be expanded as

$$\bar{W}_i = \sum_{j=0}^{\infty} \frac{\mu^j}{j!} \begin{bmatrix} \bar{W}_{i1}^{(j)} & \mu \bar{W}_{i2}^{(j)} \\ \mu \bar{W}_{i2}^{(j)\prime} & \mu \bar{W}_{i3}^{(j)} \end{bmatrix}, \quad i = 1, 2. \tag{C.2}$$

If $\bar{W}_i$ and the power series expansion for $\bar{K}_i$ are substituted into (C.1) we get at $\mu = 0$

$$\bar{W}_{i1}^{(0)}[A_{11} - B_{1f}M_{fs} - B_{1f}M_{fs}] + [A_{11} - B_{1f}M_{fs} - B_{1f}M_{fs}]'\bar{W}_{i1}^{(0)} + \bar{W}_{i2}^{(0)}[A_{21} - B_{2f}M_{fs} - B_{2f}M_{fs}] + [A_{21} - B_{2f}M_{fs} - B_{2f}M_{fs}]'\bar{W}_{i2}^{(0)\prime} = 0 \tag{C.3}$$

$$\bar{W}_{i3}^{(0)}A_{22} + A_{22}'\bar{W}_{i3}^{(0)} = 0 \tag{C.4}$$

and

$$\bar{W}_{i1}^{(0)}A_{12} + \bar{W}_{i2}^{(0)}A_{22} + [A_{21} - B_{2f}M_{fs} - B_{2f}M_{fs}]'\bar{W}_{i3}^{(0)} = 0. \tag{C.5}$$

Since A_{22} is stable (C.4) implies that

$$\bar{W}_{i3}^{(0)} = 0. \tag{C.6}$$

Substitution of (C.6) into (C.5) gives

$$\bar{W}_{i2}^{(0)} = -\bar{W}_{i1}^{(0)}A_{12}A_{22}^{-1}. \tag{C.7}$$

Finally, substitution of (C.7) into (C.3) gives

$$0 = \bar{W}_{i1}^{(0)}[A_0 - B_{01}M_{1s} - B_{02}M_{2s}] + [A_0 - B_{01}M_{1s} - B_{02}M_{2s}]'\bar{W}_{i1}^{(0)}. \tag{C.8}$$

The matrix $[A_0 - B_{01}M_{1s} - B_{02}M_{2s}]$ is the feedback matrix of the slow subsystem (6) which is stable. Hence

$$\bar{W}_{i1}^{(0)} = 0, \quad i = 1, 2, \tag{C.9}$$

which implies that

$$\bar{W}_{i2}^{(0)} = 0, \quad i = 1, 2. \tag{C.10}$$

Thus we have proven Theorem III.

Feedback and Well-Posedness of Singularly Perturbed Nash Games

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Abstract—This paper discusses linear-quadratic Nash games for systems with slow and fast modes. Singular perturbation is employed to replace a Nash game played on the full model by a game played on a reduced order model. The well-posedness of different solutions corresponding to different state feedback information structures is considered. An important relation between the feedback information structure and the well-posedness of the game is found and used to conjecture special cases when the linear memoryless closed-loop solution is well-posed.

I. Introduction

In THIS paper we consider a two-player linear-quadratic Nash game for systems with slow and fast modes. Linear time-invariant models of many physical systems contain slow and fast modes. Control problems for such models are ill-conditioned and have motivated several model simplification approaches [1], [2] which neglect fast modes. In the singular perturbation method [3] both slow and fast modes are retained, but analysis and design problems are solved in two stages, first for the fast and then for the slow. We define two subgames, one for the slow modes and one for the fast modes, and obtain their open-loop and linear memoryless closed-loop Nash solutions. Then we analyze the asymptotic behavior of the Nash solution of the original game for different assumptions of the feedback information available to the players. In particular, we show that, for a first-order approximation, the open-loop Nash solution of the original game reduces to the sum of the open-loop solutions of the slow and fast games. For a linear memoryless partially closed-loop Nash solution (closed-loop in slow variables only), it is shown that, for a first-order approximation, the solution of the original game reduces to the sum of the linear memoryless closed-loop Nash solution of the slow game and the open-loop Nash solution of the fast game. Finally, in view of this analysis and of the work of Gardner and Cruz [4] on the closed-loop Nash solution, we discuss the well-posedness of Nash games in general and illustrate the impact of the feedback information available to players on the well-posedness of the game.

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II. Slow and Fast Subproblems

We consider a singularly perturbed linear time-invariant system

$$\dot{x} = A_{11}x + A_{12}z + B_{11}u_1 + B_{12}u_2, \qquad x(0) = x_0 \quad \text{(1a)}$$

$$\mu\dot{z} = A_{21}x + A_{22}z + B_{21}u_1 + B_{22}u_2, \qquad z(0) = z_0 \quad \text{(1b)}$$

where $\dim x = n_1$, $\dim z = n_2$, $\dim u_i = m_i$. The small singular perturbation parameter $\mu > 0$ represents small time constants, inertias, masses, etc. The vector z is "fast" since its derivative $\dot{z}$ is of order $1/\mu$ which is large. The ith player chooses his strategy u_i to minimize his performance criterion

$$J_i = \frac{1}{2}\int_0^{\infty} (y_i'y_i + u_i'R_{ii}u_i + u_j'R_{ij}u_j)\,dt, \qquad i \neq j \quad (2)$$

where $R_{ii} > 0$, $R_{ij} \geq 0$, and

$$y_i = C_{i1}x + C_{i2}z. \quad (3)$$

A Nash equilibrium solution of this game is a pair (u_1^*, u_2^*) such that

$$J_i(u_i^*, u_j^*) < J_i(u_i, u_j^*), \qquad i \neq j, \; i = 1,2 \quad (4)$$

for all admissible u_i. A slow subsystem is formed by neglecting the fast modes, which is equivalent to letting $\mu = 0$ in (1),

$$\dot{x}_s = A_{11}x_s + A_{12}z_s + B_{11}u_{1s} + B_{12}u_{2s}, \qquad x_s(0) = x_0 \quad \text{(5a)}$$

$$0 = A_{21}x_s + A_{22}z_s + B_{21}u_{1s} + B_{22}u_{2s} \quad \text{(5b)}$$

$$y_{is} = C_{i1}x_s + C_{i2}z_s. \quad (6)$$

Assuming that A_{22} is nonsingular,[1] we express z_s as

$$z_s = -A_{22}^{-1}(A_{21}x_s + B_{21}u_{1s} + B_{22}u_{2s}) \quad (7)$$

and, substituting it into (5) and (6), we define the slow subsystem of (1), (3) as

$$\dot{x}_s = A_0x_s + B_{01}u_{1s} + B_{02}u_{2s}, \qquad x_s(0) = x_0 \quad (8)$$

$$y_{is} = C_{i0}x_s + D_{i1}u_{1s} + D_{i2}u_{2s} \quad (9)$$

where

[1] If A_{22} is singular, the variables in the null space of A_{22} are not fast variables. A reformulation of the problem which includes these variables in the x-vector will lead to nonsingular A_{22}.

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$$A_0 = A_{11} - A_{12}A_{22}^{-1}A_{21}, \quad B_{0i} = B_{1i} - A_{12}A_{22}^{-1}B_{2i}$$
$$C_{i0} = C_{i1} - C_{i2}A_{22}^{-1}A_{21}, \quad D_{ij} = -C_{i2}A_{22}^{-1}B_{2j}, \qquad i,j = 1,2. \tag{10}$$

A fast subsystem is derived by assuming that the slow variables are constant during fast transients, that is, $\dot{z}_s = 0$ and $x_s =$ a constant. Letting $z_f = z - z_s$, $u_{if} = u_i - u_{is}$, $y_{if} = y_i - y_{is}$ the fast subsystem of (1), (3) is defined as

$$\frac{dz_f}{d\tau} = A_{22}z_f + B_{21}u_{1f} + B_{22}u_{2f}, \qquad z_f(0) = z(0) - z_s(0) \tag{11}$$

$$y_{if} = C_{i2}z_f \tag{12}$$

where $\tau = t/\mu$ is a stretched time scale. Following the treatment of the optimal regulator problem [5], our approach is to extract from J_i two quadratic performance criteria, one for the variables of the slow subsystem (8) and the other for the variables of the fast subsystem (11). We formulate two subgames.

Slow Nash Game

Find a Nash equilibrium solution (u_{1f}^*, u_{2f}^*) of

$$J_{is} = \frac{1}{2}\int_0^\infty \big[x_s'C_{i0}'C_{i0}x_s + 2x_s'C_{i0}'(D_{ii}u_{is} + D_{ij}u_{js}) + 2u_{is}'D_{ii}'D_{ij}u_{js} + u_{is}'(R_{ii} + D_{ii}'D_{ii})u_{is} + u_{js}'(R_{ij} + D_{ij}'D_{ij})u_{js}\big]\,dt, \qquad i \neq j,\ i,j = 1,2 \tag{13}$$

for the slow subsystem (8). The expression for J_{is} is obtained by formally substituting (7) into (2).

Fast Nash Game

Find a Nash equilibrium solution (u_{1f}^*, u_{2f}^*) of

$$J_{if} = \frac{1}{2}\int_0^\infty (z_f'C_{i2}'C_{i2}z_f + u_{if}'R_{ii}u_{if} + u_{jf}'R_{ij}u_{jf})\,d\tau, \qquad i \neq j \tag{14}$$

for the fast subsystem (11).

Necessary conditions for Nash solutions have been derived in [6], [7] using variational techniques. Unlike the optimal control problem (case of one player), the Nash equilibrium strategy for two or more players has different open-loop and closed-loop solutions. Thus, the closed-loop Nash solution cannot be obtained by solving the open-loop Nash game for every initial point, which is valid in the optimal control problem. The conditions of [6], [7] would be directly applicable if the problems (13) and (14) were for a finite-time interval. Since (13), (14) are infinite-time problems, the admissible strategies are limited to those for which the state vector converges to zero as $t \to \infty$. For example, the admissible open-loop strategies for the slow Nash game are square integrable, i.e.,

$$\int_0^\infty u_{is}'(t)u_{is}(t)\,dt < \infty \tag{15}$$

such that $x_s(t) \to 0$ as $t \to \infty$ and

$$\int_0^\infty x_s'(t)x_s(t)\,dt < \infty \tag{16}$$

where $x_s(t)$ is the solution of

$$\dot{x}_s = A_0x_s + B_{01}u_{1s}(t) + B_{02}u_{2s}(t).$$

The admissible closed-loop strategies are defined similarly except that $u_i = u_i(t, x_s)$. With strategies restricted to this admissible class, the finite-time necessary conditions can be extended and used for infinite-time problems, as it is done in this paper.

An open-loop Nash solution $(u_{1s}^{OL}(t), u_{2s}^{OL}(t))$ of the slow game must satisfy

$$\dot{x}_s = A_0x_s + B_{01}u_{1s} + B_{02}u_{2s}, \qquad x_s(0) = x_0,\ x_s(t) \to 0 \text{ as } t \to \infty \tag{17}$$

$$(R_{ii} + D_{ii}'D_{ii})u_{is}^{OL} + D_{ii}'D_{ij}u_{js}^{OL} + D_{ii}'C_{i0}x_s^{OL} + B_{0i}'\lambda_{is}^{OL} = 0, \qquad i \neq j \tag{18}$$

$$-\dot{\lambda}_{is}^{OL} = A_0'\lambda_{is}^{OL} + C_{i0}'C_{i0}x_s^{OL} + C_{i0}'\big(D_{ii}u_{is}^{OL} + D_{ij}u_{js}^{OL}\big), \qquad \lambda_{is}^{OL}(t) \to 0 \text{ as } t \to \infty. \tag{19}$$

Assuming that the matrix

$$R = \begin{bmatrix} R_{11} + D_{11}'D_{11} & D_{11}'D_{12} \\ D_{22}'D_{21} & R_{22} + D_{22}'D_{22} \end{bmatrix} \tag{20}$$

is nonsingular, (17), (18), and (19) can be rewritten as

$$\frac{d}{dt}\begin{bmatrix} x_s^{OL} \\ \lambda_{1s}^{OL} \\ \lambda_{2s}^{OL} \end{bmatrix} = V\begin{bmatrix} x_s^{OL} \\ \lambda_{1s}^{OL} \\ \lambda_{2s}^{OL} \end{bmatrix}. \tag{21}$$

We assume the following.

Assumption 1: There exists a unique pair $(u_{1s}^{OL}(t), u_{2s}^{OL}(t))$ satisfying (17), (18), and (19).

Assumption 2: The matrix V has no eigenvalues with zero real part, that is,

$$|\mathrm{Re}\,\lambda(V)| > 0. \tag{22}$$

A closed-loop Nash solution $(u_{1s}^{CL}(t,x_s), u_{2s}^{CL}(t,x_s))$ of the slow game must satisfy (17), (18), and

$$-\dot{\lambda}_{is}^{CL} = A_0'\lambda_{is}^{CL} + C_{i0}'C_{i0}x_s^{CL} + C_{i0}'\big(D_{ii}u_{is}^{CL} + D_{ij}u_{js}^{CL}\big) + \left(\frac{\partial u_{js}^{CL}}{\partial x_s}\right)' \cdot\big[(R_{ij} + D_{ij}'D_{ij})u_{js}^{CL} + D_{ij}'D_{ii}u_{is}^{CL} + D_{ij}'C_{i0}x_s^{CL} + B_{0j}'\lambda_{is}^{CL}\big], \qquad i \neq j,\ \lambda_{is}^{CL}(t) \to 0 \text{ as } t \to \infty. \tag{23}$$

The solution of (17), (18), and (23), if it exists, is in general not unique [8]. To avoid this nonuniqueness problem we restrict ourselves to a class of solutions in which

$$\lambda_{is}^{CL}(t, x_s^{CL}) = K_{is}x_s^{CL}(t) + g_{is}(t) \tag{24}$$

where K_{is} is an $n_1 \times n_1$ symmetric matrix. Subsequently, u_{is}^{CL} takes the form

$$u_{is}^{CL}(t, x_s^{CL}) = F_{is} x_s^{CL}(t) + h_{is}(t). \tag{25}$$

Substituting (24), (25) into (17), (18), and (23), and separating the variables we find that K_{is}, F_{is} satisfy the algebraic equations

$$0 = (R_{ii} + D_{ii}'D_{ii})F_{is} + D_{ii}'D_{ij}F_{js} + D_{ii}'C_{i0} + B_{0i}'K_{is}, \qquad i \neq j \tag{26}$$

$$\begin{aligned} 0 = K_{is}A_0 + A_0'K_{is} + C_{i0}'C_{i0} + K_{is}(B_{0i}F_{is} + B_{0j}F_{js}) \\ + C_{i0}'(D_{ii}F_{is} + D_{ij}F_{js}) + F_{js}'[(R_{ij} + D_{ij}'D_{ij})F_{js} \\ + D_{ij}'D_{ii}F_{is} + D_{ij}'C_{i0} + B_{0j}'K_{is}], \qquad i \neq j \end{aligned} \tag{27}$$

while $g_{is}(t)$, $h_{is}(t)$ satisfy the equations

$$0 = (R_{ii} + D_{ii}'D_{ii})h_{is}(t) + D_{ii}'D_{ij}h_{js}(t) + B_{0i}'g_{is}(t), \qquad i \neq j \tag{28}$$

$$\begin{aligned} -\dot{g}_{is} = A_0'g_{is}(t) + K_{is}(B_{0i}h_{is}(t) + B_{0j}h_{js}(t)) \\ + C_{i0}'(D_{ii}h_{is}(t) + D_{ij}h_{js}(t)) \\ + F_{js}'[(R_{ij} + D_{ij}'D_{ij})h_{js}(t) + D_{ij}'D_{ii}h_{is}(t) \\ + B_{0j}'g_{is}(t)], \qquad i \neq j, \; g_{is}(t) \to 0 \text{ as } t \to \infty. \end{aligned} \tag{29}$$

Using (28) to eliminate $h_{is}(t)$ from (29) we find that $g_{1s}(t)$, $g_{2s}(t)$ satisfy a homogeneous equation

$$-\begin{bmatrix} \dot{g}_{1s} \\ \dot{g}_{2s} \end{bmatrix} = W \begin{bmatrix} g_{1s} \\ g_{2s} \end{bmatrix}, \qquad g_{is}(t) \to 0 \text{ as } t \to \infty. \tag{30}$$

We assume the following.

Assumption 3: There exists a unique solution $(K_{1s}, K_{2s}, F_{1s}, F_{2s})$ of (26) and (27) such that

$$\operatorname{Re}\{\lambda(A_0 + B_{01}F_{1s} + B_{02}F_{2s})\} < 0. \tag{31}$$

Assumption 4:

$$\operatorname{Re}\{\lambda(W)\} < 0. \tag{32}$$

Assumptions 3 and 4 guarantee that in the class of linear memoryless closed-loop solutions of the form (25) there exists a unique pair $(u_{1s}^{CL}, u_{2s}^{CL})$ given by

$$u_{is}^{CL}(x_s, t) = F_{is}x_s(t). \tag{33}$$

An open-loop Nash solution $(u_{1f}^{OL}(\tau), u_{2f}^{OL}(\tau))$ of the fast game must satisfy

$$\frac{dz_f}{d\tau} = A_{22}z_f + B_{21}u_{1f} + B_{22}u_{2f}, \qquad z_f(0) = z_0 - z_s(0), \qquad z_f(\tau) \to 0 \text{ as } \tau \to \infty \tag{34}$$

$$u_{if}^{OL} = -R_{ii}^{-1}B_{2i}'\lambda_{if}^{OL}(\tau) \tag{35}$$

$$-\frac{d\lambda_{if}^{OL}}{d\tau} = A_{22}'\lambda_{if}^{OL} + C_{i2}'C_{i2}z_f^{OL}, \qquad \lambda_{if}^{OL}(\tau) \to 0 \text{ as } \tau \to \infty. \tag{36}$$

We assume the following.

Assumption 5: There exists a unique solution $(u_{1f}^{OL}(\tau), u_{2f}^{OL}(\tau))$ of (34), (35), and (36).

Assumption 6: The matrix

$$D = \begin{bmatrix} A_{22} & -B_{21}R_{11}^{-1}B_{21}' & -B_{22}R_{22}^{-1}B_{22}' \\ -C_{12}'C_{12} & -A_{22}' & 0 \\ -C_{22}'C_{22} & 0 & -A_{22}' \end{bmatrix} \tag{37}$$

has no eigenvalues with zero real part.

We now want to investigate the relation between the Nash equilibrium solution of the original game (u_1^*, u_2^*) and the Nash equilibrium solutions of the slow and fast games defined above. To see what we want from such an investigation let us recall a similar optimal control problem. It has been shown in [5] that the optimal control $u^*(t,\mu)$ satisfies the relation $u^*(t,\mu) = u_s^*(t) + u_f^*(\tau) + O(\mu)$ for all $t \geq 0$, where $u_s^*(t)$ and $u_f^*(\tau)$ are the unique stabilizing solutions of slow and fast subproblems defined in a way similar to our slow and fast subgames. This means that for sufficiently small μ one can replace the optimal control by the sum of the slow and fast controls, thus solving two lower order, well-conditioned problems instead of solving the original ill-conditioned problem. Other advantages of this near optimal design, like allowing the ignorance of the value of μ and achieving $O(\mu^2)$ approximation in the value of the performance criterion, are discussed in [5]. Moreover, for all $t > 0$, $\lim_{\mu \to 0} u^*(t,\mu) = u_s^*(t)$, that is, for all t except $t = 0$, the solution of the original control problem tends to the solution of the slow problem as $\mu \to 0$. Thus, neglecting μ either in modeling the system or in the exact solution leads to the same lower order solution. A problem which has this property is said to be well-posed, since the design algorithm is not too sensitive to modeling errors, a property which a practical engineer intuitively requires.

The Nash solution does not always have this property. Gardner and Cruz [4] have shown by a counterexample that the closed-loop solution of a Nash game does not tend to the closed-loop solution of the slow subgame as $\mu \to 0$. It is important to understand why the closed-loop solution of the Nash game fails to possess the desired well-posedness property. The answer is to be sought in the difference between open-loop and closed-loop solutions of a Nash game. When analyzing asymptotic behavior of solutions one should be careful not to confuse solutions corresponding to different feedback information. For example, if we are looking at the open-loop solution of the original game, we should not compare its limit to the closed-loop solution of the slow game. In other words, both the original and slow games should be solved under the same assumption about the feedback information available to the players. One obvious case is the open-loop. If we solve the open-loop original game, then we expect its solution to tend to the open-loop solution of the slow game as $\mu \to 0$. This open-loop problem is investigated in Section III. Now let us assume that the slow game has been solved under the assumption that x_s is fedback, that is we have the closed-loop slow solution. We

want to identify a solution of the original game, which, in the limit as $\mu \to 0$, has the same feedback information as the closed-loop slow solution. Since we usually require the closeness of x and x_s for sufficiently small μ we expect that such a solution should assume that x is fedback. What about the feedback of z? If we assume that z also is fedback, that is, seeking a closed-loop solution of the original game, then we would have changed the feedback information assumption between the slow and original games because a feedback from z contains information about the fast dynamics of the system, something which is not available in the slow game. This means that the original problem which has the same feedback information as the closed-loop slow problem is a partially closed-loop game in which only x is fedback. This partially closed-loop Nash solution is investigated in Section IV.

III. Open-Loop Solution

An open-loop Nash equilibrium solution of the original game (1) and (2) must satisfy

$$\dot{x} = A_{11}x + A_{12}z + B_{11}u_1 + B_{12}u_2, \qquad x(0)=x_0,\ x(t)\to 0 \text{ as } t\to\infty \tag{38}$$

$$\mu\dot{z} = A_{21}x + A_{22}z + B_{21}u_1 + B_{22}u_2, \qquad z(0)=z_0,\ z(t)\to 0 \text{ as } t\to\infty \tag{39}$$

$$0 = R_{ii}u_i + B_{1i}'\lambda_i + B_{2i}'\rho_i \tag{40}$$

$$-\dot{\lambda}_i = A_{11}'\lambda_i + A_{21}'\rho_i + C_{i1}'C_{i1}x + C_{i1}'C_{i2}z, \qquad \lambda_i(t)\to 0 \text{ as } t\to\infty \tag{41}$$

$$-\mu\dot{\rho}_i = A_{12}'\lambda_i + A_{22}'\rho_i + C_{i2}'C_{i1}x + C_{i2}'C_{i2}z, \qquad \rho_i(t)\to 0 \text{ as } t\to\infty. \tag{42}$$

The relation between the solution of the two-point boundary value problem (38)–(42) and the open-loop solutions of the slow and fast games is established in the following theorem.

Theorem 1: Under Assumptions 1, 2, 5, and 6 there exists $\mu^* > 0$ such that for every $\mu \in (0,\mu^*]$, (38)–(42) possess a unique solution

$$x^{OL}(t), z^{OL}(t), u_i^{OL}(t), \lambda_i^{OL}(t), \rho_i^{OL}(t)$$

such that

$$u_i^{OL}(t) = u_{is}^{OL}(t) + u_{if}^{OL}(\tau) + O(\mu), \qquad \forall t \geq 0. \tag{43}$$

This theorem, which is proved in Appendix A, suggests that

$$u_i^{AOL}(t) = u_{is}^{OL}(t) + u_{if}^{OL}(\tau), \qquad i=1,2 \tag{44}$$

may be used as an approximation of the exact open-loop equilibrium strategy $(u_1^{OL}(t), u_2^{OL}(t))$, although it is not a Nash strategy. To justify its use as an approximation of a Nash strategy we suppose that Player 2, for example, uses the approximate strategy $u_2^{AOL}(t)$ and pose the following question. Does there exist a strategy $u_1(t) \neq u_1^{AOL}(t)$ such that $J_1(u_1, u_2^{AOL}) < J_1(u_1^{AOL}, u_2^{AOL})$? Or, in other words, can Player 1 benefit by deviating from u_1^{AOL}? The answer is given in Theorem 2 which establishes the "asymptotic Nash" property of the approximate strategy (u_1^{AOL}, u_2^{AOL}).

Theorem 2:

$$J_i(u_i^{AOL}, u_j^{AOL}) < J_i(u_i, u_j^{AOL}) + O(\mu) \tag{45}$$

for all admissible u_i, $j \neq i$, $i,j=1,2$.

In words, the theorem states that neither player can benefit by more than $O(\mu)$ if he unilaterally deviates from the approximate open-loop strategy. In the limit as $\mu\to 0$, the approximate strategy has the Nash property, i.e., it satisfies an equality similar to (4).

Proof: Using (43) it is straightforward to show that

$$J_i(u_1^{AOL}, u_2^{AOL}) = J_i(u_1^{OL}, u_2^{OL}) + O(\mu), \qquad i=1,2 \tag{46}$$

$$J_i(u_i, u_j^{AOL}) = J_i(u_i, u_j^{OL}) + O(\mu) \tag{47}$$

for all admissible u_i, $j\neq i$, $i,j=1,2$.

Consider

$$J_i(u_i^{AOL}, u_j^{AOL}) = J_i(u_i, u_j^{AOL}) + J_i(u_i^{AOL}, u_j^{AOL}) - J_i(u_i, u_j^{AOL}) + J_i(u_i^{OL}, u_j^{OL}) - J_i(u_i^{OL}, u_j^{OL})$$

for any admissible u_i. From (4) we get

$$J_i(u_i^{AOL}, u_j^{AOL}) < J_i(u_i, u_j^{AOL}) + J_i(u_i^{AOL}, u_j^{AOL}) - J_i(u_i^{OL}, u_j^{OL}) + J_i(u_i, u_j^{OL}) - J_i(u_i, u_j^{AOL}).$$

Using (46) and (47) proves (45).

IV. Partially Closed-Loop Solution

When only x is available for measurement we seek a partially closed-loop Nash equilibrium solution of the original game (1), (2). It is assumed that each player has access to the current value of $x(t)$ with no recall of past values, a memoryless partially closed-loop information structure. To complete our definition of the information structure we need to state which part of the initial conditions x_0, z_0 will be known to the players. Let us recall that in open-loop solutions the players know the initial conditions x_0, z_0, while in memoryless closed-loop solutions they do not. In partially closed-loop solutions the players should be provided with open-loop information concerning z. Such information includes not only z_0 but also x_0, because the initial conditions $z_f(0)$ of the fast subsystem depend on both x_0 and z_0 as it can be seen from (11). In summary, $u_i^{PCL} = u_i(t, x(t), x_0, z_0)$.

The partially closed-loop solution must satisfy (38), (39), (40), and (42) as in the open-loop case, while (41) is replaced by

$$-\dot{\lambda}_i = A_{11}'\lambda_i + A_{21}'\rho_i + C_{i1}'C_{i1}x + C_{i1}'C_{i2}z + \left(\frac{\partial u_j}{\partial x}\right)'\left[R_{ij}u_j + B_{1j}'\lambda_{i1} + B_{2j}'\lambda_{i2}\right], \qquad i\neq j. \tag{48}$$

The difference between (48) and (41) is in the presence of

the term $\partial u_j/\partial x$ which allows for the possible dependence of u_j on x. We restrict ourselves to a solution in which λ_i, ρ_i take the form

$$\lambda_i(t,\mu,x,x_0,z_0) = K_i(\mu)x(t,\mu) + g_i(t,\mu,x_0,z_0) \quad (49a)$$

$$\rho_i(t,\mu,x,x_0,z_0) = P_i(\mu)x(t,\mu) + q_i(t,\mu,x_0,z_0). \quad (49b)$$

Equation (40) shows that u_i takes the form

$$u_i(t,\mu,x,x_0,z_0) = F_i(\mu)x(t,\mu) + h_i(t,\mu,x_0,z_0) \quad (50)$$

where F_i and h_i are functions of K_i, P_i, g_i, and q_i. Equations (38), (39), (40), (42), and (48) can now be replaced by

$$\dot{x} = \left(A_{11} + \sum_{i=1}^{2} B_{1i}F_i\right)x + A_{12}z + \sum_{i=1}^{2} B_{1i}h_i, \qquad x(0) = x_0, \quad x(t)\to 0 \text{ as } t\to\infty \quad (51)$$

$$\mu\dot{z} = \left(A_{21} + \sum_{i=1}^{2} B_{2i}F_i\right)x + A_{22}z + \sum_{i=1}^{2} B_{2i}h_i, \qquad z(0) = z_0, \quad z(t)\to 0 \text{ as } t\to\infty \quad (52)$$

$$0 = (R_{ii}F_i + B'_{1i}K_i + B'_{2i}P_i)x + R_{ii}h_i + B'_{1i}g_i + B'_{2i}q_i \quad (53)$$

$$\begin{aligned} -(K_i\dot{x} + \dot{g}_i) = {} & (A'_{11}k_i + A'_{21}P_i)x + A'_{11}g_i + A'_{21}q_i \\ & + C'_{i1}C_{i1}x + C'_{i1}C_{i2}z \\ & + F'_j\big[(R_{ij}F_j + B'_{1j}K_i + B'_{2j}P_i)x \\ & + R_{ij}h_j + B'_{1j}g_i + B'_{2j}q_i\big], \qquad g_i(t)\to 0 \text{ as } t\to\infty \end{aligned} \quad (54)$$

$$\begin{aligned} -\mu(P_i\dot{x} + \dot{q}_i) = {} & (A'_{12}K_i + A'_{22}P_i)x + A'_{12}g_i + A'_{22}q_i \\ & + C'_{i2}C_{i1}x + C'_{i2}C_{i2}z, \qquad q_i(t)\to 0 \text{ as } t\to\infty. \end{aligned} \quad (55)$$

Investigation of (51)–(55), which is carried out in Appendix B, leads to the following theorem.

Theorem 3: Under the assumptions of Lemmas B1 and B2 there exists $\mu^* > 0$ such that for every $\mu\in(0,\mu^*]$, (51)–(55) possess a solution $x^{PCL}(t), z^{PCL}(t), u_i^{PCL}(t, x^{PCL})$ such that for all $t>0$

$$x^{PCL}(t,\mu) = x_s^{CL}(t) + O(\mu) \quad (56a)$$

$$z^{PCL}(t,\mu) = z_s^{CL}(t) + z_f^{OL}(\tau) + O(\mu) \quad (56b)$$

$$u_i^{PCL}(t,\mu,x^{PCL}) = F_{is}x^{PCL}(t,\mu) + u_{if}^{OL}(\tau) + O(\mu). \quad (56c)$$

Theorem 3 establishes the existence of a partially closed-loop solution. We cannot claim the uniqueness because the information available to each player includes the initial conditions x_0, z_0 allowing each player to generate $x(t)$ in the open-loop term h_i of (50). Thus, the separation of variables technique used in Appendix B is one but not necessarily the only way to construct the solution [8].

Theorem 3 suggests that

$$u_i^{APC} = F_{is}x + u_{if}^{OL}(\tau), \qquad i = 1,2 \quad (57)$$

may be used as an approximation of the constructed partially closed-loop solution. The validity of such approximation is established in Theorem 4 which shows that the approximate strategy (u_1^{APC}, u_2^{APC}) has the "asymptotic Nash" property.

Theorem 4:

$$J_i(u_i^{APC}, u_j^{APC}) < J_i(u_i, u_j^{APC}) + O(\mu) \quad (58)$$

for all admissible u_i, $j\neq i$, $i,j = 1,2$.

Proof: Let $x^A(t), z^A(t)$ be the trajectory resulting from applying u_1^{APC}, u_2^{APC} to (1), then x^A, z^A satisfy

$$\frac{dx^A}{dt} = \left(A_{11} + \sum_{i=1}^{2} B_{1i}F_{is}\right)x^A(t) + A_{12}z^A(t) + \sum_{i=1}^{2} B_{1i}u_{if}^{OL}(\tau), \qquad x^A(0) = x_0 \quad (59a)$$

$$\mu\frac{dz^A}{dt} = \left(A_{21} + \sum_{i=1}^{2} B_{1i}F_{is}\right)x^A(t) + A_{22}z^A(t) + \sum_{i=1}^{2} B_{2i}u_{if}^{OL}(\tau), \qquad z^A(0) = z_0. \quad (59b)$$

Using standard singular perturbation techniques [5] it can be shown that for all $t>0$

$$x^A(t) = x_s^{CL}(t) + O(\mu) \quad (60a)$$

$$z^A(t) = z_s^{CL}(t) + z_f^{OL}(\tau) + O(\mu). \quad (60b)$$

Hence,

$$x^A(t) = x^{PCL}(t) + O(\mu) \quad (61a)$$

$$z^A(t) = z^{PCL}(t) + O(\mu) \quad (61b)$$

$$u_i^{APC}(t) = u_i^{PCL}(t) + O(\mu). \quad (61c)$$

Now it is straightforward to show that

$$J_i(u_1^{APC}, u_2^{APC}) = J_i(u_1^{PCL}, u_2^{PCL}) + O(\mu), \qquad i = 1,2. \quad (62)$$

Using a similar argument it can be shown that

$$J_i(u_i, u_j^{APC}) = J_i(u_i, u_j^{PCL}) + O(\mu) \quad (63)$$

for all admissible u_i, $j\neq i$, $i,j = 1,2$. The rest of the proof is similar to the proof of Theorem 2, and is omitted.

In many engineering applications only feedback strategies are allowed. For such applications the approximate strategy (57) is not satisfactory because it contains an open-loop term $u_{if}^{OL}(\tau)$. However, in an important special case this term can be neglected.

Lemma: If the assumptions of Lemmas B1 and B2 are satisfied and if

$$\mathrm{Re}\{\lambda(A_{22})\} < 0, \quad (64)$$

then the application of the reduced strategy

$$u_{ir} = F_{is}x, \qquad i = 1,2 \quad (65)$$

to the actual system (1) results in

$$J_i(u_{1r}, u_{2r}) = J_i(u_1^{PCL}, u_2^{PCL}) + O(\mu), \qquad i = 1,2. \quad (66)$$

Moreover, the reduced strategy has the "asymptotic Nash" property

$$J_i(u_{ir},u_{jr}) < J_i(u_i,u_{jr}) + 0(\mu) \tag{67}$$

for all admissible u_i, $j \neq i$, $i,j = 1,2$.

Proof: Let $x_r(t), z_r(t)$ be the trajectory resulting from applying u_{1r}, u_{2r} to (1), then x_r, z_r satisfy

$$\dot{x}_r = \left(A_{11} + \sum_{i=1}^{2} B_{1i}F_{is}\right)x_r(t) + A_{12}z_r(t), \qquad x_r(0) = x_0 \tag{68a}$$

$$\mu\dot{z}_r = \left(A_{21} + \sum_{i=1}^{2} B_{2i}F_{is}\right)x_r(t) + A_{22}z_r(t), \qquad z_r(0) = z_0. \tag{68b}$$

Using standard singular perturbation techniques [3] it can be shown that for all $t > 0$

$$x_r(t) = x_s^{\mathrm{CL}}(t) + 0(\mu) \tag{69a}$$

$$z_r(t) = z_s^{\mathrm{CL}}(t) + \exp\left[A_{22}\tau\right]z_f(0) + 0(\mu). \tag{69b}$$

Hence,

$$x_r(t) = x^{\mathrm{PCL}}(t) + 0(\mu) \tag{70a}$$

$$z_r(t) = z^{\mathrm{PCL}}(t) + \exp\left[A_{22}\tau\right]z_f(0) + z_f^{\mathrm{OL}}(\tau) + 0(\mu) \tag{70b}$$

$$u_{ir}(t) = u_i^{\mathrm{PCL}}(t) - u_{if}^{\mathrm{OL}}(\tau) + 0(\mu). \tag{70c}$$

Lemmas B1 and B2 and (64) guarantee that

$$\int_0^{t_1} \exp\left[A_{22}\frac{t}{\mu}\right]dt = 0(\mu) \tag{71a}$$

$$\int_0^{t_1} z_f^{\mathrm{OL}}\left(\frac{t}{\mu}\right)dt = 0(\mu) \tag{71b}$$

$$\int_0^{t_1} u_{if}^{\mathrm{OL}}\left(\frac{t}{\mu}\right)dt = 0(\mu) \tag{71c}$$

for any fixed $t_1 > 0$, which implies (66). The rest of the proof is similar to that of Theorem 4.

This lemma says that if A_{22} is a stable matrix, a designer can neglect the fast modes of the system, solve a closed-loop Nash game for the slow part, and apply a reduced strategy (65) to the actual system (1). Doing this he will be sure that in the limit as $\mu \to 0$ he is approaching a Nash equilibrium point, however, in the space of linear memoryless partially closed-loop strategies rather than in the space of linear memoryless closed-loop strategies.

V. Example

Consider the second-order system

$$\dot{x} = z, \qquad x(0) = \sqrt{3} \tag{72a}$$

$$\mu\dot{z} = -x - z + u_1 + u_2, \qquad z(0) = 1 \tag{72b}$$

with performance criteria

$$J_1 = \frac{1}{2}\int_0^{\infty}\left(z^2 + u_1^2\right)dt \tag{73}$$

$$J_2 = \frac{1}{2}\int_0^{\infty}\left(z^2 + u_2^2\right)dt. \tag{74}$$

We investigate the well-posedness of open-loop, closed-loop, and partially closed-loop Nash solutions of the two-player game (72) to (74). The open-loop solution is given by

$$u_1^{\mathrm{OL}}(t) = u_2^{\mathrm{OL}}(t) = \left[0, -\frac{1}{2}(\sqrt{3}-1)\right] \cdot \left[\exp t\begin{bmatrix} 0 & 1 \\ \frac{-1}{\mu} & \frac{-\sqrt{3}}{\mu}\end{bmatrix}\right]\begin{bmatrix}\sqrt{3} \\ 1\end{bmatrix}. \tag{75}$$

For sufficiently small μ, (71) can be expressed as

$$u_1^{\mathrm{OL}}(t) = u_2^{\mathrm{OL}}(t) = -\frac{1}{2}(\sqrt{3}-1)\left[-e^{-t/\sqrt{3}} + 2e^{-\sqrt{3}t/\mu}\right] + 0(\mu). \tag{76}$$

On the other hand, if the slow and fast dynamics of the system (72) are considered separately, slow and fast games can be defined as in Section I. The open-loop Nash solution of the slow game is

$$u_{1s}^{\mathrm{OL}}(t) = u_{2s}^{\mathrm{OL}}(t) = \frac{1}{2}(\sqrt{3}-1)e^{-t/\sqrt{3}} \tag{77}$$

while the open-loop Nash solution of the fast game is

$$u_{1f}^{\mathrm{OL}}(t/\mu) = u_{2f}^{\mathrm{OL}}(t/\mu) = -(\sqrt{3}-1)e^{-\sqrt{3}t/\mu}. \tag{78}$$

Equations (76), (77), and (78) show that

$$u_i^{\mathrm{OL}}(t) = u_{is}^{\mathrm{OL}}(t) + u_{if}^{\mathrm{OL}}(t/\mu) + 0(\mu). \qquad \forall t > 0, \ i = 1,2. \tag{79}$$

In particular, we get that for all $t > 0$, $u_i^{\mathrm{OL}}(t) \to u_{is}^{\mathrm{OL}}(t)$ as $\mu \to 0$. Hence, the open-loop Nash strategy is robust against the inaccuracies caused by neglecting higher order dynamics. For this reason the open-loop Nash solution is said to be well-posed. As a consequence of this well-posedness the sum of the open-loop Nash solutions of the slow and fast games can be used as an approximation of the open-loop Nash solution of the full game in the sense of Theorem 2.

The linear closed-loop Nash solution of the game (72), (73) and (74) is given by

$$u_1^{\mathrm{CL}} = u_2^{\mathrm{CL}} = -\frac{1}{3}z \tag{80}$$

and the corresponding values of the performance criteria are

$$J_1^{\mathrm{CL}} = J_2^{\mathrm{CL}} = \frac{1}{2}\left(1 + \frac{\mu}{3}\right). \tag{81}$$

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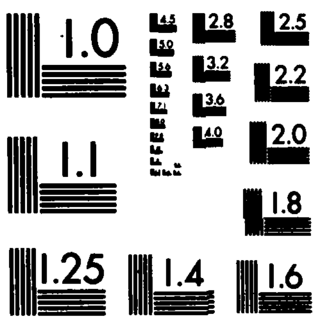

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A slow game defined as in Section I would have the linear closed-loop Nash solution

$$u_{1s}^{CL}=u_{2s}^{CL}=\frac{1}{7}(3-\sqrt{2})x_s$$

with

$$J_{1s}^{CL}=J_{2s}^{CL}=\frac{3}{14}(3\sqrt{2}-2).$$

It is seen that the limit of J_i^{CL} as $\mu\to 0$ is different from the value of J_{is}^{CL} at $\mu=0$. This discrepancy between the J's in the neighborhood of $\mu=0$ indicates that the closed-loop Nash strategy yields a performance criterion which is not robust.

Finally, the linear partially closed-loop Nash solution of the full game (72) to (74) can be expressed as

$$u_1^{PCL}=u_2^{PCL}$$
$$=\left[\frac{1}{7}(3-\sqrt{2})+O(\mu)\right]x-(\sqrt{3}-1)e^{-3t/\mu}+O(\mu)$$

with

$$\lim_{\mu\to 0}J_i^{PCL}=\frac{3}{14}(3\sqrt{2}-2),\qquad i=1,2$$

showing that the partially closed-loop Nash solution of the full game tends to the closed-loop Nash solution of the slow game as $\mu\to 0$.

VI. Discussion

We have pointed out that a study of the well-posedness of Nash equilibrium solutions should take into consideration the feedback information available to the players in both the original and slow games. We have identified two cases where the players of the original game and those of the slow game have the same information. The first case is when both games are played under open-loop assumptions. The second case is when the slow game is solved under closed-loop memoryless assumption, while the original game is solved under partial closed-loop memoryless assumption, closed-loop in x. In both cases it has been demonstrated that the Nash equilibrium solution is well-posed. This leads to the conclusion that Nash equilibrium solutions are well-posed as long as the original game and the slow game are solved under the same assumption of feedback information available to the players.

In their investigation of the closed-loop solution of the original game [4], Gardner and Cruz have shown that the linear memoryless closed-loop solution of the original game (u_1^{CL},u_2^{CL}) satisfies $u_i^{CL}=u_{is}+u_{if}^{CL}$ where u_{if}^{CL} is the linear memoryless closed-loop solution of the fast game. However, u_{is} is not, in general, the closed-loop solution of the slow game defined in this paper. It is the linear memoryless closed-loop solution of another game called the modified slow game [4]. In fact, in view of the result of Section IV we should not expect u_{is}^{CL} to be the limit of u_i^{CL} since we have shown that it is the limit of u_i^{PCL}. However, there are special cases when u_i^{CL} tends to u_{is}^{CL} as $\mu\to 0$. Gardner and Cruz [4] have reported two cases. The first case is the zero sum game, and the second one is the case when $C_{i2}=0$. Khalil [9] has reported a third case in which the fast vector z has been partitioned into two weakly-coupled subvectors with each player controlling only his subvector. An obvious fourth special case is the identical goal game, since it reduces to an optimal control problem. The existence of such special cases raises a question about additional conditions which guarantee that u_i^{CL} tends to u_{is}^{CL} as $\mu\to 0$. Since u_{is}^{CL} is the limit of u_i^{PCL}, then if u_i^{CL} tends to u_{is}^{CL}, it should be true that in the limit as $\mu\to 0$ the closed-loop solution and the partial closed-loop solution are equivalent. In other words, the fast game should have the same Nash cost under both open-loop and closed-loop memoryless information structures. All the four cases reported above have this property. We make a conjecture that whenever the fast game (11), (14) has the property that its Nash cost is the same under both open-loop and closed-loop memoryless information structures, the linear memoryless closed-loop solution of the original game tends to the linear memoryless closed-loop solution of the slow game as $\mu\to 0$.

Appendix A

Proof of Theorem 1: Using (40) to eliminate u_i, we write (38) to (42) as

$$\begin{bmatrix}\dot{x}\\ \dot{\lambda}_1\\ \dot{\lambda}_2\end{bmatrix}=A\begin{bmatrix}x\\ \lambda_1\\ \lambda_2\end{bmatrix}+B\begin{bmatrix}z\\ \rho_1\\ \rho_2\end{bmatrix},\qquad x(0)=x_0,\ x(t)\to 0,$$
$$\text{and } \lambda_i(t)\to 0 \text{ as } t\to\infty \tag{A1}$$

$$\mu\begin{bmatrix}\dot{z}\\ \dot{\rho}_1\\ \dot{\rho}_2\end{bmatrix}=C\begin{bmatrix}x\\ \lambda_1\\ \lambda_2\end{bmatrix}+D\begin{bmatrix}z\\ \rho_1\\ \rho_2\end{bmatrix},\qquad z(0)=z_0,\ z(t)\to 0,$$
$$\text{and } \rho_i(t)\to 0 \text{ as } t\to\infty \tag{A2}$$

where

$$A=\begin{bmatrix}A_{11} & -B_{11}R_{11}^{-1}B_{11}' & -B_{12}R_{22}^{-1}B_{12}'\\ -C_{11}'C_{11} & -A_{11}' & 0\\ -C_{21}'C_{21} & 0 & -A_{11}'\end{bmatrix},$$

$$B=\begin{bmatrix}A_{12} & -B_{11}R_{11}^{-1}B_{21}' & -B_{12}R_{22}^{-1}B_{22}'\\ -C_{11}'C_{12} & -A_{21}' & 0\\ -C_{21}'C_{22} & 0 & -A_{21}'\end{bmatrix},$$

$$C=\begin{bmatrix}A_{21} & -B_{21}R_{11}^{-1}B_{11}' & -B_{22}R_{22}^{-1}B_{12}'\\ -C_{12}'C_{11} & -A_{12}' & 0\\ -C_{22}'C_{21} & 0 & -A_{12}'\end{bmatrix},$$

and D is given in (37). We transform (A1), (A2) into the separate slow and fast parts

$$\begin{bmatrix} \dot{v} \\ \dot{\xi}_1 \\ \dot{\xi}_2 \end{bmatrix} = (A - BT) \begin{bmatrix} v \\ \xi_1 \\ \xi_2 \end{bmatrix} \tag{A3}$$

$$\mu \begin{bmatrix} \dot{w} \\ \dot{\eta}_1 \\ \dot{\eta}_2 \end{bmatrix} = (D + \mu TB) \begin{bmatrix} w \\ \eta_1 \\ \eta_2 \end{bmatrix}. \tag{A4}$$

The transformation used is

$$\begin{bmatrix} v \\ \xi_1 \\ \xi_2 \end{bmatrix} = (I - \mu ST) \begin{bmatrix} x \\ \lambda_1 \\ \lambda_2 \end{bmatrix} - \mu S \begin{bmatrix} z \\ \rho_1 \\ \rho_2 \end{bmatrix}, \tag{A5}$$

$$\begin{bmatrix} w \\ \eta_1 \\ \eta_2 \end{bmatrix} = T \begin{bmatrix} x \\ \lambda_1 \\ \lambda_2 \end{bmatrix} + \begin{bmatrix} z \\ \rho_1 \\ \rho_2 \end{bmatrix} \tag{A6}$$

where T and S satisfy the algebraic equations

$$0 = DT - C - \mu T(A - BT) \tag{A7}$$

$$0 = -S(D + \mu TB) + B + \mu(A - BT)S. \tag{A8}$$

Assumption 6 guarantees that for sufficiently small μ there exist unique T and S satisfying (A7) and (A8). Furthermore,

$$T = D^{-1}C + 0(\mu). \tag{A9}$$

Thus, we end up with two problems:

$$\begin{bmatrix} \dot{v} \\ \dot{\xi}_1 \\ \dot{\xi}_2 \end{bmatrix} = (A - BD^{-1}C + 0(\mu)) \begin{bmatrix} v \\ \xi_1 \\ \xi_2 \end{bmatrix}, \quad v(0) = v_0, \; v(t) \to 0, \quad \text{and } \xi_i(t) \to 0 \text{ as } t \to \infty \tag{A10}$$

$$\mu \begin{bmatrix} \dot{w} \\ \dot{\eta}_1 \\ \dot{\eta}_2 \end{bmatrix} = (D + 0(\mu)) \begin{bmatrix} w \\ \eta_1 \\ \eta_2 \end{bmatrix}, \quad w(0) = w_0, \; w(t) \to 0, \quad \text{and } \eta_i(t) \to 0 \text{ as } t \to \infty. \tag{A11}$$

Using the matrix identity

$$\begin{bmatrix} R_{11}^{-1}B_{21}' & 0 \\ 0 & R_{22}^{-1}B_{22}' \end{bmatrix} \cdot \begin{bmatrix} A_{22}' - C_{12}'D_{11}R_{11}^{-1}B_{21}' & -C_{12}'D_{12}R_{22}^{-1}B_{22}' \\ -C_{22}'D_{21}R_{11}^{-1}B_{21}' & A_{22}' - C_{22}'D_{22}R_{22}^{-1}B_{22}' \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} R_{11} + D_{11}'D_{11} & D_{11}'D_{12} \\ D_{22}'D_{21} & R_{22} + D_{22}'D_{22} \end{bmatrix}^{-1} \begin{bmatrix} B_{21}'A_{22}'^{-1} & 0 \\ 0 & B_{22}'A_{22}'^{-1} \end{bmatrix} \tag{A12}$$

it can be shown, after straightforward calculation, that

$$A - BD^{-1}C = V, \tag{A13}$$

$$v_0 = x_0 + 0(\mu), \tag{A14}$$

$$w_0 = z_f(0) + 0(\mu). \tag{A15}$$

Now we establish the existence of the solution of (A10) based on Assumptions 1 and 2. Assumption 1 implies the existence of a unique $2n_1 \times n_1$ matrix E such that (21) and $x_s(0) = x_0$, $(\lambda_{1s}'(0), \lambda_{2s}'(0)) = x_0'E'$ have an exponentially decaying solution. By Assumption 2 the eigenvalue structure of $A - BD^{-1}C + 0(\mu)$ is the same as that of $A - BD^{-1}C$. Thus, there exists a unique matrix $\tilde{E} = E + 0(\mu)$ such that (A10) and $v(0) = v_0$, $(\xi_1'(0), \xi_2'(0)) = v_0'\tilde{E}'$ have an exponentially decaying solution. This establishes the existence of a unique solution of (A10). Moreover, (A14) implies that

$$v(t) = x_s^{OL}(t) + 0(\mu), \tag{A16}$$

$$\xi_i(t) = \lambda_{is}^{OL}(t) + 0(\mu), \qquad \forall t > 0. \tag{A17}$$

By a similar argument we can show that there exists a unique solution of (A11) which satisfies

$$w(\tau) = z_f^{OL}(\tau) + 0(\mu) \tag{A18}$$

$$\eta_i(\tau) = \lambda_{if}^{OL}(\tau) + 0(\mu), \qquad \forall \tau > 0. \tag{A19}$$

Using the inverse transformation of (A5), (A6)

$$\begin{bmatrix} x \\ \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} v \\ \xi_1 \\ \xi_2 \end{bmatrix} + \mu S \begin{bmatrix} w \\ \eta_1 \\ \eta_2 \end{bmatrix} \tag{A20}$$

$$\begin{bmatrix} z \\ \rho_1 \\ \rho_2 \end{bmatrix} = -T \begin{bmatrix} v \\ \xi_1 \\ \xi_2 \end{bmatrix} + (I - \mu TS) \begin{bmatrix} w \\ \eta_1 \\ \eta_2 \end{bmatrix} \tag{A21}$$

completes the proof.

Appendix B

We investigate the necessary conditions of partially closed-loop solutions, (51)–(55).

In its present form we cannot separate from (51)–(55) a set of algebraic equations that should be satisfied by K_i, P_i, F_i. The reason is the dependency of z on x. To overcome this difficulty we introduce the following transformation:

$$\begin{bmatrix} \tilde{v} \\ \tilde{w} \end{bmatrix} = \begin{bmatrix} I - \mu ML & -\mu M \\ L & I \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} \tag{B1}$$

where L and M satisfy the algebraic equations

$$0 = A_{22}L - \left(A_{21} + \sum_{i=1}^{2} B_{2i}F_i\right) - \mu L\left(A_{11} + \sum_{i=1}^{2} B_{1i}F_i - A_{12}L\right) \tag{B2}$$

$$0 = -M(A_{22} + \mu LA_{12}) + A_{12} + \mu\left(A_{11} + \sum_{i=1}^{2} B_{1i}F_i - A_{12}L\right)M. \tag{B3}$$

Now we can separate the variables by equating the coef-

ficients of $\tilde{v}$ to zero. Doing this we find that K_i, P_i, F_i, and L must satisfy (B2) and

$$0 = R_{ii}F_i + B'_{1i}K_i + B'_{2i}P_i \tag{B4}$$

$$\begin{aligned} 0 = K_i\left(A_{11} + \sum_{i=1}^{2} B_{1i}F_i - A_{12}L\right) + A'_{11}K_i + A'_{21}P_i \\ + C'_{i1}C_{i1} - C'_{i1}C_{i2}L \\ + F'_j(R_{ij}F_j + B'_{1j}K_i + B'_{2j}P_i), \quad i \neq j \end{aligned} \tag{B5}$$

$$\begin{aligned} 0 = \mu P_i\left(A_{11} + \sum_{i=1}^{2} B_{1i}F_i - A_{12}L\right) + A'_{12}K_i + A'_{22}P_i \\ + C'_{i2}C_{i1} - C'_{i2}C_{i2}L. \end{aligned} \tag{B6}$$

On the other hand, g_i, q_i, and h_i are obtained by solving the equations

$$\mu\dot{\tilde{w}} = (A_{22} + \mu LA_{12})\tilde{w} + \sum_{i=1}^{2}(B_{2i} + \mu LB_{1i})h_i,$$
$$\tilde{w}(0) = Lx_0 + z_0, \quad \tilde{w}(t) \to 0 \text{ as } t \to \infty \tag{B7}$$

$$0 = R_{ii}h_i + B'_{1i}g_i + B'_{2i}q_i \tag{B8}$$

$$\begin{aligned} -\dot{g}_i = K_i\left[\sum_{i=1}^{2} B_{1i}h_i + M(A_{22} + \mu LA_{12})\tilde{w}\right] \\ + \mu(A'_{11}K_i + A'_{21}P_i + C'_{i1}C_{i1})M\tilde{w} \\ + A'_{11}g_i + A'_{21}q_i + C'_{i1}C_{i2}(I - \mu LM)\tilde{w} \\ + F'_j[\mu(R_{ij}F_j + B'_{1j}K_i + B'_{2j}P_i)M\tilde{w} \\ + R_{ij}h_j + B'_{1j}g_i + B'_{2j}q_i], \quad g_i(t) \to 0 \text{ as } t \to \infty \end{aligned} \tag{B9}$$

$$\begin{aligned} -\mu\dot{q}_i = \mu P_i\left[\sum_{i=1}^{2} B_{1i}h_i + M(A_{22} + \mu LA_{12})\tilde{w}\right] \\ + \mu(A'_{12}K_i + A'_{22}P_i + C'_{i2}C_{i1})M\tilde{w} \\ + A'_{12}g_i + A'_{22}q_i + C'_{i2}C_{i2}(I - \mu LM)\tilde{w}, \\ q_i(t) \to 0 \text{ as } t \to \infty. \end{aligned} \tag{B10}$$

Let us investigate first the set of algebraic equations (B2) and (B4)–(B6).

Setting $\mu = 0$, it can be easily shown that $k_i(0)$ and $F_i(0)$ satisfy equations (26), (27) replacing K_{is} and F_{is}, respectively. Then Assumption 3 implies that

$$K_i(0) = K_{is}, \qquad F_i(0) = F_{is}. \tag{B11}$$

We also obtain that

$$P_i(0) = -A'^{-1}_{22}[A'_{12}K_{is} + C'_{i2}C_{i0} + C'_{i2}(D_{ii}F_{is} + D_{ij}F_{js})], \quad i \neq j \tag{B12}$$

$$L(0) = A^{-1}_{22}\left(A_{21} + \sum_{i=1}^{2} B_{2i}F_{is}\right). \tag{B13}$$

Equation (B2) and (B4)–(B6) can be interpreted as

$$\pi(K, \mu) = 0 \tag{B14}$$

where $K = (K_1, K_2, P_1, P_2, F_1, F_2, L)$. Evaluating the Frechet differential of π at $(K(0), 0)$ with increment δK we get the following set of linear equations in δK_i, δF_i:

$$\mathcal{R}_{ii} = (R_{ii} + D'_{ii}D_{ii})\delta F_i + D'_{ii}D_{ij}\delta F_j + B'_{0i}\delta K_i, \qquad i \neq j \tag{B15}$$

$$\begin{aligned} \mathcal{R}_{2i} = \delta K_i(A_0 + B_{0i}F_{is} + B_{0j}F_{js}) + (A_0 + B_{0j}F_{js})'\delta K_i \\ + (C'_{i0}D_{ii} + K_{is}B_{0i} + F'_{js}D'_{ij}D_{ii})\delta F_i \\ + [C'_{i0}D_{ij} + K_{is}B_{0j} + F'_{js}(R_{ij} + D'_{ij}D_{ij})]\delta F_j \\ + \delta F'_j[(R_{ij} + D'_{ij}D_{ij})F_{js} \\ + D'_{ij}D_{ii}F_{is} + D'_{ij}C_{i0}] \end{aligned} \tag{B16}$$

where $\mathcal{R}_{ii}$ and $\mathcal{R}_{2i}$ are some known quantities. The increments δP_i, δL are obtained as linear functions of δK_i, δF_i. If (B15), (B16) have a unique solution $(\delta K_1, \delta K_2, \delta F_1, \delta F_2)$, then using the implicit function theorem we get the following result.

Lemma B1: If Assumption 3 is satisfied, and if (B15), (B16) have a unique solution, then there exists $\mu^* > 0$ such that for every $\mu \in (0, \mu^*]$, (B2) and (B4)–(B6) possess a unique solution such that

$$K_i = K_{is} + O(\mu), \qquad F_i = F_{is} + O(\mu),$$
$$P_i = P_i(0) + O(\mu), \qquad L = L(0) + O(\mu). \tag{B17}$$

One way to check the existence of a unique solution of the matrix equation (B15), (B16) is to rewrite it in lexicographic notation as a vector equation, using the Kronecker product, and check that the corresponding matrix is nonsingular.

Now we turn to investigating the equations for g_i, q_i, and h_i, (B7)–(B10). Using (B8) we eliminate h_i to obtain a homogeneous two-point boundary value problem. We use a transformation similar to (A5), (A6) to separate the slow and fast parts. Based on Assumption 6 we get that the slow variables satisfy

$$-\begin{bmatrix} \dot{v}_1 \\ \dot{v}_2 \end{bmatrix} = (W + O(\mu))\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}, \qquad v_i(t) \to 0 \text{ as } t \to \infty, \tag{B18}$$

while the fast variables satisfy the equation

$$\mu\begin{bmatrix} \varphi \\ \dot{\Psi}_1 \\ \dot{\Psi}_2 \end{bmatrix} = (D + O(\mu))\begin{bmatrix} \varphi_1 \\ \Psi_1 \\ \Psi_2 \end{bmatrix}, \qquad \varphi(0) = \varphi_0, \ \varphi(t) \to 0,$$
$$\text{and } \Psi_i(t) \to 0 \text{ as } t \to \infty. \tag{B19}$$

Assumption 4 implies that, for sufficiently small μ, the unique solution of (B18) is

$$v_i(t) \equiv 0, \qquad \forall t > 0, \tag{B20}$$

while Assumptions 5 and 6 imply that, for sufficiently small μ, there exists a unique exponentially decaying solution of (B19) which satisfies

$$\varphi(t)=z_f^{OL}(\tau)+0(\mu), \qquad \forall t>0 \tag{B21}$$

$$\Psi_i(t)=\lambda_{if}^{OL}(\tau)+0(\mu), \qquad \forall t>0. \tag{B22}$$

Using the inverse transformation completes the proof of the following lemma.

Lemma B2: Under Assumptions 4–6 there exists $\mu^*>0$ such that for every $\mu\in(0,\mu^*]$, (B7)–(B10) possess a unique exponentially decaying solution. Moreover, for all $t>0$

$$g_i(t)=0(\mu) \tag{B23}$$

$$\tilde{\omega}(t)=z_f^{OL}(\tau)+0(\mu) \tag{B24}$$

$$q_i(t)=\lambda_{if}^{OL}(\tau)+0(\mu). \tag{B25}$$

Using Lemmas B1 and B2 we obtain Theorem 3 whose proof is straightforward using the inverse of transformation (B1).

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Closed-Loop Stackelberg Strategies for Singularly Perturbed Linear Quadratic Problems

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Abstract—Linear closed-loop Stackelberg strategies for systems with slow and fast modes are considered. The asymptotic behavior of the solution is studied and used to derive near-optimal strategies which do not require the knowledge of the small singular perturbation parameter.

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I. Introduction

Stackelberg strategies have been defined for sequential decision-making problems in which one decision maker, called the leader, announces his strategy before the second decision maker, called the follower, selects his own strategy. The leader anticipates an optimal reaction on the part of the follower and determines his own optimal Stackelberg strategy by solving a nonclassical control problem that takes into account the reaction of the follower. Consideration of Stackelberg solutions of differential games [1]–[3] has led to three different types of Stackelberg strategies: 1) open-loop strategies, 2) closed-loop strategies, and 3) equilibrium strategies. Closed-loop strategies for linear quadratic games are studied in [4]. It is shown there that if the initial conditions of the state are randomized and the performance criteria are averaged over the initial conditions, then there is an optimal pair of Stackelberg strategies that are linear in the state and necessary conditions characterizing these strategies are derived.

In this paper we consider closed-loop Stackelberg strategies for linear quadratic games when the system in singularly perturbed, that is, when the system contains slow and fast modes. Investigation of singular perturbations of differential games have been initiated in [5], [6] for Nash equilibrium strategies. It has been found that the problem of determining closed-loop Nash strategies by singular perturbation techniques is generally ill-posed. Ill-posedness results from the dependence of the solution on the available feedback structure, which is in particular exhibited through the fact that closed-loop solutions are generally distinct from open-loop solutions. Well-posedness is achieved only with specific feedback structures as discussed in [6]. This has motivated the investigation of closed-loop Stackelberg strategies in singularly perturbed systems since the sequential decision-making problem also exhibits different open-loop and closed-loop solutions. In Section II we formulate the problem and show that linear closed-loop strategies as defined in [4] are not appropriate for singularly perturbed systems since for infinitely fast modes there may be no solution even though a solution exists if the fast modes are neglected. In Section III we restrict the closed-loop strategy of the leader by allowing him feedback from the slow variables only, and show that in this strategy space the closed-loop solution is well-posed. As a result of this well-posedness the closed-loop strategies are approximated in Section III by near-optimal strategies which are easier to compute and do not require the knowledge of the value of the small singular perturbation parameter.

II. Problem Statement

We consider a singularly perturbed linear time-invariant system

$$\dot{x} = A_{11}x + A_{12}z + B_{11}u_1 + B_{12}u_2, \qquad x(0) = x_0 \tag{1a}$$

$$\mu\dot{z} = A_{21}x + A_{22}z + B_{21}u_1 + B_{22}u_2, \qquad z(0) = z_0 \tag{1b}$$

where $x \in R^{n_1}$, $z \in R^{n_2}$, $u_i \in R^{m_i}$. The small singular perturbation parameter $\mu > 0$ represents small time constants, inertias, masses, etc. The vector z is "fast" since its derivative $\dot{z}$ is of order $1/\mu$ which is large. The ith player choose his strategy[1] u_i from admissible strategy set U_i to minimize his performance criterion

$$J_i = \frac{1}{2}\int_0^\infty \left(\begin{bmatrix} x \\ z \end{bmatrix}' \begin{bmatrix} C_{i1}'C_{i1} & C_{i1}'C_{i2} \\ C_{i2}'C_{i1} & C_{i2}'C_{i2} \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} + u_i'R_{ii}u_i + u_j'R_{ij}u_j \right) dt,$$

$$R_{ii} > 0, \quad i \neq j. \tag{2}$$

A strategy set (u_1^s, u_2^s) is called a Stackelberg strategy with Player 2 as leader and Player 1 as follower if for any $u_2 \in U_2$ and $u_1 \in U_1$

$$J_2(u_1^s, u_2^s) \le J_2(u_1^0(u_2), u_2) \tag{3}$$

where

$$J_1(u_1^0(u_2), u_2) = \min_{u_1} J_1(u_1, u_2) \tag{4}$$

and

$$u_1^s = u_1^0(u_2^s). \tag{5}$$

Closed-loop Stackelberg strategies for linear quadratic problems have been studied in [4]. In that paper U_1 and U_2 are restricted to contain linear feedback strategies of the form

$$u_1(y,t) = -L_1 y(t), \quad u_2(y,t) = -L_2 y(t) \tag{6}$$

where $y'(t) = (x'(t), z'(t))$. It is shown that optimal gains L_1 and L_2 are dependent on the initial state of the system y_0. To eliminate this dependence on y_0, the initial state is considered to be randomly distributed, so that

$$E\{y_0\} = 0, \quad E\{y_0 y_0'\} = V_0. \tag{7}$$

The matrix V_0 is a design parameter which may be used by the leader if he has *a priori* knowledge of the initial state statistics. If no such information is available, the choice

$$V_0 = I \text{ (identity)} \tag{8}$$

has been suggested and will be used in this paper. With this viewpoint the performance criterion is modified as

$$\bar{J}_i(L_1, L_2) = \underset{y_0}{E}\{J_i(L_1, L_2)\}, \qquad i = 1, 2. \tag{9}$$

The leader optimal gain L_2 is obtained by solving the following set of algebraic matrix equations:

$$A_c'M_1 + M_1A_c + M_1S_{11}M_1 + L_2'R_{12}L_2 + Q_1 = 0 \tag{10a}$$

$$A_c'M_2 + M_2A_c + M_1S_{21}M_1 + L_2'R_{22}L_2 + Q_2 = 0 \tag{10b}$$

$$N_1A_c' + A_cN_1 - S_{11}M_2N_2 - N_2M_2S_{11} + S_{21}M_1N_2 + N_2M_1S_{21} = 0 \tag{10c}$$

$$N_2A_c' + A_cN_2 + I = 0 \tag{10d}$$

$$R_{12}L_2N_1 + R_{22}L_2N_2 - B_2'(M_1N_1 + M_2N_2) = 0 \tag{10e}$$

while the follower's optimal gain L_1 is given by

$$L_1 = R_{11}^{-1}B_1'M_1 \tag{11}$$

where

$$A_c = A - S_{11}M_1 - B_2L_2, \quad S_{ij} = B_jR_{jj}^{-1}R_{ij}R_{jj}^{-1}B_j'.$$

For our problem (1), (2) we have

$$A = \begin{bmatrix} A_{11} & A_{12} \\ \frac{A_{21}}{\mu} & \frac{A_{22}}{\mu} \end{bmatrix}, \quad B_i = \begin{bmatrix} B_{1i} \\ \frac{B_{2i}}{\mu} \end{bmatrix}, \quad Q_i = \begin{bmatrix} C_{i1}'C_{i1} & C_{i1}'C_{i2} \\ C_{i2}'C_{i1} & C_{i2}'C_{i2} \end{bmatrix}.$$

Knowing the value of μ one would seek the solution of (10), (11) to obtain the optimal gains L_1, L_2.

In practice a designer usually neglects some small time-constants, masses, moments of inertia, and other parameters whose presence increases the order of the system. He therefore bases his design on a reduced-order model formed by neglecting the fast modes [7]. This is equivalent to setting $\mu = 0$ in (1), and results in the model

$$\dot{x}_s = A_{11}x_s + A_{12}z_s + B_{11}u_{1s} + B_{12}u_{2s}, \qquad x_s(0) = x_0 \tag{12a}$$

$$0 = A_{21}x_s + A_{22}z_s + B_{21}u_{1s} + B_{22}u_{2s}. \tag{12b}$$

Assuming that A_{22} is nonsingular, we express z_s as

$$z_s = -A_{22}^{-1}(A_{21}x_s + B_{21}u_{1s} + B_{22}u_{2s}) \tag{13}$$

and substituting it into (12) and (2), we obtain a reduced game. The ith

[1] Notice that the same symbol u is used for both the control values and control laws of the players.

player now chooses his strategy u_{is} from admissible strategy set U_{is} to minimize

$$J_{is}=\frac{1}{2}\int_0^{\infty}\Big[x_s'C_{i0}'C_{i0}x_s+2x_s'C_{i0}(D_{ii}u_{is}+D_{ij}u_{js})+2u_{is}'D_{ii}'D_{ij}u_{js}+u_{is}'R_{i0}u_{is}+u_{js}'R_{ij0}u_{js}\Big]dt,\qquad i\neq j \quad (14)$$

subject to the reduced system

$$\dot{x}_s=A_0x_s+B_{01}u_{1s}+B_{02}u_{2s},\qquad x_s(0)=x_0 \quad (15)$$

where

$$A_0=A_{11}-A_{12}A_{22}^{-1}A_{21},\quad B_{0i}=B_{1i}-A_{12}A_{22}^{-1}B_{2i},$$
$$C_{i0}=C_{i1}-C_{i2}A_{22}^{-1}A_{21},\quad D_{ij}=-C_{i2}A_{22}^{-1}B_{2j},\qquad i,j=1,2,$$
$$R_{i0}=R_{ii}+D_{ii}'D_{ii},\quad R_{ij0}=R_{ij}+D_{ij}'D_{ij},\qquad i\neq j.$$

A closed-loop Stackelberg strategy of the reduced game (14), (15) is sought with the restriction that U_{1s} and U_{2s} contain linear feedback strategies of the form

$$u_{1s}(x_s,t)=-L_{1s}x_s(t),\quad u_{2s}(x_s,t)=-L_{2s}x_s(t) \quad (16)$$

and the performance criterion (14) is modified to

$$\bar{J}_{is}(L_{1s},L_{2s})=\underset{x_0}{E}\{J_{is}(L_{1s},L_{2s})\},\quad E\{x_0\}=0,\quad E\{x_0x_0'\}=I. \quad (17)$$

In the Appendix we give the equations that must be satisfied by the optimal gains L_{1s}, L_{2s}. Throughout this paper we assume that these equations possess a unique stabilizing solution.

In this paper we analyze the asymptotic behavior of the solution $L_1(\mu)$, $L_2(\mu)$ of (10), (11) as $\mu\to 0$. Of particular interest is whether the limit $\lim_{\mu\to 0}\bar{J}_i=\lim_{\mu\to 0}\frac{1}{2}\operatorname{tr}M_i$ exists, and if it exists does it equal to $\bar{J}_{is}=\frac{1}{2}\operatorname{tr}M_{is}$? If $\lim_{\mu\to 0}\bar{J}_i=\bar{J}_{is}$, the singularly perturbed closed-loop Stackelberg problem is said to be well-posed since assuming μ to be small, and neglecting it altogether results in small error in the design procedure. An example of a well-posed design problem is the optimal control problem [8], while an example of an ill-posed problem is the closed-loop solution of the Nash equilibrium strategy [5]. Investigation of the ill-posedness of Nash games [6] shows that the cause of ill-posedness stems from the difference in the information structures of the limiting full problem as $\mu\to 0$ and the reduced problem for $\mu=0$. Nash games have the property that the solution depends on the information available to the players (e.g., they in general have different open-loop and closed-loop solutions). Allowing the strategy in the original game to be a function of the full state means that the solution depends on feedback information on the fast modes z of the system. This information is not available in the reduced game. Thus, the closed-loop solution of a Nash game does not tend, in general, to the closed-loop solution of its reduced game. In consistency with this argument it is shown in [6] that if we solve a full Nash game assuming only x is available for measurement, the solution as $\mu\to 0$ tends to the closed-loop solution of the reduced Nash game. In the optimal control problem it turns out that allowing the control in the full solution to also be a function of z is not crucial for this limiting behavior and the problem is well-posed. This is a consequence of the fact that for the optimal control problem the open-loop and the closed-loop solutions lead to identical controls, trajectories, and cost functions. The above conclusions concerning the relation between the information structure and the well-posedness of singularly perturbed problems hold for any problem in which the open-loop and closed-loop solutions are different. Since Stackelberg strategies have different open-loop and closed-loop solutions [3] we expect that seeking a solution of the full game in the form (6) might lead to an ill-posed problem. Analyzing the asymptotic behavior of (10) shows that the difficulty with solutions of the form (6) is more serious than having a limit which is different from the reduced solution. In fact (10) may have no solution in the limit as $\mu\to 0$ even if the reduced game has a solution. To see this notice that when $R_{12}=0$ for (10) to have a unique solution it is necessary that the matrix N_2 be positive definite (nonsingular) as it is obvious from (10e). In our case N_2 tends to a singular matrix as $\mu\to 0$. The reason is that as $\mu\to 0$, $z(t)=Kx(t)$ for all $t>0$ where K is some constant matrix, while $N_2=\int_0^{\infty}E\{y(t)y'(t)\}dt$. Thus, it can be easily shown that N_2 has rank n_1 as $\mu\to 0$. To illustrate this fact consider the following example.

Example:

$$A=\begin{bmatrix}0 & 1\\ -\frac{1}{\mu} & -\frac{1}{\mu}\end{bmatrix},\quad B_1=B_2=\begin{bmatrix}0\\ \frac{1}{\mu}\end{bmatrix},\quad Q_1=Q_2=\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix},$$
$$R_{11}=R_{22}=1,\quad R_{12}=R_{21}=0. \quad (18)$$

We seek the solution of (10) in the form

$$M_i=\begin{bmatrix}m_{i1} & \mu m_{i2}\\ \mu m_{i2} & \mu m_{i3}\end{bmatrix},\quad N_i=\begin{bmatrix}n_{i1} & n_{i2}\\ n_{i2} & n_{i3}\end{bmatrix},\quad L_2=[l_{21}\ \ l_{22}] \quad (19)$$

where the form of M_i is chosen to avoid unboundedness in the limit as $\mu\to 0$, as usual in singularly perturbed Riccati equations [8]. Setting $\mu=0$ in (10) we obtain the limit values

$$m_{11}=m_{12}(1+l_{22}),\quad m_{12}=\sqrt{1+(1+l_{21})^2}-(1+l_{21}),\quad m_{13}=0 \quad (20a)$$

$$m_{21}=m_{22}(1+l_{22})+m_{23}\sqrt{1+(1+l_{21})^2}-l_{21}l_{22},$$
$$m_{22}=\frac{1+l_{21}^2}{2\sqrt{1+(1+l_{21})^2}},\quad m_{23}=\frac{l_{22}^2}{2(1+l_{22})} \quad (20b)$$

$$n_{11}=\frac{(1+l_{21}^2)-v^2l_{22}^2}{4v^3(1+l_{22})},\quad n_{12}=0,\quad n_{13}=-n_{11}v^2 \quad (20c)$$

$$n_{21}=-\frac{1}{2v},\quad n_{22}=-\frac{1}{2},\quad n_{23}=-\frac{v}{2} \quad (20d)$$

$$l_{21}=-1-\frac{1}{2n_{11}}-\frac{1}{2vn_{11}},\quad l_{22}=-1+\frac{1}{2vn_{11}}+\frac{1}{2v^2n_{11}} \quad (20e)$$

where

$$v=-\frac{\sqrt{1+(1+l_{21})^2}}{1+l_{22}} \quad (21)$$

Notice first from (20d) that N_2 is singular. This means that the existence of a solution depends on whether (10e) is consistent or not. In this particular example it turns out that there is no solution as we can see by eliminating n_{11} between the two equations of (20e) and substituting (21) for v to get

$$\sqrt{1+(1+l_{21})^2}-(1+l_{21})=0 \quad (22)$$

which is satisfied only if $l_{21}\to\infty$. On the other hand, the reduced problem is given by

$$A_0=-1,\quad B_{01}=B_{02}=1,\quad C_{10}=C_{20}=1,\quad D_{ij}=0,\qquad i,j=1,2 \quad (23)$$

and the corresponding equations (A1)–(A5) have a unique solution

$$L_{2s}=0.25992,\quad M_{1s}=0.3486,\quad M_{2s}=0.33184. \quad (24)$$

Thus, we have a situation in which the reduced problem has a unique solution while the full problem has no solution in the limit as $\mu\to 0$. Thus, based on the discussion of the information structures and of the possibility of having ill-behaved limits as $\mu\to 0$, we reach the conclusion that the set of strategies defined by (6) is not the appropriate set to obtain closed-loop Stackelberg strategies for singularly perturbed systems.

Our task now is to choose new strategy sets U_1 and U_2 for the solution of the singularly perturbed Stackelberg game. Our choice is guided by the utility and simplicity of linear feedback strategies and the above conclusions on the relation of well-posedness and information structure. To this end we discuss the effect of allowing the players to know the

current value of z. Here we discriminate between the leader and the follower as a result of the different roles they have in the game. The follower merely solves an optimal control problem, so if the leader strategy set is well defined, the follower may be allowed to use the current value of z without affecting the well-posedness of the resulting optimal control problem. Thus, the follower strategy set U_1 need not be modified and will continue to be defined by (6). The leader, however, solves a control problem which has different open-loop and closed-loop solutions. Thus, to have the problem well-posed we exclude from his information set the knowledge of the current value of z and his strategy set U_2^c will now be defined by

$$u_2(x,t) = -L_{21}x(t). \tag{25}$$

This choice of the U_2^c guarantees that the full game solution involves, in the limit as $\mu \to 0$, the same information available in the reduced problem. It will be shown that this choice overcomes the ill-posedness of the problem and in particular overcomes problems relating to the singularity of N_2 since only the positive definite part of N_2 will be used.

III. Constrained Closed-Loop Stackelberg Strategy

In this section we analyze the asymptotic behavior of linear Stackelberg strategies when the leader is restricted to take feedback from the slow variables only, and thus when L_2 is restricted to have the form $L_2 = (L_{21}, 0)$. We will refer to this solution as a constrained Stackelberg strategy.

By employing the matrix minimum principle [9] to derive the necessary conditions for this problem we find that the leader optimal gain L_{21} is obtained by solving (10a)–(10d) together with

$$R_{12}L_{21}N_{11} + R_{22}L_{21}N_{21} - B_{12}'(M_{11}N_{11} + M_{21}N_{21}) - B_{22}'(M_{12}'N_{11} + M_{22}'N_{21})$$
$$- \mu B_{12}'(M_{12}N_{12}' + M_{22}N_{22}') - B_{22}'(M_{13}N_{12}' + M_{23}N_{22}') = 0 \tag{26}$$

where M_i, N_i, L_2 have been partitioned as

$$M_i = \begin{bmatrix} M_{i1} & \mu M_{i2} \\ \mu M_{i2}' & \mu M_{i3} \end{bmatrix}, \quad N_i = \begin{bmatrix} N_{i1} & N_{i2} \\ N_{i2}' & N_{i3} \end{bmatrix}, \quad L_2 = (L_{21} \quad 0). \tag{27}$$

Without any loss of generality the matrices M_i are assumed to have the form (27) in order to obtain a set of well defined equations in partitioned form, as $\mu \to 0$. The relation between (26) and (10e) becomes evident if we write L_2 in (10e) as $L_2 = (L_{21} L_{22})$, and partition (10e) into two equations. The first corresponds to taking the partial derivative of the criterion with respect to L_{21} the second with respect to L_{22}. The first equation for the case now considered, when $L_{22} = 0$, reduces to (26), while the second equation does not appear because L_{22} is constrained to be zero. We note that the follower strategy is still given by (11).

We now investigate (10a)–(10d) and (26) as $\mu \to 0$. Assuming that the triple (A_{22}, B_{21}, C_{12}) is stabilizable-detectable, it can be shown,[2] after lengthy manipulation, that $L_{21}(0)$, $M_{11}(0)$, $M_{21}(0)$, $N_{11}(0)$, and $N_{21}(0)$ satisfy (A1)–(A5) replacing L_{2s}, M_{1s}, M_{2s}, N_{1s}, and N_{2s}, respectively. Assuming that (A1)–(A5) possess a unique stabilizing solution we get that

$$\begin{aligned} L_{21}(0) = L_{2s}, \quad M_{11}(0) = M_{1s}, \quad & M_{21}(0) = M_{2s}, \\ N_{11}(0) = N_{1s}, \quad & N_{21}(0) = N_{2s}. \end{aligned} \tag{28}$$

Subsequently,

$$\lim_{\mu \to 0} \bar{J}_i = \lim_{\mu \to 0} \tfrac{1}{2} \operatorname{tr} \begin{bmatrix} M_{i1} & \mu M_{i2} \\ \mu M_{i2}' & \mu M_{i3} \end{bmatrix} = \tfrac{1}{2} \operatorname{tr} M_{is} = \bar{J}_{is}, \qquad i = 1, 2. \tag{29}$$

Thus, the constrained closed-loop solution of the Stackelberg game (1), (2), (9), (25) is well-posed in the sense that it tends to the closed-loop solution of the reduced-order Stackelberg game (14), (15), (17) as $\mu \to 0$.

[2]Complete derivation can be found in [10].

IV. Near-Optimal Strategy

The well-posedness property will now be used to obtain a near-optimal strategy for the constrained Stackelberg game which is easier to compute as compared to the exact strategy. Motivation for this is that although the constrained Stackelberg strategy is well behaved as $\mu \to 0$, solving (10a)–(10d) and (26) may not be easy because of the high order of the system or because of the numerical difficulties arising in solving the Riccati equation (10a) when μ is very small. These reasons motivate, while the results of Section III on the limiting behavior of the constrained strategy as $\mu \to 0$, allow the construction of an appropriate approximate strategy which is easier to compute than the exact full solution. In this section we postulate such an approximate strategy which takes into account the slow and fast modes of the system. We then explain in what sense this strategy is near-optimal.

Suppose that the players solve the reduced order Stackelberg game (14), (15), (17) for the slow modes of the system and their optimal gains are L_{1s}, L_{2s}. Suppose also that the follower solves the following optimization problem for the fast modes of the system: find u_{1f} to minimize

$$J_{1f} = \frac{1}{2}\int_0^\infty (z_f' C_{12}' C_{12} z_f + u_{1f}' R_{11} u_{1f})\, d\tau, \qquad \tau = \frac{t}{\mu} \tag{30}$$

subject to

$$\frac{dz_f}{d\tau} = A_{22} z_f + B_{21} u_{1f}. \tag{31}$$

Assuming that (A_{22}, B_{21}, C_{12}) is stabilizable-detectable, the solution of this optimization problem is

$$u_{1f} = -R_{11}^{-1} B_{21}' M_{1f} z_f \tag{32}$$

where M_{1f} is the unique positive semidefinite solution of the Riccati equation

$$M_{1f}A_{22} + A_{22}'M_{1f} + C_{12}'C_{12} - M_{1f}B_{21}R_{11}^{-1}B_{21}'M_{1f} = 0. \tag{33}$$

The follower then composes his approximate strategy as the sum of his slow strategy $u_{1s} = -L_{1s}x_s$ and his fast strategy $u_{1f} = -R_{11}^{-1}B'_{21}M_{1f}z_f$. He replaces z_f by $z - z_s$, where from (3) $z - z_s = z + A_{22}^{-1}(A_{21} - B_{21}L_{1s} - B_{22}L_{2s})x_s$. Then, replacing x_s by x, he uses the approximate strategy

$$\begin{aligned} u_{1app} &= -L_{1s}x - R_{11}^{-1}B_{21}'M_{1f}[z + A_{22}^{-1}(A_{21} - B_{21}L_{1s} - B_{22}L_{2s})x] \\ &= -R_{11}^{-1}\begin{pmatrix} B_{11}' & \dfrac{B_{21}'}{\mu} \end{pmatrix} \begin{bmatrix} M_{1s} & 0 \\ \mu M_{12}'(0) & \mu M_{1f} \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} \\ &\triangleq -R_{11}^{-1}B_1'K_1 y. \end{aligned} \tag{34}$$

The leader continues to use his slow strategy replacing only x_s by x so that

$$u_{2app} = -L_{2s}x = -(L_{2s} \quad 0)y = -F_2 y. \tag{35}$$

We now investigate the effect of using the approximate strategies u_{1app}, u_{2app} instead of the exact constrained strategies $u_1^c = L_1 y$, $u_2^c = L_2 y = L_{21}x$. Since in a Stackelberg strategy the leader announces his strategy first it must be shown that the approximate strategy u_{2app} is near optimal from the leader's point of view. This means that if the leader uses u_{2app} and the follower strategy lies on the reaction curve, i.e., $u_1 = u_1^0(u_{2app})$, then the leader's performance criterion $\bar{J}_2(u_1^0(u_{2app}), u_{2app})$ tends to $\bar{J}_2(u_1^c, u_2^c)$ as $\mu \to 0$. Next, if the leader uses u_{2app}, and the follower uses the approximate strategy u_{1app}, then u_{1app} must be near optimal from the follower's point of view,[3] i.e., the follower's performance criterion $\bar{J}_1(u_{1app}, u_{2app})$ tends to $\bar{J}_1(u_1^0(u_{2app}), u_{2app})$ as $\mu \to 0$. It is, however, noted that by using the approximate strategy u_{1app} the follower deviates from the reaction curve $u_1 = u_1^0(u_2)$, and, strictly speaking, violates the rules of the game. Such deviation from the reaction curve, i.e., from the strict

[3]The use of u_{1app} rather than $u_1^0(u_{2app})$ is computationally attractive since solving for $u_1^0(u_{2app})$ would involve a Riccati equation similar to (10a).

definition of the solution, will be tolerated in computing the approximate solution, just as it is in optimal control problems [8], or Nash games [5],[6], if the resulting effect of the deviation on the leader's value function from the true Stackelberg value remains small. In other words, the use of the approximate strategy u_{1app} by the follower will be tolerated if $\bar{J}_2(u_{1app}, u_{2app})$ tends to $\bar{J}_2(u_1^0(u_{2app}), u_{2app})$ as $\mu \to 0$.

The near optimality of the approximate strategy u_{1app}, u_{2app} is established in the following theorem.

Theorem: If (A1)–(A5) possess a unique stabilizing solution and if (A_{22}, B_{21}, C_{12}) is stabilizable-detectable, then

$$\lim_{\mu\to 0}\left[\bar{J}_2\left(u_1^0(u_{2app}), u_{2app}\right) - \bar{J}_2(u_1^*, u_2^*)\right] = 0 \tag{36}$$

$$\lim_{\mu\to 0}\left[\bar{J}_1(u_{1app}, u_{2app}) - \bar{J}_1\left(u_1^0(u_{2app}), u_{2app}\right)\right] = 0 \tag{37}$$

$$\lim_{\mu\to 0}\left[\bar{J}_2(u_{1app}, u_{2app}) - \bar{J}_2\left(u_1^0(u_{2app}), u_{2app}\right)\right] = 0. \tag{38}$$

Moreover,

$$\lim_{\mu\to 0}\left[\bar{J}_i(u_{1app}, u_{2app}) - \bar{J}_i(u_1^*, u_2^*)\right] = 0, \qquad i = 1, 2. \tag{39}$$

Proof: When the exact constrained strategies $u_1^* = -L_1 y$, $u_2^* = -L_2 y = -(L_{21} \;\; 0)y$ are used, the resulting values of the performance criteria are $\bar{J}_i(u_1^*, u_2^*) = \frac{1}{2}\operatorname{tr} M_i$ where M_1 and M_2 are given by (10a) and (10b), respectively. Suppose now that the leader uses the approximate strategy $u_{2app} = -F_2 y$, and let the follower respond optimally by using

$$u_1 = u_1^0(u_{2app}) = -R_{11}^{-1}B_1' V_1 y \tag{40}$$

where V_1 is the stabilizing solution of the Riccati equation

$$V_1(A - S_{11}V_1 - B_2F_2) + (A - S_{11}V_1 - B_2F_2)'V_1 + Q_1 + V_1S_{11}V_1 + F_2'R_{12}F_2 = 0. \tag{41}$$

The resulting values of the performance criteria are $\bar{J}_i(u_1^0(u_{2app}), u_{2app}) = \frac{1}{2}\operatorname{tr} V_i$, $i = 1, 2$, where V_1 satisfies (41) and V_2 satisfies the equation

$$V_2(A - S_{11}V_1 - B_2F_2) + (A - S_{11}V_1 - B_2F_2)'V_2 + Q_2 + V_1S_{21}V_1 + F_2'R_{22}F_2 = 0. \tag{42}$$

We seek V_1 and V_2 in the form

$$V_i = \begin{bmatrix} V_{i1} & \mu V_{i2} \\ \mu V_{i2}' & \mu V_{i3} \end{bmatrix}, \qquad i = 1, 2. \tag{43}$$

Substituting (43) in (41), setting $\mu = 0$, using that the matrix $(A_{22} - B_{21}R_{11}^{-1}B_{21}'M_{1f})$ is stable and that (A1)–(A5) possess a unique stabilizing solution it can be shown that

$$V_{1j}(0) = M_{1j}(0), \qquad j = 1, 2, 3. \tag{44}$$

Using (44) we can now prove (36). To do this we substitute (43) in (42), subtract (10b) from (42), set $\mu = 0$, use (44) and employ the stability of the matrices A_s and $(A_{22} - B_{21}R_{11}^{-1}B_{21}'M_{1f})$. It follows that

$$V_{2j}(0) = M_{2j}(0), \qquad j = 1, 2, 3 \tag{45}$$

which proves (36). If, instead of responding optimally, the follower uses the approximate strategy $u_{1app} = -R_{11}^{-1}B_1'K_1 y$, the values of the performance criteria will be $\bar{J}_i(u_{1app}, u_{2app}) = \frac{1}{2}\operatorname{tr} W_i$, where W_1 and W_2 satisfy the equations

$$W_1(A - S_{11}K_1 - B_2F_2) + (A - S_{11}K_1 - B_2F_2)'W_1 + Q_1 + K_1'S_{11}K_1 + F_2'R_{12}F_2 = 0 \tag{46}$$

$$W_2(A - S_{11}K_1 - B_2F_2) + (A - S_{11}K_1 - B_2F_2)'W_2 + Q_2 + K_1'S_{21}K_1 + F_2'R_{22}F_2 = 0. \tag{47}$$

Seeking W_i in the form (43) and subtracting (41) from (46) and (42) from (47), it can be shown that

$$W_{ij}(0) = V_{ij}(0), \qquad i = 1, 2,\; j = 1, 2, 3 \tag{48}$$

which proves (37) and (38). The proof of (39) is now straightforward.

V. Conclusions

We have shown that allowing the leader to use feedback from z may result in the full problem having no solution for sufficiently small μ, even though the reduced problem at $\mu = 0$ has a solution. We avoided this difficulty by excluding z from the information available to the leader. This information structure is reasonable when the state vector x is available for measurement. However, in many physical systems one has no access to the state vector x and can only measure some linear combinations of x and z. For such cases alternative ways of overcoming the ill-posedness of the problem when output feedback rather than state feedback is available must be explored.

Appendix

The set of necessary conditions that the solution of the reduced Stackelberg game (14), (15), (17) must satisfy are derived by employing the matrix minimum principle [9]. The leader optimal gain L_{2s} is obtained by solving the following set of algebraic equations:

$$\begin{aligned} &M_{1s}A_s + A_s'M_{1s} + C_{10}'C_{10} - C_{10}'D_{11}R_{10}^{-1}D_{11}'C_{10} \\ &- C_{10}'(I - D_{11}R_{10}^{-1}D_{11}')D_{12}L_{2s} - L_{2s}'D_{12}'(I - D_{11}R_{10}^{-1}D_{11}')C_{10} \\ &+ L_{2s}'(R_{120} - D_{12}'D_{11}R_{10}^{-1}D_{11}'D_{12})L_{2s} + M_{1s}B_{01}R_{10}^{-1}B_{01}'M_{1s} = 0 \end{aligned} \tag{A1}$$

$$\begin{aligned} &M_{2s}A_s + A_s'M_{2s} + C_{20}'C_{20} - C_{20}'D_{21}R_{10}^{-1}D_{11}'C_{10} - C_{10}'D_{11}R_{10}^{-1}D_{21}'C_{20} \\ &\quad + C_{10}'D_{11}R_{10}^{-1}R_{210}R_{10}^{-1}D_{11}'C_{10} + (-C_{20}'D_{22} + C_{20}'D_{21}R_{10}^{-1}D_{11}'D_{12} \\ &\quad - C_{10}'D_{11}R_{10}^{-1}R_{210}R_{10}^{-1}D_{11}'D_{12} + C_{10}'D_{11}R_{10}^{-1}D_{21}'D_{22})L_{2s} \\ &\quad + L_{2s}'(-D_{22}'C_{20} + D_{12}'D_{11}R_{10}^{-1}D_{21}'C_{20} - D_{12}'D_{11}R_{10}^{-1}R_{210}R_{10}^{-1}D_{11}'C_{10} \\ &\quad + D_{22}'D_{21}R_{10}^{-1}D_{11}'C_{10}) + L_{2s}'(R_{20} + D_{12}'D_{11}R_{10}^{-1}R_{210}R_{10}^{-1}D_{11}'D_{12} \\ &\quad - D_{22}'D_{21}R_{10}^{-1}D_{11}'D_{12} - D_{12}'D_{11}R_{10}^{-1}D_{21}'D_{22})L_{2s} \\ &\quad - (C_{20}'D_{21} - C_{10}'D_{11}R_{10}^{-1}R_{210})R_{10}^{-1}B_{01}'M_{1s} \\ &\quad - M_{1s}B_{01}R_{10}^{-1}(D_{21}'C_{20} - R_{210}R_{10}^{-1}D_{11}'C_{10}) \\ &\quad + M_{1s}B_{01}R_{10}^{-1}(D_{21}'D_{22} - R_{210}R_{10}^{-1}D_{11}'D_{12})L_{2s} \\ &\quad + L_{2s}'(D_{22}'D_{21} - D_{12}'D_{11}R_{10}^{-1}R_{210})R_{10}^{-1}B_{01}'M_{1s} \\ &\quad + M_{1s}B_{01}R_{10}^{-1}R_{210}R_{10}^{-1}B_{01}'M_{1s} = 0 \end{aligned} \tag{A2}$$

$$\begin{aligned} &N_{1s}A_s' + A_sN_{1s} - B_{01}R_{10}^{-1}B_{01}'M_{2s}N_{2s} - N_{2s}M_{2s}B_{01}R_{10}^{-1}B_{01}' \\ &- B_{01}R_{10}^{-1}(D_{21}'C_{20} - R_{210}R_{10}^{-1}D_{11}'C_{10})N_{2s} \\ &- N_{2s}(C_{20}'D_{21} - C_{10}'D_{11}R_{10}^{-1}R_{210})R_{10}^{-1}B_{01}' \\ &+ B_{01}R_{10}^{-1}R_{210}R_{10}^{-1}B_{01}'M_{1s}N_{2s} \\ &+ N_{2s}M_{1s}B_{01}R_{10}^{-1}R_{210}R_{10}^{-1}B_{01}' \\ &- N_{2s}L_{2s}'(D_{12}'D_{11}R_{10}^{-1}R_{210} - D_{22}'D_{21})R_{10}^{-1}B_{01}' \\ &- B_{01}R_{10}^{-1}(R_{210}R_{10}^{-1}D_{11}'D_{12} - D_{21}'D_{22})L_{2s}N_{2s} = 0 \end{aligned} \tag{A3}$$

$$N_{2s}A_s' + A_sN_{2s} + I = 0 \tag{A4}$$

$$\begin{aligned} &(R_{120} - D_{12}'D_{11}R_{10}^{-1}D_{11}'D_{12})L_{2s}N_{1s} \\ &+ (R_{20} + D_{12}'D_{11}R_{10}^{-1}R_{210}R_{10}^{-1}D_{11}'D_{12} - D_{22}'D_{21}R_{10}^{-1}D_{11}'D_{12} \\ &- D_{12}'D_{11}R_{10}^{-1}D_{21}'D_{22})L_{2s}N_{2s} - D_{12}'(I - D_{11}R_{10}^{-1}D_{11}')C_{10}N_{1s} \\ &- (D_{22}'C_{20} - D_{12}'D_{11}R_{10}^{-1}D_{21}'C_{20} + D_{12}'D_{11}R_{10}^{-1}R_{210}R_{10}^{-1}D_{11}'C_{10} \\ &- D_{22}'D_{21}R_{10}^{-1}D_{11}'C_{10})N_{2s} - (B_{02} - B_{01}R_{10}^{-1}D_{11}'D_{12})'(M_{1s}N_{1s} + M_{2s}N_{2s}) \\ &- (D_{12}'D_{11}R_{10}^{-1}R_{210} - D_{22}'D_{21})R_{10}^{-1}B_{01}'M_{1s}N_{2s} = 0 \end{aligned} \tag{A5}$$

where

$$A_s = A_0 - B_{01}R_{10}^{-1}B_{01}'M_{1s} - B_{01}R_{10}^{-1}D_{11}'C_{10} - (B_{02} - B_{01}R_{10}^{-1}D_{11}'D_{12})L_{2s}. \tag{A6}$$

The follower optimal gain L_{1s} is given by

$$L_{1s} = R_{10}^{-1}(B_{01}'M_{1s} + D_{11}'C_{10} - D_{11}'D_{12}L_{2s}). \tag{A7}$$

Thus, given L_{2s} the follower needs to solve (A1) to be able to compute his optimal gain L_{1s}.

The solution of (A1)–(A5) makes sense only if the closed-loop matrix A_s is stable. So we assume that (A1)–(A5) possess a unique stabilizing solution.

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Well-Posedness of Linear Closed-Loop Stackelberg Strategies for Singularly Perturbed Systems†

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ABSTRACT: *This paper is concerned with a linear closed-loop Stackelberg strategy for singularly perturbed systems. A procedure to obtain a well-posed formulation of the problem, where both fast and slow modes are available for measurements, is given.*

I. Introduction

The Stackelberg strategy (1–3) is the solution concept for a broad class of decision making problems in which one decision-maker, called the *leader*, announces his strategy before the other decision-maker, called the *follower*, selects his strategy. There are different types of Stackelberg strategies: (a) open-loop strategies, (b) closed-loop strategies, (c) feedback strategies. For more information about these types the reader is referred to (11, 13). The closed-loop Stackelberg strategy appears more favorable to the leader than the other two kinds (4, 5), but the disadvantage in using the closed-loop Stackelberg strategy is that it does not satisfy the principle of optimality (4).

When the space of closed-loop Stackelberg strategies is constrained to be a linear function of the state variables, it was found (6) that such linear strategies do not exist because some gain matrices depend on the initial conditions. But by assuming that the initial conditions are randomly distributed and averaging the performance indices over these initial conditions, linear closed-loop Stackelberg strategies were obtained.

When the system contains slow and fast modes the control problem is numerically stiff. To alleviate this numerical stiffness and to decrease computational manipulation the singular perturbation method has been used (7).

Applying the theory of differential games to singularly perturbed systems was initiated in (8, 9), in which it was found that the usual formulation using singular perturbation techniques to find closed-loop Nash and Stackelberg strategies is generally ill-posed. In (8) a method was shown to obtain a well-posed formulation for Nash games, when both the slow and the fast

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variables are available for measurements. In (9) a linear closed-loop Stackelberg strategy as described in (6), was considered and it was shown that if we restrict the space of strategies to be taken from the slow variable only, we obtain a well-posed formulation.

In this paper, we consider the linear closed-loop Stackelberg strategy when both the slow and the fast variables are available for measurements. This information structure is different from the one in (9). We describe a method by which we find strategies using reduced order systems such that if we apply these strategies to the full order system, the resulting cost functions will have the same limits as the cost functions for the same full order systems if the full order optimal strategies are applied.

II. *Formulation of the Problem*

Let us consider the singularly perturbed system:

$$\dot{x} = A_{11}x + A_{12}z + B_{11}u_1 + B_{12}u_2; \quad x(0) = x_0,$$
$$\mu\dot{z} = A_{21}x + A_{22}z + B_{21}u_1 + B_{22}u_2; \quad z(0) = z_0,$$

where $x \in R^{n_1}$, $z \in R^{n_2}$; $u_i \in R^{m_i}$ and μ is a small positive parameter. Assume that the cost function associated with player i is

$$\bar{J}_i = \underset{y_0}{E}\left[\tfrac{1}{2}\int_0^\infty \{y'Q_iy + u_i'R_{ii}u_i + u_j'R_{ij}u_j\}\,\mathrm{d}t\right],$$

where

$$y = \begin{bmatrix} x \\ z \end{bmatrix}; \quad Q_i = \begin{bmatrix} Q_{i1} & Q_{i2} \\ Q_{i2}' & Q_{i3} \end{bmatrix}.$$

R_{ii}, R_{ij} are symmetric, positive definite matrices, $E(y_0) = 0$; $E(y_0y_0') = I$ where I is the identity matrix.

A strategy set (u_1^*, u_2^*) is called a Stackelberg strategy with player 2 as a leader and player 1 as follower if for any $u_1 \in U_1$, $u_2 \in U_2$.

$$\bar{J}_2(u_1^*, u_2^*) \leq \bar{J}_2(Tu_2, u_2),$$

where

$$\bar{J}_1(Tu_2, u_2) \leq \bar{J}_1(u_1, u_2)$$

and

$$u_1^* = Tu_2^*.$$

Closed-loop linear Stackelberg strategy was considered by Medanic (6). In his paper the controls were assumed to be of the form

$$u_1 = -F_1y, \qquad u_2 = -F_2y$$

and F_2, the gain of the leader is found by solving the following equations.

$$A_c'M_1 + M_1A_c + M_1S_{11}M_1 + F_2'R_{12}F_2 + Q_1 = 0. \tag{1a}$$

$$A_c'M_2 + M_2A_c + M_1S_{21}M_1 + F_2'R_{22}F_2 + Q_2 = 0. \tag{1b}$$

$$N_1A_c' + A_cN_1 - S_{11}M_2N_2 - N_2M_2S_{11} + S_{21}M_1N_2 + N_2M_1S_{21} = 0, \quad (1c)$$

$$N_2A_c' + A_cN_2 + I = 0, \quad (1d)$$

$$R_{12}F_2N_1 + R_{22}F_2N_2 - B_2'(M_1N_1 + M_2N_2) = 0, \quad (1e)$$

where

$$F_1 = R_{11}^{-1}B_1'M_1,$$

$$A_c = A - S_{11}M_1 - B_2F_2,$$

$$S_{ij} = B_jR_{jj}^{-1}R_{ij}R_{jj}^{-1}B_j',$$

$$A = \begin{bmatrix} A_{11} & A_{12} \\ \dfrac{A_{21}}{\mu} & \dfrac{A_{22}}{\mu} \end{bmatrix}, B_i = \begin{bmatrix} B_{1i} \\ \dfrac{B_{2i}}{\mu} \end{bmatrix}, Q_i = \begin{bmatrix} Q_{i1} & Q_{i2} \\ Q_{i2}' & Q_{i3} \end{bmatrix}.$$

In general by letting $\mu \to 0$ in the full order system we change the meaning of the vector z from a state variable to a variable which depends on x. So if we solve the resulting slow optimization problem, we will have a change in information structure. To avoid this change in information structure, we solve the problem as an output feedback problem, where we constrain the feedback to be taken from x and z. This is clearly shown in Section IV.

In the following sections we will show a procedure to get a well-posed solution of the problem depending on reduced order systems while both x and z are available for measurements for both players. Let

$$u_{1op} = -L_{11}x - L_{12}z,$$

$$u_{2op} = -L_{21}x - L_{22}z.$$

The follower will find L_{12} by minimizing the fast part of his optimization function while the fast part of the system is given and he will find L_{11} by minimizing his modified slow optimization function. The leader will find his gains L_{21} and L_{22} by minimizing his slow part of the optimization function under the constraints that the follower applies the above procedure and the slow part of the system is given.

III. The Fast Optimization Problem for the Follower

The follower can find the gain L_{12} by minimizing the fast part of his performance index which is

$$\bar{J}_{1f} = \underset{z_f(0)}{E}\left[\frac{1}{2}\int_0^\infty (z_f'Q_{13}z_f + u_{1f}'R_{11}u_{1f} + u_{2f}'R_{12}u_{2f})\,dt\right],$$

given that

$$\mu\frac{dz_f}{dt} = A_{22}z_f + B_{21}u_{1f} + B_{22}u_{2f}$$

and

$$u_{1f} = -L_{12}z_f,$$
$$u_{2f} = -L_{22}z_f.$$

Substituting for u_{1f}, u_{2f}, we get

$$\bar{J}_{1f} = \underset{z_f(0)}{E}\left[\tfrac{1}{2}\int_0^\infty z_f'(Q_{13} + L_{12}'R_{11}L_{12} + L_{22}'R_{12}L_{22})z_f\,dt\right].$$
$$\mu\dot{z}_f = (A_{22} - B_{21}L_{12} - B_{22}L_{22})z_f = \hat{A}_{22}z_f.$$

Solving the problem, we get the following necessary conditions

$$L_{12} = R_{11}^{-1}B_{21}'K_{13}, \tag{2}$$
$$\hat{A}_{22}'K_{13} + K_{13}\hat{A}_{22} + K_{13}\bar{S}_{13}K_{13} + L_{22}'R_{12}L_{22} + Q_{13} = 0, \tag{3}$$

where

$$\bar{S}_{13} = B_{21}R_{11}^{-1}B_{21}'.$$

IV. Slow Optimization Problem for the Follower

The follower can find L_{11} by the following procedure. Letting $\mu \to 0$ in the system considered we obtain

$$\dot{x}_s = A_{11}x_s + A_{12}z_s + B_{11}u_{1s} + B_{12}u_{2s},$$
$$0 = A_{21}x_s + A_{22}z_s + B_{21}u_{1s} + B_{22}u_{2s}$$

and if we constrain the controls to be of the form

$$u_{1s} = -L_{11}x_s - L_{12}z_s,$$
$$u_{2s} = -L_{21}x_s - L_{22}z_s$$

and substitute for u_{1s}, u_{2s}, we obtain

$$z_s = -(A_{22} - B_{21}L_{12} - B_{22}L_{22})^{-1}(A_{21} - B_{21}L_{11} - B_{22}L_{21})x_s.$$

Assuming that $(A_{22} - B_{21}L_{12} - B_{22}L_{22})$ is non-singular and substituting for u_{1s}, u_{2s}, z_s in the differential equation, we obtain

$$\dot{x}_s = [A_{11} - B_{11}L_{11} - B_{12}L_{21} - (A_{12} - B_{11}L_{12} - B_{12}L_{22})(A_{22} - B_{21}L_{12} - B_{22}L_{22})^{-1}$$
$$-(A_{21} - B_{21}L_{11} - B_{22}L_{21})]x_s$$

or

$$\dot{x}_s = A_0x_s, \qquad x_s(t) = \phi(t, 0)x(0),$$

where

$$A_0 = \hat{A}_{11} - \hat{A}_{12}\hat{A}_{22}^{-1}\hat{A}_{21},$$
$$\hat{A}_{11} = A_{11} - B_{11}L_{11} - B_{12}L_{21}, \quad \hat{A}_{12} = A_{12} - B_{11}L_{12} - B_{12}L_{22},$$
$$\hat{A}_{21} = A_{21} - B_{21}L_{11} - B_{22}L_{21}, \quad \hat{A}_{22} = A_{22} - B_{21}L_{12} - B_{22}L_{22}.$$

Substituting for u_{1s}, u_{2s}, z_s in the optimization function of the follower, we obtain

$$\bar{J}_{1s} = \underset{x_0}{E}\left[\tfrac{1}{2}\int_0^\infty x_0'\{\phi'(t,0)[Q_{11} + L_{11}'R_{11}L_{11} + L_{21}'R_{12}L_{21}\right.$$
$$-(Q_{12} + L_{11}'R_{11}L_{12} + L_{21}'R_{12}L_{22})$$
$$\times \hat{A}_{22}^{-1}\hat{A}_{21} - \hat{A}_{21}'(\hat{A}_{22}^{-1})'(Q_{12}' + L_{12}'R_{11}L_{11} + L_{22}'R_{12}L_{21}) + \hat{A}_{21}'(\hat{A}_{22}^{-1})'$$
$$\left.\times(Q_{13} + L_{12}'R_{11}L_{12} + L_{22}'R_{12}L_{22})\hat{A}_{22}^{-1}\hat{A}_{21}]\phi(t,0)\}x_0\,dt\right].$$

Applying the same procedure as in the output regulator problem (10), while using the assumption that $E(x_0x_0') = I$, we obtain

$$\bar{J}_{1s}(L_{11}, L_{12}) = \tfrac{1}{2}\,\mathrm{trace}\int_0^\infty \phi'(t,0)[\tilde{Q}_{11} - \tilde{Q}_{12}\hat{A}_{22}^{-1}\hat{A}_{21} - \hat{A}_{21}'(\hat{A}_{22}^{-1})'\tilde{Q}_{12}'$$
$$+ \hat{A}_{21}(\hat{A}_{22}^{-1})'\tilde{Q}_{13}\hat{A}_{22}^{-1}\hat{A}_{21}]\phi(t,0)\,dt,$$

where

$$\tilde{Q}_{11} = Q_{11} + L_{11}'R_{11}L_{11} + L_{21}'R_{12}L_{21},$$
$$\tilde{Q}_{12} = Q_{12} + L_{11}'R_{11}L_{12} + L_{21}'R_{12}L_{22},$$
$$\tilde{Q}_{13} = Q_{13} + L_{12}'R_{11}L_{12} + L_{22}'R_{12}L_{22}.$$

Finding $\partial\bar{J}_{1s}/\partial L_{11}$ and putting it equal to zero we obtain

$$R_{11}L_{11} - R_{11}L_{12}\hat{A}_{22}^{-1}\hat{A}_{21} + B_{21}'(\hat{A}_{22}^{-1})'\tilde{Q}_{12}' - B_{21}'(\hat{A}_{22}^{-1})'\tilde{Q}_{13}\hat{A}_{22}^{-1}\hat{A}_{21} = \tilde{B}_1'K_{11}, \tag{4}$$

where

$$\tilde{B}_1' = B_{11}' - B_{21}'(\hat{A}_{22}^{-1})'(\hat{A}_{12})'$$

and K_{11} is the solution of

$$K_{11}A_0 + A_0'K_{11} + \tilde{Q}_{11} - \tilde{Q}_{12}\hat{A}_{22}^{-1}\hat{A}_{21} - \hat{A}_{21}'(\hat{A}_{22}^{-1})'\tilde{Q}_{12}'$$
$$+ \hat{A}_{21}'(\hat{A}_{22}^{-1})'\tilde{Q}_{13}\hat{A}_{22}^{-1}\hat{A}_{21} = 0. \tag{5}$$

Substituting for L_{12} obtained from Eqs. (2), (3) in Eqs. (4), (5), we can find L_{11}.

Comments

(1) Finding $\partial\bar{J}_{1s}/\partial L_{12}$ and letting it equal to zero will lead to the same equations as (4), (5). This is due to the fact that $z_s(t)$ is a linear function of $x_s(t)$.

(2) If we constrain $u_{1s} = -L_{11}x_s$ and apply the same procedure, the formulation is ill-posed.

V. The Leader Problem

Before describing how the leader can find L_{21}, L_{22}, it is advantageous to change the form of Eqs. (4) and (5) by using Eqs. (2) and (3) and by letting L_{11} to be of the following form

$$L_{11} = R_{11}^{-1}(B'_{11}K_{11} + B'_{21}K'_{12}).$$

Then after some straightforward but lengthy algebra Eq. (4) leads to

$$K_{12} = -[Q_{12} + L'_{21}R_{12}L_{22} + K_{11}A_{12} - K_{11}B_{12}L_{22} + \hat{A}'_{21}K_{13}]\hat{A}_{22}^{-1}, \qquad (6)$$

where

$$\hat{A}_{21} = A_{21} - \bar{S}'_{12}K_{11} - B_{22}L_{21},$$
$$\bar{S}_{12} = B_{11}R_{11}^{-1}B'_{21}, \qquad \bar{S}_{11} = B_{11}R_{11}^{-1}B'_{11}$$

and Eq. (5) becomes

$$K_{11}A_{11} + A'_{11}K_{11} - K_{11}\bar{S}_{11}K_{11} - K_{11}B_{12}L_{21} - L'_{21}B'_{12}K_{11} + Q_{11} - K_{12}\bar{S}_{13}K'_{12} + K_{12}\hat{A}_{21} + \hat{A}'_{21}K'_{12} + L'_{21}R_{12}L_{21} = 0. \qquad (7)$$

By substituting for u_{1s}, u_{2s}, z_s in the slow part of the leader's optimization function, we obtain

$$\bar{J}_{2s} = \tfrac{1}{2} \mathop{E}_{x_0}\left[\int_0^\infty x_s'[\hat{Q}_{21} - \hat{Q}_{22}\hat{A}_{22}^{-1}\hat{A}_{21} - \hat{A}'_{21}(\hat{A}_{22}^{-1})'\hat{Q}'_{22} + \hat{A}'_{21}(\hat{A}_{22}^{-1})'\hat{Q}_{23}\hat{A}_{22}^{-1}\hat{A}_{21}]x_s(t)\,dt\right],$$

where

$$\hat{Q}_{21} = Q_{21} + L'_{21}R_{22}L_{22} + K_{12}\bar{S}'_{22}K_{11} + K_{12}\bar{S}_{23}K'_{12} + K_{11}\bar{S}_{21}K_{11} + K_{11}\bar{S}_{22}K'_{12},$$
$$\hat{Q}_{22} = Q_{22} + L'_{21}R_{22}L_{22} + K_{12}\bar{S}_{23}K_{13} + K_{11}\bar{S}_{22}K_{13},$$
$$\hat{Q}_{23} = Q_{23} + L'_{22}R_{22}L_{22} + K_{13}\bar{S}_{23}K_{13},$$
$$\bar{S}_{21} = B_{11}R_{11}^{-1}R_{21}R_{11}^{-1}B'_{11}, \bar{S}_{22} = B_{11}R_{11}^{-1}R_{21}R_{11}^{-1}B'_{21},$$
$$S_{23} = B_{21}R_{11}^{-1}R_{21}R_{11}^{-1}B'_{21}.$$

Let

$$\bar{J}_{2s} = \mathop{E}_{x(0)}[\tfrac{1}{2}x'(0)K_2x(0)],$$

where K_2 satisfies

$$A'_0K_2 + K_2A_0 + \hat{Q}_{21} - \hat{Q}_{22}\hat{A}_{22}^{-1}\hat{A}_{21} - \hat{A}'_{21}(\hat{A}_{22}^{-1})'\hat{Q}'_{22} + \hat{A}'_{21}(\hat{A}_{22}^{-1})'\hat{Q}_{23}\hat{A}_{22}^{-1}\hat{A}_{21} = 0. \qquad (8)$$

So the leader has to minimize $\bar{J}_{2s}$ under the following constraints:

$$\hat{A}'_{22}K_{13} + K_{13}\hat{A}_{22} + K_{13}\bar{S}_{13}K_{13} + L'_{22}R_{12}L_{22} + Q_{13} = 0,$$
$$K_{11}A_{11} + A'_{11}K_{11} - K_{11}\bar{S}_{11}K_{11} - K_{11}B_{12}L_{21} - L'_{21}B'_{12}K_{11} + Q_{11} - K_{12}\bar{S}_{12}K'_{12} + K_{12}\hat{A}_{21} + \hat{A}'_{21}K'_{12} + L'_{21}R_{12}L_{21} = 0,$$
$$A'_0K_2 + K_2A_0 + \hat{Q}_{21} - \hat{Q}_{22}\hat{A}_{22}^{-1}\hat{A}_{21} - \hat{A}'_{21}(\hat{A}_{22}^{-1})'\hat{Q}'_{22} + \hat{A}'_{21}(\hat{A}_{22}^{-1})'\hat{Q}_{23}\hat{A}_{22}^{-1}\hat{A}_{21} = 0,$$

where

$$K_{12} = -[Q_{12} + L'_{21}R_{12}L_{22} + K_{11}A_{12} - K_{11}B_{12}L_{22} + \hat{A}'_{21}K_{13}]\hat{A}_{22}^{-1}.$$

The reader is referred to Appendix A for the derivation of the necessary conditions for the leader's minimization problem.

IV. Full Order Problem

In Eq. (1) we assume

$$M_i = \begin{bmatrix} M_{i1} & \mu M_{i2} \\ \mu M'_{i2} & \mu M_{i3} \end{bmatrix}, \quad N_i = \begin{bmatrix} N_{i1} & N_{i2} \\ N'_{i2} & N_{i3} \end{bmatrix}.$$

Substituting for M_1 in Eq. (1a) and letting $\mu \to 0$, we obtain the following

$$\begin{aligned} A'_{c11}M_{11}(0) + M_{11}(0)A_{c11} + A'_{c21}M'_{12}(0) + M_{12}(0)A_{c21} + M_{11}(0) \\ \times(\bar{S}_{11}M_{11}(0) + \bar{S}_{12}M'_{12}(0)) + M_{12}(0)(\bar{S}'_{12}M_{11}(0) \\ + \bar{S}_{13}M'_{12}(0)) + F'_{21}(0)R_{12}F_{21}(0) + Q_{11} = 0, \quad (9) \end{aligned}$$

$$\begin{aligned} M_{12}(0) = -[Q_{12} + F'_{21}(0)R_{12}F_{21}(0) + (A_{21} - \bar{S}_{12}M_{11}(0) - B_{22}F_{21}(0))'M_{13}(0) \\ + M_{11}A_{12} - M_{11}(0)B_{12}F_{22}(0)]A_{c22}^{-1}, \quad (10) \end{aligned}$$

$$A'_{c22}M_{13}(0) + M_{13}(0)A_{c22} + M_{13}(0)\bar{S}_{13}M_{13}(0) + F'_{22}(0)R_{12}F_{22}(0) + Q_{13} = 0, \quad (11)$$

where

$$\begin{aligned} A_{c11} &= A_{11} - \bar{S}_{11}M_{11}(0) - \bar{S}_{12}M'_{12}(0) - B_{12}F_{21}(0), \\ A_{c12} &= A_{12} - \bar{S}_{12}M_{13}(0) - B_{12}F_{22}(0), \\ A_{c21} &= A_{21} - \bar{S}'_{12}M_{11}(0) - \bar{S}_{13}M'_{12}(0) - B_{22}F_{21}(0), \\ A_{c22} &= A_{22} - \bar{S}_{13}M_{13}(0) - B_{22}F_{22}(0), \end{aligned}$$

assuming that A_{c22} is non-singular.

It is noticed that Eqs. (3), (6) and (7) are identical to Eqs. (11), (10) and (9) respectively where $M_{11}(0)$, $M_{12}(0)$, $M_{13}(0)$, $F_{21}(0)$, $F_{22}(0)$ replace K_{11}, K_{12}, K_{13}, L_{21}, L_{22}.

Substituting for M_2 in Eq. (1b) and letting $\mu \to 0$, we obtain

$$\begin{aligned} M_{21}(0)A_{c11} + A'_{c11}M_{21}(0) + M_{22}(0)A_{c21} + A'_{c22}M'_{22}(0) + Q_{21} + F'_{21}(0)R_{22}F_{21}(0) \\ + M_{11}(0)(\bar{S}_{21}M_{11}(0) + \bar{S}_{22}M'_{12}(0)) + M_{12}(0)(\bar{S}'_{22}M_{11}(0) + \bar{S}_{23}M'_{12}(0)) = 0, \quad (12) \end{aligned}$$

$$\begin{aligned} M_{21}(0)A_{c12} + M_{22}(0)A_{c22} + A_{c21}M_{23}(0) + M_{11}(0)\bar{S}_{22}M_{13}(0) + M_{12}(0)\bar{S}_{23}M_{13}(0) \\ + F'_{21}(0)R_{22}F_{22}(0) + Q_{22} = 0, \quad (13) \end{aligned}$$

$$A'_{c22}M_{23}(0) + M_{23}(0)A_{c22} + M_{13}(0)\bar{S}_{23}M_{13}(0) + F'_{22}(0)R_{22}F_{22}(0) + Q_{23} = 0. \quad (14)$$

From (13) we have

$$\begin{aligned} M_{22} = -[A'_{c21}M_{23}(0) + M_{21}(0)A_{c12} + M_{11}(0)S_{22}M_{13}(0) + M_{12}(0)S_{23}M_{13}(0) \\ + F'_{21}(0)R_{22}F_{22}(0) + Q_{22}]A_{c22}^{-1}. \end{aligned}$$

Substituting for M_{22} in (12) and using Eq. (14) we obtain an equation identical to (8), where $M_{21}(0)$, $M_{11}(0)$, $M_{12}(0)$, $M_{13}(0)$, $F_{21}(0)$, $F_{22}(0)$ replace K_2, K_{11}, K_{12}, K_{13}, L_{21}, L_{22} respectively.

Decomposing Eq. (1d) and letting $\mu \to 0$, we obtain

$$A_{c11}N_{21}(0)+N_{21}(0)A'_{c11}+A_{c12}N'_{22}(0)+N_{22}(0)A'_{c12}+I=0, \tag{15}$$

$$N_{21}(0)A'_{c21}+N_{22}(0)A'_{c22}=0, \tag{16}$$

$$A_{c21}N_{22}+N'_{22}A'_{c21}+A_{c22}N_{23}+N_{23}A'_{c22}=0. \tag{17}$$

From (16), we get $N_{22}(0)=-N_{21}(0)A'_{c21}(A^{-1}_{c22})'=N_{21}(0)V'$. From (17), we get $N_{23}(0)=VN_{21}(0)V'$, where

$$V=-A^{-1}_{c22}A_{c21}.$$

Substituting for N_{22} in (15), we obtain

$$(A_{c11}-A_{c12}A^{-1}_{c22}A_{c21})N_{21}(0)+N_{21}(0)(A_{c11}-A_{c12}A^{-1}_{c22}A_{c21})'+I=0,$$

which is identical to (A2), where $N_{21}(0)$, $M_{11}(0)$, $M_{12}(0)$, $M_{13}(0)$, $F_{21}(0)$, $F_{22}(0)$ replace P_2, K_{11}, K_{12}, K_{13}, L_{21}, L_{22} respectively. After decomposing (1c) and letting $\mu \to 0$, we obtain

$$\begin{aligned}
&[A_{c11}N_{11}(0)+A_{c12}N'_{12}(0)-(\bar{S}_{11}M_{21}(0)+\bar{S}_{12}M'_{22}(0))N_{21}(0)-\bar{S}_{12}M_{23}(0)N'_{22}(0)\\
&\quad+(\bar{S}_{21}M_{11}(0)+\bar{S}_{22}M'_{12}(0))N_{21}(0)+\bar{S}_{22}M_{13}(0)N'_{22}(0)]+[N_{11}(0)A'_{c11}\\
&\quad+N_{12}(0)A'_{c12}-N_{21}(0)(M_{21}(0)\bar{S}_{11}+M_{22}(0)\bar{S}'_{12})-N_{22}(0)M_{23}(0)\bar{S}'_{12}+N_{21}(0)\\
&\quad\times(M_{11}(0)\bar{S}_{21}+M_{12}(0)\bar{S}'_{22})+N_{22}(0)M_{13}(0)\bar{S}'_{22}]=0,
\end{aligned} \tag{18}$$

$$\begin{aligned}
&N_{11}(0)A'_{c21}+N_{12}(0)A'_{c22}+N_{21}(0)(M_{21}(0)\bar{S}_{21}+M_{22}(0)\bar{S}_{13})-N_{22}(0)M_{23}(0)\bar{S}_{13}\\
&\quad+N_{21}(0)(M_{11}(0)\bar{S}_{22}+M_{12}(0)\bar{S}_{23})+N_{22}(0)M_{13}(0)\bar{S}_{23}=0,
\end{aligned} \tag{19}$$

$$\begin{aligned}
&A_{c21}N_{12}(0)+A_{c22}N_{13}(0)+N'_{12}(0)A'_{c21}+N_{13}(0)A'_{c22}-T_3N_{22}(0)-N'_{22}(0)T'_3\\
&\quad-T_4N_{23}(0)-N_{23}(0)T'_4=0.
\end{aligned} \tag{20}$$

From (19) we obtain

$$N_{12}(0)=N_{21}(0)W'+N_{11}(0)V',$$

where

$$\begin{aligned}
W&=A^{-1}_{c22}(T_3+T_4V),\\
T_3&=\bar{S}_{13}M'_{22}(0)+\bar{S}'_{12}M_{21}(0)-\bar{S}_{23}M'_{12}(0)-\bar{S}'_{22}M_{11}(0),\\
T_4&=\bar{S}_{13}M_{23}(0)-\bar{S}_{23}M_{13}(0).
\end{aligned}$$

Substituting for $N_{12}(0)$, $M_{22}(0)$, $N_{22}(0)$ in (18), we get an equation identical to (A1) where $N_{11}(0)$, $N_{21}(0)$, $M_{21}(0)$, $M_{11}(0)$, $F_{21}(0)$, $F_{22}(0)$, $M_{13}(0)$, $M_{12}(0)$ replace P_1, P_2, K_2, K_{11}, L_{21}, L_{22}, K_{13}, K_{12}, respectively.

Substituting for $N_{12}(0)$, $N_{23}(0)$, $M_{22}(0)$ in (20) and using Eq. (14), we get an equation identical to (A5), where $N_{13}(0)$, $N_{11}(0)$, $N_{21}(0)$, $M_{21}(0)$, $M_{11}(0)$, $M_{12}(0)$, $M_{13}(0)$, $F_{21}(0)$, $F_{22}(0)$ replace P_3, P_1, P_2, K_2, K_{11}, K_{12}, K_{13}, L_{21}, L_{22} respectively.

Decomposing (1e) and letting $\mu \to 0$, we have

$$R_{12}(F_{21}(0)N_{11}(0)+F_{22}(0)N'_{12}(0))+R_{22}(F_{21}(0)N_{21}(0)$$
$$+F_{22}(0)N'_{22}(0))-B'_{12}M_{11}(0)N_{11}(0)-B'_{22}(M'_{12}(0)N_{11}(0)+M_{13}(0)N'_{12}(0))$$
$$-B'_{12}M_{21}(0)N_{21}(0)-B'_{22}(M'_{22}(0)N_{21}(0)+M_{23}(0)N'_{22}(0))=0, \quad (21)$$

$$(R_{12}F_{21}(0)-B'_{12}M_{11}(0)-B'_{22}M'_{12}(0))N_{12}(0)$$
$$+(R_{22}F_{21}(0)-B'_{12}M_{21}(0)-B'_{22}M'_{22}(0))N_{22}(0)$$
$$+(R_{12}F_{22}(0)-B'_{22}M_{13}(0))N_{13}(0)+(R_{22}F_{12}\ 0)-B'_{22}M_{23}(0)]N_{23}(0)=0. \quad (22)$$

Substituting for $N_{12}(0)$, $N_{22}(0)$, $M_{22}(0)$ and $N_{23}(0)$ in Eqs. (21) and (22) and using Eq. (14), we get Eq. (21) identical to (A3) and Eq. (22) identical to (A4), where $N_{11}(0)$, $N_{21}(0)$, $M_{21}(0)$, $M_{11}(0)$, $M_{12}(0)$, $M_{13}(0)$, $F_{21}(0)$ $F_{22}(0)$ and $N_{13}(0)$ replace P_1, P_2, K_2, K_{11}, K_{12}, K_{13}, L_{21}, L_{22} and P_3 respectively.

To compare the performance indices resulting from solving the full order problem with the ones resulting from the reduced order solution we need the following assumptions:

(a) The fast optimization problem of the follower has a unique stabilizing solution. In other words there exists a unique K_{13} which is a solution of Eq. (3) for each L_{22} applied such that $\lambda(\hat{A}_{22})<0$.

(b) The slow optimization problem of the follower has a unique solution after substituting for L_{12} from the fast problem, i.e. Eqs. (6) and (7) have a unique solution for K_{11}, K_{12}.

(c) The leader optimization problem has a unique stabilizing solution, i.e. there exists a unique pair L_{21} and L_{22} as a solution of the set of Eqs. (3), (6), (7), (8) and (A1)–(A5) such that $\lambda(A_0)<0$.

Theorem

If assumptions (a), (b) and (c) are satisfied then:

(1) $\lim_{\mu \to 0} (u_i - u_{i_{op}}) = 0, \quad \text{for} \quad i = 1, 2,$

(2) $\lim_{\mu \to 0} (J_i - J_i^*) = 0,$

where

$u_{i_{op}} = -L_{i1}x - L_{i2}z, \quad u_i = -F_{i1}x - F_{i2}z,$
J_i^* is the performance index when u_1 and u_2 are used,
J_i is the performance index when $u_{1_{op}}$ and $u_{2_{op}}$ are used.

Proof: (1) It was shown that $M_{11}(0)$, $M_{12}(0)$, $M_{13}(0)$, $M_{21}(0)$, $N_{11}(0)$, $N_{21}(0)$, $N_{13}(0)$, $F_{21}(0)$, $F_{22}(0)$ replace K_{11}, K_{12}, K_{13}, K_2, P_1, P_2, P_3, L_{21}, L_{22} respectively in the equations and if the uniqueness assumptions are satisfied, then we have unique values of K_{11}, K_{12}, K_{13}, K_2, P_1, P_2, P_3, L_{21}, L_{22} and $K_{11}=M_{11}(0)$, $K_{12}=M_{12}(0)$, $K_{13}=M_{13}(0)$, $F_{21}(0)=L_{21}$, $F_{22}(0)=L_{22}$, $K_2=M_{21}(0)$, $P_1=N_{11}(0)$, $P_2=N_{21}(0)$, $P_3=N_{13}(0)$.

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For the follower:

$$u_{1ap} = -L_{11}x - L_{12}z.$$

Substituting for L_{11}, L_{12}, we obtain

$$u_{1ap} = -R_{11}^{-1}(B'_{11}K_{11} + B'_{21}K'_{12})x - R_{11}^{-1}B'_{21}K_{13}z.$$

But the exact control of the follower is:

$$u_1 = -R_{11}^{-1}\left[B'_{11}\ \frac{B'_{21}}{\mu}\right]\begin{bmatrix} M_{11} & \mu M_{12} \\ \mu M'_{12} & \mu M_{13} \end{bmatrix}\begin{bmatrix} x \\ z \end{bmatrix},$$
$$= -R_{11}^{-1}[B'_{11}(M_{11}x + \mu M_{12}z) + B'_{21}(M'_{12}x + M_{13}z)],$$
$$= -R_{11}^{-1}[(B'_{11}M_{11} + B'_{21}M'_{12})x + \mu B'_{11}M_{12}z + B'_{21}M_{13}z].$$

Clearly,

$$\lim_{\mu \to 0} u_1 = \lim_{\mu \to 0} u_{1ap}.$$

For the leader:

$$u_2 = -F_{21}x - F_{22}z,$$
$$u_{2ap} = -L_{21}x - L_{22}z.$$

Clearly,

$$\lim_{\mu \to 0} u_2 = \lim_{\mu \to 0} u_{2ap}.$$

(2) When the exact controls, $u_1 = -F_1 y$ and $u_2 = -F_2 y$ are used the resulted performance index $J_i^* = \frac{1}{2}y'_0 M_i y_0$ where M_i is given by Eqs. (1a), (1b). If u_{1ap} and u_{2ap} are used, where

$$u_{1ap} = -R_{11}^{-1}B'_1K_1 y, \quad K_1 = \begin{bmatrix} k_{11} & 0 \\ \mu K'_{12} & \mu K_{13} \end{bmatrix},$$
$$u_{2ap} = -L_2 y,$$

we will have J_i as the performance index, where $J_i = \frac{1}{2}y'_0 W_i y_0$ and W_1, W_2 satisfy the following equations:

$$W_1(A - S_{11}K_1 - B_2L_2) + (A - S_{11}K_1 - B_2L_2)'W_1 + Q_1 + K_1S_{11}K_1 + L'_2R_{12}L_2 = 0, \tag{23}$$

$$W_2(A - S_{11}K_1 - B_2L_2) + (A - S_{11}K_1 - B_2L_2)'W_2 + Q_2 + K_1S_{21}K_1 + L'_2R_{22}L_2 = 0. \tag{24}$$

Subtracting (23) from (24) and (1a) from (1b), we find that

$$P_1 = W_1 - M_1 \text{ and } P_2 = W_2 - M_2$$

satisfy

$$P_1(A - S_{11}K_1 - B_2L_2) + (A - S_{11}K_1 - B_2L_2)'P_1 + (K_1 - M_1)'S_{11}(K - M_1)$$
$$+ M_1B_2(L_2 - F_2) + (L_2 - F_2)'B'_2M_1 + L'_2R_{12}L_2 - F'_2R_{12}F_2 = 0. \tag{25}$$

$$P_2(A-S_{11}K_1-B_2L_2)+(A-S_{11}K_1-B_2L_2)'P_2+K_1S_{21}K_1-M_1S_{21}M_1$$
$$-M_2S_{11}(K_1-M_1)-(K_1-M_1)'S_{11}M_2-M_2B_2(L_2-F_2)$$
$$-(L_2-F_2)'B_2'M_2+L_2'R_{22}L_2-F_2'R_{22}F_2=0. \quad (26)$$

Taking

$$P_i=\begin{bmatrix} P_{i1} & \mu P_{i2} \\ \mu P_{i2}' & \mu P_{i3} \end{bmatrix}$$

and substituting for P_1 and P_2 in Eqs. (25) and (26) respectively and setting $\mu \to 0$, we obtain

$$P_{i1}\hat{A}_{11}+P_{i2}\hat{A}_{21}+\hat{A}_{11}'P_{i1}+\hat{A}_{21}'P_{i2}'=0, \quad (27)$$

$$P_{i1}\hat{A}_{12}+P_{i2}\hat{A}_{22}+\hat{A}_{21}'P_{i3}=0 \quad (28)$$

$$P_{i3}\hat{A}_{22}+\hat{A}_{22}'P_{i3}=0. \quad (29)$$

Since $\hat{A}_{22}$ is stable, $P_{i3}=0$ is the unique solution of Eq. (29). From (28) we have

$$P_{i2}=-[P_{i1}\hat{A}_{12}]\hat{A}_{22}^{-1}.$$

Substituting P_{i2} in (27), we obtain

$$P_{i1}(\hat{A}_{11}-\hat{A}_{12}\hat{A}_{22}^{-1}\hat{A}_{21})+(\hat{A}_{11}-\hat{A}_{12}\hat{A}_{22}^{-1}\hat{A}_{21})'P_{i1}=0$$

or $P_{i1}A_0+A_0P_{i1}=0$. Since A_0 is stable, $P_{i1}=0$. Thus

$$P_{ij}=0 \quad \text{for} \quad i=1,2, \quad j=1,2,3.$$

Conclusion

In the method described in this paper we managed to decrease the amount of computation and alleviate the numerical stiffness which we would have if we solve the full order problem. In this method the follower finds his gains in a way similar to the hierarchical reduction scheme described in (8). That is, the fast gain is obtained from solving a fast optimization problem independent of the slow information and the slow gain is obtained from solving a modified slow optimization problem. The leader finds his gains by minimizing the slow part of his cost function under the constraint that the follower applies the above procedure.

It was shown that if we apply the strategies obtained from this method to the full order system, the resulting cost functions will have the same limits as the cost functions for the same full order system if the full order optimal strategies are applied.

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Appendix A

Applying the matrix minimum principle (12) to the leader optimization problem we get the following set of matrix algebraic equations:

$$
\begin{aligned}
&P_1A'_{11} - P_1K_{11}\bar{S}_{11} - P_1L'_{21}B'_{12} - P_1K_{12}\bar{S}_{13}\pi'_1 - P_1K_{12}\bar{S}'_{12} + P_1\bar{A}'_{21}\pi'_1 + P_2K_2\pi_2 \\
&\quad + P_2K_{12}\bar{S}'_{22} + P_2K_{12}\bar{S}_{23}\pi'_1 + P_2K_{11}\bar{S}_{21} + P_2K_{11}\bar{S}_{22}\pi'_1 + P_2\hat{Q}_{22}\hat{A}_{22}^{-1}(\bar{S}'_{12} + \bar{S}_{13}\pi'_1) \\
&\quad - P_2\hat{A}'_{21}(\hat{A}_{22}^{-1})'\cdot(K_{13}\bar{S}'_{23} + K_{13}\bar{S}_{23}\pi'_1) - P_2\hat{A}'_{21}(\hat{A}_{22}^{-1})'\hat{Q}_{23}\hat{A}_{22}^{-1}(\bar{S}'_{12} + \bar{S}_{13}\pi'_1) \\
&\quad + A_{11}P_1 - \bar{S}_{11}K_{11}P_1 - B_{12}L_{21}P_1 - \pi_1\bar{S}_{13}K'_{12}P_1 - \bar{S}_{12}K_{12}P_1 + \pi_1\bar{A}_{21}P_1 \\
&\quad + \pi'_2K_2P_2 + \pi_1\bar{S}'_{22}K_{11}P_2 + \pi_1\bar{S}_{23}K'_{12}P_2 + \bar{S}_{21}K_{11}P_2 + \bar{S}_{22}K'_{12}P_2 \\
&\quad - (\pi_1\bar{S}_{23}K_{13} + \bar{S}_{22}K_{13})\hat{A}_{22}^{-1}A_{21}P_2 + (\pi_1\bar{S}_{13} + \bar{S}_{12})\cdot(\hat{A}_{22}^{-1})'\hat{Q}'_{22}P_2 \\
&\quad - (\bar{S}_{12} + \pi_1\bar{S}_{13})(\hat{A}_{22}^{-1})'\hat{Q}_{23}(\hat{A}_{22}^{-1})\hat{A}_{21}P_2 = 0,
\end{aligned} \tag{A1}
$$

$$
P_2A'_0 + A_0P_2 = 0, \tag{A2}
$$

$$
\begin{aligned}
&-B'_{12}K_{11}P_1 - \pi_3\bar{S}_{13}K'_{12}P_1 + \pi_3\bar{A}_{21}P_1 - B'_{22}K'_{12}P_1 + \pi'_4K_2P_2 + R_{22}L_{21}P_2 \\
&\quad + \pi_3\bar{S}_{23}K'_{12}P_2 + \pi_3\bar{S}'_{22}K_{11}P_2 + R_{12}L_{21}P_1 - (R_{22}L_{22} + \pi_3\bar{S}_{23}K_{13})\hat{A}_{22}^{-1}\hat{A}_{21}P_2 \\
&\quad + (\pi_3\bar{S}_{13} + B'_{22})(\hat{A}_{22}^{-1})'\hat{Q}'_{22}P_2 - (B'_{22} + \pi_3\bar{S}_{13})(\hat{A}_{22}^{-1})'\hat{Q}_{23}\hat{A}_{22}^{-1}\hat{A}_{21}P_2 = 0,
\end{aligned} \tag{A3}
$$

$$
\begin{aligned}
&-\pi'_5P_1K_{12}\bar{S}_{13}(\hat{A}_{22}^{-1})' + \pi'_5P_1\bar{A}'_{21}(\hat{A}_{22}^{-1})' + \pi'_5P_2K_2\pi_6 + \hat{B}'_2K_2P_2\hat{A}'_{21}(\hat{A}_{22}^{-1})' \\
&\quad + \pi'_5P_2K_{11}\bar{S}_{22}(\hat{A}_{22}^{-1})' + \pi'_5P_2K_{12}\bar{S}_{23}(\hat{A}_{22}^{-1})' - R_{22}L_{21}P_2\hat{A}_{21}\ (A_{22}^{-1})' \\
&\quad - \pi'_5P_2\hat{A}'_{21}(\hat{A}_{22}^{-1})'K_{13}\bar{S}_{23}(\hat{A}_{22}^{-1})' - B'_{22}(\hat{A}_{22}^{-1})'\hat{Q}'_{22}P_2\hat{A}'_{21}(\hat{A}_{22}^{-1})'
\end{aligned}
$$

$$
\begin{aligned}
&+\pi_5' P_2 \hat{Q}_{22} \hat{A}_{22}^{-1} \bar{S}_{13} (\hat{A}_{22}^{-1})' - \pi_5' P_2 \hat{A}_{21}' (\hat{A}_{22}^{-1})' \hat{Q}_{23}' (\hat{A}_{22}^{-1}) \bar{S}_{13} (\hat{A}_{22}^{-1})' \\
&+ B_{22}' (\hat{A}_{22}^{-1})' \hat{Q}_{23} (\hat{A}_{22}^{-1}) \hat{A}_{21} P_2 \hat{A}_{21}' \hat{A}_{22}')' + R_{22} L_{22} (\hat{A}_{22}^{-1}) \hat{A}_{21} P_2 \hat{A}_{21}^{-1} (\hat{A}_{22}^{-1})' \\
&+ [R_{12} L_{22} - B_{22}' K_{13}] P_3 = 0,
\end{aligned} \tag{A4}
$$

$$
\begin{aligned}
&-\hat{A}_{22}^{-1} \hat{A}_{21} P_1 \hat{A}_{21}' + \hat{A}_{22}^{-1} \bar{S}_{13} K_{12}' P_1 \hat{A}_{21}' + \hat{A}_{21} P_2 K_2' (\bar{S}_{12} - \hat{A}_{12} \hat{A}_{22}^{-1} \bar{S}_{13}) (\hat{A}_{22}^{-1})' \\
&+ \hat{A}_{22}^{-1} \hat{A}_{21} P_2 K_2 (\bar{S}_{12} - \hat{A}_{12} \hat{A}_{22}^{-1} \bar{S}_{13}) - \hat{A}_{22}^{-1} \hat{A}_{21} P_2 \hat{Q}_{22} \hat{A}_{22}^{-1} \bar{S}_{13} - \hat{A}_{21} P_2 \hat{Q}_{22} \hat{A}_{22}^{-1} \bar{S}_{13} (\hat{A}_{22}^{-1})' \\
&- \hat{A}_{22}^{-1} \hat{A}_{21} P_2 K_{11} \bar{S}_{22} - \hat{A}_{22}^{-1} \hat{A}_{21} P_2 K_{12} \bar{S}_{23} + \hat{A}_{21} P_2 \hat{A}_{21}' (\hat{A}_{22}^{-1})' K_{13} \bar{S}_{23} (\hat{A}_{22}^{-1})' \\
&+ \hat{A}_{21} P_2 \hat{A}_{21}' (\hat{A}_{22}^{-1})' \hat{Q}_{23} \hat{A}_{22}^{-1} \bar{S}_{13} (\hat{A}_{22}^{-1})' + (\hat{A}_{22}^{-1}) \hat{A}_{21} P_2 \hat{A}_{21}' (\hat{A}_{22}^{-1})' \hat{Q}_{23} \hat{A}_{22}^{-1} \bar{S}_{13} + (\hat{A}_{22}^{-1} \hat{A}_{21} P_2 \\
&\times \hat{A}_{21}' (\hat{A}_{22}^{-1})' K_{13} \bar{S}_{23} + \hat{A}_{22} P_3 - (\hat{A}_{22}^{-1}) \bar{S}_{22}' K_{11} P_2 \hat{A}_{21}' - \hat{A}_{22}^{-1} \bar{S}_{23} K_{12}' P_2 \hat{A}_{21}' - \hat{A}_{21} P_1 \bar{\bar{A}}_{21}' (\hat{A}_{22}^{-1})' \\
&+ \hat{A}_{21} P_1 K_{12} (\hat{A}_{22}^{-1})' + (\hat{A}_{22}^{-1}) (\bar{S}_{12} - \hat{A}_{12} \hat{A}_{22}^{-1} \bar{S}_{13})' K_2 P_2 \hat{A}_{21}' + (\bar{S}_{12} - \hat{A}_{12} \hat{A}_{22}^{-1} \bar{S}_{13})' K_2 P_2 \hat{A}_{21}' \\
&\times (A_{22}^{-1})' - \bar{S}_{13} (\hat{A}_{22}^{-1})' \hat{Q}_{22}' P_2 \hat{A}_{11}' (\hat{A}_{22}^{-1})' - (\hat{A}_{22}^{-1}) \bar{S}_{13} (\hat{A}_{22}^{-1})' \hat{Q}_{22}' P_2 \hat{A}_{21}' - \bar{S}_{22}' K_{11} P_2 \hat{A}_{21}' (\hat{A}_{22}^{-1})' \\
&- \bar{S}_{23} K_{12}' P_2 \hat{A}_{21}' (\hat{A}_{22}^{-1})' + (\hat{A}_{22}^{-1}) \bar{S}_{23} K_{13} (\hat{A}_{22}^{-1}) \hat{A}_{21} P_2 \hat{A}_{21}' + (\hat{A}_{22}^{-1}) \bar{S}_{13} (\hat{A}_{22}^{-1})' \hat{Q}_{23} (\hat{A}_{22}^{-1}) \\
&\times \hat{A}_{21} P_2 \hat{A}_{21}' + \bar{S}_{13} (\hat{A}_{22}^{-1})' \hat{Q}_{23} (\hat{A}_{22}^{-1}) \hat{A}_{21} P_2 \hat{A}_{21}' (\hat{A}_{22}^{-1})' + \bar{S}_{23} K_{13} (\hat{A}_{22}^{-1}) \hat{A}_{21} P_2 \hat{A}_{21}' (\hat{A}_{22}^{-1})' \\
&+ P_3 \hat{A}_{22}' - \hat{A}_{21} P_2 K_{11} \bar{S}_{22} (\hat{A}_{22}^{-1})' - \hat{A}_{21} P_2 K_{12} \bar{S}_{23} (\hat{A}_{22}^{-1})' = 0,
\end{aligned} \tag{A5}
$$

where

$\pi_1 = (-A_{12} + B_{12} L_{22} + \bar{S}_{12} K_{13}) \hat{A}_{22}^{-1}$,

$\pi_2 = -\bar{S}_{11} - \bar{S}_{12} \pi_1' + \hat{A}_{12} \hat{A}_{22}^{-1} (\bar{S}_{12}' + \bar{S}_{13} \pi_1')$,

$\pi_3 = (-R_{12} L_{22} + B_{22}' K_{13}) \hat{A}_{22}^{-1}$,

$\pi_4 = [-\bar{S}_{12} \pi_3' - B_{12} + \hat{A}_{12} \hat{A}_{22}^{-1} (B_{22} + \bar{S}_{12} \pi_3')]$,

$\pi_5 = -[L_{21}' R_{12} - K_{11} B_{12} + (Q_{12} + L_{21}' R_{12} L_{22} + K_{11} A_{12} - K_{11} B_{12} L_{22}$
$\quad + \hat{A}_{21}' K_{13}) \hat{A}_{22}^{-1} B_{22}]$,

$\pi_6 = -\bar{S}_{12} (\hat{A}_{22}^{-1})' + \hat{A}_{12} \hat{A}_{22}^{-1} \bar{S}_{13} (\hat{A}_{22}^{-1})'$, $\quad \bar{B}_2 = B_{12} - \hat{A}_{12} \hat{A}_{22}^{-1} B_{22}$.

So in order for the leader to find L_{21}, L_{22} he has to solve Eqs. (3), (6), (7), (8), and (A1)–(A5).